Quasimodes for the Stationary Damped Wave Operator: Sharp Decay from Schrödinger Observability

Perry Kleinhenz

If you have any ideas or questions about this, please feel free to email me at pbklein@ilstu.edu

Question 1. Find a function $W \in L^\infty(\mathbb{T}^2)$, $W \geq 0, W \not\equiv 0$, a constant $C > 0$, and sequences $\{u_j\}_{j=1}^\infty \subset L^2(\mathbb{T}^2), \{\lambda_j\}_{j=1}^\infty \subset \mathbb{R}$, such that

$$\left|\left|(-\Delta + i\lambda_j W - \lambda_j^2)u_j\right|\right|_{L^2} \leq \frac{C}{|\lambda_j|} \left|\left|u_j\right|\right|_{L^2}, \quad \text{and } |\lambda_j| \to \infty.$$

These u_j are called quasimodes.

The context of this question is energy decay rates for the damped wave equation on the two-torus. For $W \in L^\infty(\mathbb{T}^2)$, the damped wave equation is

$$\begin{cases} (\partial_t^2 - \Delta + W\partial_t)u = 0, & (x,t) \in \mathbb{T}^2 \times \mathbb{R} \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) & \in H^2(\mathbb{T}^2) \times H^1(\mathbb{T}^2). \end{cases} \tag{1}$$

The energy of a solution is given by

$$E(u,t) = \frac{1}{2}\int_M |\partial_t u|^2 + |\nabla u|^2 dx.$$

The energy is non-increasing, so it is natural to ask for a rate $r(t) \to 0$ such that $E(u,t)^{1/2} \leq r(t)$ for all solutions u.

On the two-torus, when W is not identically zero, energy decay at a polynomial rate is guaranteed.

Theorem 0.1. *[AL14, Theorem 2.3] If $W \not\equiv 0$, then for all solutions of* (1)

$$E(u,t)^{1/2} \leq \frac{1}{\sqrt{t}} \left(\left|\left|u_0\right|\right|_{H^2} + \left|\left|u_1\right|\right|_{H^1}\right).$$

The result is actually more general, it can be stated in terms of Schödinger observability, which is satisfied when $W \not\equiv 0$ on $\mathbb{T}^2$. These energy decay rates are proved via resolvent estimates for the associated stationary damped wave operator $(-\Delta + i\lambda W - \lambda^2)$ via the following semigroup theory result.

Theorem 0.2. *[BT10][AL14, Proposition 2.4] The following are equivalent:*

1. *For all solutions of* (1)

$$E(u,t)^{1/2} \leq \frac{1}{t^{\alpha}} \left(||u_0||_{H^2} + ||u_1||_{H^1} \right).$$

2. *There exist* $C > 0, \lambda_0 \geq 0$ *such that*

$$\left|\left|(-\Delta + i\lambda W - \lambda^2)^{-1}\right|\right|_{L^2 \to L^2} \leq C|\lambda|^{\frac{1}{\alpha}-1}, \text{ for all } \lambda \in \mathbb{R}, |\lambda| \geq \lambda_0.$$

The result of [AL14] follows from a resolvent estimate of this form, but it is not known if the resolvent estimate is sharp. Finding the quasimodes described in Question One would show the resolvent estimate is sharp, and thus that the decay rate of $1/\sqrt{t}$ is sharp.

Hints

1. It is likely that W can be taken invariant in one direction, which permits the reduction of the problem to one of ordinary differential equations using Fourier series. When the boundary of the support of the damping is strictly convex it is sometimes possible to obtain improved resolvent estimates over the y-invariant case. Compare [Sun23, Kle19, DK20]. Said another way, y-invariant damping currently provides the best known quasimodes.

2. $W = \mathbb{1}_{[a,b]}(x)$ does not solve the open question. By [AL14, Sta17], the sharp resolvent estimate in that case is

$$\left|\left|(-\Delta + i\lambda \mathbb{1}_{[a,b]}(x) - \lambda^2)^{-1}\right|\right|_{L^2 \to L^2} \leq |\lambda|^{1/2}.$$

3. Taking a more regular damping likely will not work. When $W = (|x| - \sigma)_+^{\beta}$, $\beta \geq 0$, by [Kle19, DK20] the sharp resolvent estimate is

$$\left|\left|(-\Delta + i\lambda(|x| - \sigma)_+^{\beta} - \lambda^2)^{-1}\right|\right|_{L^2 \to L^2} \leq |\lambda|^{\frac{1}{\beta+2}}.$$

4. Taking a more singular damping might work, but remember the damping must be in L^{∞}. When $W = (|x| - \sigma)_+^{\beta}$, $\beta \in (-1, 0)$ the sharp resolvent estimate still grows like $|\lambda|^{\frac{1}{\beta+2}}$, and the desired $|\lambda|$ growth is approached as $\beta \to -1$. However, the a priori estimate of [AL14] only holds for bounded damping. Only a slower decay rate is guaranteed via Schrödinger observability when the damping is unbounded [KW22, KW23].

5. The sequence $\{\lambda_j\}_{j=1}^{\infty}$ can actually be taken complex, as long as $|\text{Im } (\lambda_j)| \leq \frac{1}{C|\text{Re } (\lambda_j)|^2}$.

References

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