OPEN PROBLEM: VIOLATION OF LOCALITY FOR SCHRÖDINGER OPERATORS WITH COMPLEX POTENTIALS

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Abstract. We explain in which sense Schrödinger operators with complex potentials appear to violate locality (or Weyl's asymptotics), and we pose three open problems related to this phenomenon.

1. Background

We are interested in the semiclassical Schrödinger operator

$$-\hbar^2\Delta + V \quad \text{in} \quad L^2(\mathbb{R}^d)$$

with *complex-valued* potential V and semiclassical parameter $\hbar \in (0,1]$. We will assume throughout that the potential V is locally sufficiently regular, so that the operator can be defined as an m-sectorial operator, and that it decays at infinity (at least in some averaged sense), so that the spectrum of the operator in $\mathbb{C}\setminus[0,\infty)$ consists of isolated eigenvalues of finite algebraic multiplicities. We denote these eigenvalues by E_j, repeating an eigenvalue according to its algebraic multiplicity. We seek bounds on sums of the E_j in terms of integrals of the potential V.

1.1. Real-valued potentials.

We begin by reviewing the case of real-valued potentials V. If V is, say, continuous and compactly supported, then Weyl's asymptotic formula says that, for $\gamma \geq 0$,

$$\lim_{\hbar\to 0} \hbar^d \operatorname{Tr}(-\hbar^2\Delta + V)_-^\gamma = L_{\gamma,d}^{\mathrm{cl}} \int_{\mathbb{R}^d} V(x)_-^{\gamma+d/2}\mathrm{d}x, \tag{1}$$

where $L^{\mathrm{cl}}_{\gamma,d} = \int_{\mathbb{R}^d}(|\xi|^2-1)_-^\gamma \frac{\mathrm{d}\xi}{(2\pi)^d}$ is the semiclassical constant (see [10]) and $V(x)_- = \max(0,-V(x))$ is the negative part of $V(x)$. For $\gamma = 0$, the left-hand side of (1) is interpreted as the number of negative eigenvalues of $-\hbar^2\Delta + V$, and for $\gamma > 0$ it is called the Riesz mean of order γ.

A non-asymptotic bound that captures the correct order of magnitude of the Riesz means (1) in the asymptotic regime would be of the form

$$\operatorname{Tr}(-\hbar^2\Delta + V)_-^\gamma = \sum_j |E_j|^\gamma \leq L_{\gamma,d}\hbar^{-d}\int_{\mathbb{R}^d} V(x)_-^{\gamma+d/2}\mathrm{d}x, \tag{2}$$

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where $L_{\gamma,d}$ is some positive constant depending on γ, d, but independent of V and $\hbar > 0$. Note that necessarily $L_{\gamma,d} \geq L^{\mathrm{cl}}_{\gamma,d}$. For the sake of the following discussion, we have stated the inequality (2) for the semiclassical Schrödinger operator $-\hbar^2\Delta + V$, but a simple scaling argument shows that it is equivalent to the corresponding bound for $\hbar = 1$.

The validity of the bound (2) for $\gamma > 1/2$ for $d = 1$ and for $\gamma > 0$ in $d \geq 2$ is a celebrated result of Lieb and Thirring. The case $\gamma = 0$ in $d \geq 3$ is due Cwikel, Lieb and Rozenblum, and the case $\gamma = 1/2$ in $d = 1$ is due to Weidl. We refer to [10] for more background and a discussion of optimal constants. (For complex-valued potentials, we will have nothing to say about constants.)

1.2. **Complex-valued potentials.** We now turn our attention to complex-valued potentials. We are interested in bounds of a similar nature as (2). However, it is not at all obvious what the most natural analogues of these bounds should look like.

