# The spectrum of second order multi-point problems

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# Multi-point problems

Consider the eigenvalue problem

$$-u'' = \lambda r u \quad \text{on } (-1,1), \tag{1}$$

where $r \in C^1[-1,1]$, $r > 0$ and $\lambda \in \mathbb{R}$, together with the multi-point boundary conditions

$$u(\pm 1) = \sum_{i=1}^{m^\pm} \alpha_i^\pm u(\eta_i^\pm), \tag{2}$$

where $m^\pm \geqslant 1$ are integers, and, for $i = 1, \ldots, m^\pm$:

$$\alpha_i^\pm \in \mathbb{R}, \quad \eta_i^\pm \in (-1,1).$$

An eigenvalue is a (real) number $\lambda$ for which (1)-(2) has a non-trivial solution $u$, called an eigenfunction.

# Functional setting, nodal sets

More precisely, we consider solutions as elements of

$$X := \{u \in C^2[-1,1] : u \text{ satisfies } (2)\}.$$

Similarly to Sturm-Liouville theory with separated boundary conditions, we get eigenfunctions with prescribed nodal properties.

For $\nu = \pm$ and $k = 1, 2, \ldots$, we define $T_k^\nu \subset X$ by: $u \in T_k^\nu \iff$

(a) $u'(\pm 1) \neq 0$ and $\nu u'(-1) > 0$;

(b) $u'$ has only simple zeros, and exactly $k$ zeros in $(-1,1)$;

(c) $u$ has a zero strictly between each consecutive zero of $u'$.

We also define $T_k = T_k^+ \cup T_k^-$, $k = 1, 2, \ldots$

NB A function $u \in T_k$ has at least $k-1$ and at most $k$ zeros in $(-1,1)$.

# Main result

It is convenient to use the notation

$$\alpha^{\pm} := (\alpha_1^{\pm}, \dots, \alpha_{m^{\pm}}^{\pm}) \in \mathbb{R}^{m^{\pm}}, \quad |\alpha^{\pm}| := \sum_{i=1}^{m^{\pm}} |\alpha_i^{\pm}|.$$

## Theorem 1 (Rynne-G. NA 2011)

*For any $r \in C^1[-1,1]$, $r > 0$, there exists $\gamma = \gamma(r) \in (0,1]$ such that if $|\alpha^{\pm}| < \gamma$ then all the eigenvalues are real and simple, and they form a strictly increasing sequence $\lambda_k = \lambda_k(r) > 0$, $k \geqslant 1$. Furthermore, each eigenvalue $\lambda_k$ has an eigenfunction $u_k \in T_k^+$, and we have $\lim_{k\to\infty} \lambda_k = \infty$.*

# Proof

Let $w(\lambda,\theta)$ be the solution of $-u'' = \lambda r u$ satisfying

$$w(\lambda,\theta)(0) = \sin\theta, \quad w(\lambda,\theta)'(0) = \lambda^{1/2}\cos\theta,$$

[case $r \equiv 1$: $w(\lambda,\theta)(x) = \sin(\lambda^{1/2}x + \theta)$]

and consider the $C^1$ functions $\Gamma^\pm : (0,\infty)\times\mathbb{R}\times\mathbb{R}^{m^\pm} \to \mathbb{R}$ defined by

$$\Gamma^\pm(\lambda,\theta,\alpha^\pm) := w(\lambda,\theta)(\pm 1) - \sum_{i=1}^{m^\pm} \alpha_i^\pm w(\lambda,\theta)(\eta_i^\pm).$$

Now, $\lambda$ eigenvalue of (1)-(2) $\iff \Gamma^\pm(\lambda,\theta,\alpha^\pm) = 0$

and then $u = w(\lambda,\theta)$ is a corresponding eigenfunction.

$$\Gamma^{\pm}(\lambda, \theta, \alpha^{\pm}) := w(\lambda, \theta)(\pm 1) - \sum_{i=1}^{m^{\pm}} \alpha_i^{\pm} w(\lambda, \theta)(\eta_i^{\pm}) = 0 \qquad (3)$$

For $\alpha^{\pm} = 0$ (Dirichlet b.c.) Sturm-Liouville theory yields

$$\lambda_k^0, \; u_k^0 = w(\lambda_k^0, \theta_k^0), \quad k = 1, 2, \dots,$$

with the usual properties.

