ESTIMATES FOR THE RESOLVENT NEAR THE SPECTRUM

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Let A be a linear operator on a Banach space. Let K be a compact perturbation of A. The approximation numbers of K are defined by

$$\alpha_N(K) := \inf\{\|K-F\|\,,\ \mathrm{rank}(F) < N\}.$$

We consider only compact operators K with $\lim_{N\to\infty} \alpha_N(K) = 0$.

The objective is to estimate the numbers of eigenvalues of the perturbed operator $B := A + K$ in certain regions of the complex plane.

Let $\Omega_t = \{\lambda \in \mathbb{C}\,,\ |\lambda| > t\}$. Denote $\mathrm{spr}(A) := \max\{|\lambda|\,,\ \lambda \in \sigma(A)\}$ and assume $\mathrm{spr}(A) < t < s$. Denote by $n_B(s)$ the number of eigenvalues of B in Ω_s. In [**?**] we obtained

$$n_B(s) \le \frac{(2e)^{\frac{p}{2}}}{\log\frac{s}{t}} \frac{\sup_{\lambda\in\Omega_t}\|(\lambda-A)^{-1}\|^p}{\left(1-\alpha_{N+1}(K)\sup_{\lambda\in\Omega_t}\|(\lambda-A)^{-1}\|\right)^p} \sum_{j=1}^{N} \left(\alpha_{N+1}(K)+\alpha_j(K)\right)^p . \tag{1}$$

Here N has to be so large that

$$\alpha_{N+1}(K) \sup_{\lambda\in\Omega_t} \|(\lambda-A)^{-1}\| < 1.$$

The optimal result depends on the behavior of $\|(\lambda-A)^{-1}\|$ near the spectrum of A, i.e. on Ω_s. This is typical for many spectral considerations. It is also related to the pseudospectrum of A. For instance if $|\lambda| > \|A\|$ then

$$\|(\lambda-A)^{-1}\| \le \frac{1}{|\lambda|-\|A\|}$$

and therefore (1) becomes

$$n_B(s) \le \frac{(2e)^{\frac{p}{2}}}{\log\frac{s}{t}\,(t-(\|A\|+\alpha_{N+1}(K)))^p} \sum_{j=1}^{N} \left(\alpha_{N+1}(K)+\alpha_j(K)\right)^p .$$

Open problems:

i) Classify the operators for which the resolvent is polynomially bounded if $\lambda \to \sigma(A)$?

ii) Classify the operators for which one can find an $M \ge 1$ such that

$$\|(\lambda-A)^{-1}\| \le \frac{M}{\mathrm{dist}\,(\lambda,\sigma(A))}$$

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for all $\lambda \in \operatorname{res}(A)$ or

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda| - \operatorname{spr}(A)}$$

for $|\lambda| > \operatorname{spr}(A)$.
Remark: For instance in Hilbert spaces, the bound in ii) is true for normal operators with $M = 1$.