- The naive generalization
$$\sum_j |E_j|^\gamma \leq C_{\gamma,d}\hbar^{-d} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2} \mathrm{d}x \tag{3}$$
fails for $\gamma > 1/2$, even for a single eigenvalue [2].
- Frank, Laptev, Lieb and Seiringer [9] proved that (3) is valid for all eigenvalues outside an arbitrary fixed cone with a constant that becomes unbounded as the angle of the cone tends to zero. More precisely, they proved that for $\gamma \geq 1$, $\kappa > 0$,
$$\sum_{|\operatorname{Im} E_j| \geq \kappa \operatorname{Re} E_j} |E_j|^\gamma \leq C_{\gamma,d}(1+\kappa^{-1})^{\gamma+d/2}\hbar^{-d} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2} \mathrm{d}x. \tag{4}$$
Bögli [4] has shown that for $d = 1$ as $\kappa \to 0$ the order of divergence $\kappa^{-\gamma-d/2}$ of the constant in (4) is optimal.
- Averaging the bound (4) with respect to the opening angle κ, Demuth–Hansmann–Katriel [6] established the following inequality, valid for *all* eigenvalues,
$$\sum_j |E_j|^{-\sigma}\delta(E_j)^{\gamma+\sigma} \leq C_{\gamma,\sigma,d}\hbar^{-d} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2} \mathrm{d}x, \tag{5}$$
where $\gamma \geq 1$, $\sigma > d/2$ and where we set
$$\delta(E) := \operatorname{dist}(E, [0,\infty))$$
Since $\delta(E_j)/|E_j| \leq 1$, the bound (5) becomes stronger the smaller σ. Demuth–Hansmann–Katriel [7] asked whether (5) remains true for $\sigma = d/2$ and $\gamma > 0$ (with $\gamma \geq 1/2$ if $d = 1$). Bögli and Štampach [3] answered this question in the negative for $d = 1$ and $\gamma \geq 1/2$.

We note that the inequalities (4) and (5) reduce to the standard Lieb–Thirring inequalities when V is real-valued since in this case, $E_j < 0$ and thus $\delta(E_j) = |E_j|$.

Our first question concerns the assumption $\gamma \geq 1$ in (5).

Question 1. *Let $d \geq 1$, $0 < \gamma < 1$ (with $\gamma \geq 1/2$ if $d = 1$) and $\sigma > d/2$. Does there exist a constant $C_{\gamma,\sigma,d}$ such that for all $V \in L^{\gamma+d/2}(\mathbb{R}^d)$ the inequality*

$$\sum_j |E_j|^{-\sigma} \delta(E_j)^{\gamma+\sigma} \leq C_{\gamma,\sigma,d} \hbar^{-d} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2} \mathrm{d}x$$

holds for all $\hbar > 0$?

Note that in this question one may assume without loss of generality that $\hbar = 1$.

2. Violation of locality

For real-valued potentials, a key feature of the asymptotics (1), which is captured by the Lieb-Thirring inequalities (2) and is one of the reasons for their usefulness in applications, is *locality*: sums of eigenvalues are estimated by an integral involving the potential. Hence, two disjoint pieces of V contribute additively to the asymptotics (1) and to the upper bound (2). This is closely related to the appearance of $\hbar^{-d}$ on the right side of the inequality. For complex-valued potentials, these features are retained by the inequalities (4) and (5).

In this section we first discuss some examples in one dimension where eigenvalue sums are $\gg \hbar^{-d}$ as $\hbar \to 0$ and then we present Lieb–Thirring-type bounds of a different nature than (4) and (5), which, to some extent, capture a growth that is faster than $\hbar^{-d}$. In the following discussion, we will focus on the power of $\hbar^{-1}$ as an indication of the validity or the degree of violation of locality.

2.1. Examples of eigenvalue sums that are $\gg \hbar^{-d}$.

We consider eigenvalue sums as on the left side of (5), that is, of the form $\sum_j |E_j|^{-\sigma} \delta(E_j)^{\beta}$, and recall some examples from the literature where they are not bounded from above by a constant times $\hbar^{-d}$. So far, these examples are limited to dimension $d = 1$.

It is shown in [5, Theorem 4] that there exists $V \in L^1(\mathbb{R})$ and a constant $c > 0$ such that for all sufficiently small $\hbar > 0$,

$$\sum_j \delta(E_j) \geq c \left(\frac{\hbar^{-1}}{\log \frac{1}{\hbar}} \right)^2. \tag{6}$$

In the same vein, the example in [3] shows that there exists $V \in L^1 \cap L^\infty(\mathbb{R})$ (in fact, i times the characteristic function of an interval) such that for every $\beta \geq 1$ there is a constant $c_\beta > 0$ such that for all sufficiently small $\hbar > 0$,

$$\sum_j |E_j|^{-1/2} \delta(E_j)^{\beta} \geq c_\beta \hbar^{-1} \log \frac{1}{\hbar}. \tag{7}$$

In fact, by looking at the computations in [3] we see that for any $\beta \geq 1$, any $0 \leq \sigma < 1/2$ and any $\varepsilon > 0$ there is a constant $c_{\beta,\sigma,\varepsilon} > 0$ such that

$$\sum_j |E_j|^{-\sigma} \delta(E_j)^{\beta} \geq c_{\beta,\sigma,\varepsilon} \hbar^{-2+2\sigma+\varepsilon}. \tag{8}$$

For $\sigma = 0$ and $\beta = 1$ this is similar, but slightly worse than (6).

In particular, for any $\gamma \geq 1/2$ there is a potential $V \in L^{\gamma+1/2}(\mathbb{R})$ such that sums $\sum_j |E_j|^{-\sigma} \delta(E_j)^{\gamma+\sigma}$ are $\gg h^{-1}$ provided $\sigma \leq 1/2$. For $\gamma \geq 1$ this should be contrasted with the bound (5), which implies that this sum is $\lesssim \hbar^{-1}$ for any $\sigma > 1/2$. In this sense, $\sigma = 1/2$ appears to be a threshold for $\gamma \geq 1$. If the answer to Question 1 is affirmative in dimension $d = 1$, it is also a threshold for $1/2 \leq \gamma < 1$.

It is natural to wonder whether the above examples have analogues in higher dimensions.

Question 2. *Let $d \geq 2$, $\gamma > 0$ and $0 \leq \sigma \leq d/2$. Does there exist a $V \in L^{\gamma+d/2}(\mathbb{R}^d)$ such that*

$$\sum_j |E_j|^{-\sigma} \delta(E_j)^{\gamma+\sigma} \gg \hbar^{-d}$$

as $\hbar \to 0$?

Sabine Bögli has informed us that she has made progress towards an answer to this question.

One can also ask (for any $d \geq 1$) whether the violation of the $\hbar^{-d}$-bound is power-like (as in (6) and (8)) or logarithmic (as in (7)). In the first case, one might ask for the optimal power. We will see below that the lower bound in (6) is optimal up to powers of logarithms.

One may wonder whether the violation of locality holds for any generic choice of V or only for certain specially chosen ones. Similarly, to which extent does the power of $\hbar$ depend on the choice of the potential? Also, it would be interesting to understand the role of the purported threshold $\sigma = d/2$ for the validity/violation of an $\hbar^{-d}$-bound on $\sum_j |E_j|^{-\sigma}\delta(E_j)^{\gamma+\sigma}$.

An even vaguer question is to understand from a more conceptual point of view the deeper reason behind a potential violation of locality.

2.2. **Nonlocal Lieb–Thirring bounds.** A family of eigenvalue bounds that is different in nature from (4) and (5) has been obtained by Frank and Sabin [11], as well as by Frank [8]. These bounds only retain the scale-invariance but lose locality. They are of the form

$$\sum_j |E_j|^{\alpha} \left(\frac{\delta(E_j)}{|E_j|}\right)^{\beta} \leq C_{\alpha,\beta,\gamma,d}\Big(\hbar^{-d}\int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2}\mathrm{d}x\Big)^{\alpha/\gamma} \tag{9}$$

for certain values of α, β, γ, d (see below). As before, the result is equivalent to the corresponding result for $\hbar = 1$.

A general observation is that all known bounds of the type (9) have $\alpha/\gamma > 1$. This means that a sum of eigenvalues is bounded by a *power* of an integral, and this power is strictly greater than 1. This is what we mean by a *loss of locality*. Disjoint pieces of the potential no longer contribute additively to the bound on an eigenvalue sum. It also means that the power of $\hbar^{-1}$ that appears in the bound on the eigenvalue sum is strictly larger than the semiclassical power d.

An example from [8] is the following bound in $d = 1$,

$$\sum_j \delta(E_j) \leq C\Big(\hbar^{-1}\int_{\mathbb{R}} |V(x)|\mathrm{d}x\Big)^2, \tag{10}$$

Comparing this upper bound with the lower bound in (6), we see that (10) is sharp as $\hbar \to 0$ up to logarithms.

The following instances of (9) have been proved by Frank–Sabin [11] and Frank [8]. Notice that in (c), (d), (e), (f), the sum is restricted to j satisfying either $|E_j|^{\gamma} \leq \hbar^{-d}\int_{\mathbb{R}^d}|V|^{\gamma+d/2}\mathrm{d}x$ or $|E_j| \geq \hbar^{-d}\int_{\mathbb{R}^d}|V|^{\gamma+d/2}\mathrm{d}x$.

(a) $\alpha = 1/2$, $\beta = 1$, $0 < \gamma < d/(2(2d-1))$, $d \geq 2$ ([11, Thereom 16], see also [8, (1.5)]).

(b) $\alpha > (d-1)\gamma/(d/2-\gamma)$, $\beta = 1$, $d/(2(2d-1)) \leq \gamma \leq 1/2$, $d \geq 2$ ([11, Thereom 16], see also [8, (1.5)]).
(c) $\alpha = \beta > 2\gamma$, $\gamma > 1/2$, truncation $|E_j|^\gamma \leq \hbar^{-d} \int_{\mathbb{R}^d} |V|^{\gamma+d/2} \mathrm{d}x$ [8, (1.7)]).
(d) $0 < \alpha < \gamma(\gamma+d/2)$, $\beta > 2\gamma$, $\gamma > 1/2$, truncation $|E_j|^\gamma \geq \hbar^{-d} \int_{\mathbb{R}^d} |V|^{\gamma+d/2} \mathrm{d}x$ [8, (1.8)]).
(e) $\alpha = \beta = \gamma + d/2$, $\gamma \geq 1/2$ ($d = 1$) or $\gamma > 0$ ($d \geq 2$), truncation $|E_j|^\gamma \leq \hbar^{-d} \int_{\mathbb{R}^d} |V|^{\gamma+d/2} \mathrm{d}x$ [8, (1.9)]).
(f) $\alpha > \gamma$, $\beta = \gamma + d/2$, $\gamma \geq 1/2$ ($d = 1$) or $\gamma > 0$ ($d \geq 2$), truncation $|E_j|^\gamma \geq \hbar^{-d} \int_{\mathbb{R}^d} |V|^{\gamma+d/2} \mathrm{d}x$ [8, (1.10)]).

Note that (10) is a particular case of (e) in $d = 1$ since $|E_j|^{1/2} \leq C\hbar^{-1} \int_{\mathbb{R}} |V(x)| \mathrm{d}x$ by [1]. As we have seen, this inequality is sharp up to logarithms.

Question 3. *For which values of $d \geq 1$ and α, β, γ listed in (a)–(f) is* (9) *sharp up to a factor of $\hbar^\varepsilon$, for arbitrary $\varepsilon > 0$? Can one increase the parameter region where bounds of the type* (9) *are valid?*

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