[case $r \equiv 1$: $\lambda_k^0 = (\frac{k\pi}{2})^2$ and $u_k^0(x) = \sin(\frac{k\pi}{2}x + \frac{k\pi}{2})$, i.e. $\theta_k^0 = \frac{k\pi}{2}$]

We solve (3) for $\alpha^{\pm} \neq 0$ by continuation from the case $\alpha^{\pm} = 0$.

By the implicit function theorem, we get solutions

$$\lambda_k(\alpha^{\pm}), \; \theta_k(\alpha^{\pm}), \; u_k(\alpha^{\pm}) = w(\lambda_k(\alpha^{\pm}), \theta_k(\alpha^{\pm})), \quad k = 1, 2, \dots,$$

for $|\alpha^{\pm}| < \gamma(r)$ with an appropriate $\gamma(r) \in (0, 1]$.

$$\Gamma^{\pm}(\lambda,\theta,\alpha^{\pm}) := w(\lambda,\theta)(\pm 1) - \sum_{i=1}^{m^{\pm}} \alpha_i^{\pm} w(\lambda,\theta)(\eta_i^{\pm}) = 0$$

This involves checking that the Jacobian determinant

$$J(\lambda,\theta,\alpha^{\pm}) := \begin{vmatrix} \Gamma_{\lambda}^{-}(\lambda,\theta,\alpha^{-}) & \Gamma_{\theta}^{-}(\lambda,\theta,\alpha^{-}) \\ \Gamma_{\lambda}^{+}(\lambda,\theta,\alpha^{+}) & \Gamma_{\theta}^{+}(\lambda,\theta,\alpha^{+}) \end{vmatrix} \neq 0. \qquad (4)$$

A key step is to prove that $w(\lambda,\theta)'(\pm 1) \neq 0$ if $|\alpha^{\pm}| < \gamma(r)$.

A priori estimates for $w(\lambda,\theta)$, $w_{\lambda}(\lambda,\theta)$ and $w_{\theta}(\lambda,\theta)$ also contribute to the definition of $\gamma(r)$.

# Open problems

In case $r \equiv 1$, $w(\lambda, \theta)(x) = \sin(\lambda^{1/2} x + \theta)$, and the analysis is easier. In fact, we can take $\gamma(1) = 1$.

Inspecting the simple 3-point problem

$$-u'' = \lambda u \quad \text{on } (0,1), \qquad u(0) = 0, \quad u(1) = \alpha u(\eta),$$

for various values of $\alpha \in \mathbb{R}$ and $\eta \in (0,1)$, one can observe that:

(i) for $\alpha = 1$, we may have $u'(1) = 0$, so $u \notin T_k$ for any $k \geqslant 1$;

(ii) for $\alpha > 1$, there may be no eigenfunctions in the sets $T_l, T_{l+1}, \ldots, T_{l+n}$, for arbitrarily large $n$.

We have only been interested in real eigenvalues here.

## Problem 1

*What happened to the 'missing eigenvalues' in (ii), have they become complex?*

For $r \equiv 1$, Theorem 1 still holds [Rynne NA 2010] for the '$p$-linear' problem

$$-(|u'|^{p-2}u')' = \lambda r|u|^{p-2}u, \qquad u(\pm 1) = \sum_{i=1}^{m^\pm} \alpha_i^\pm u(\eta_i^\pm). \qquad (5)$$

There has been a lot work on this, in particular in connection with bifurcation for fully nonlinear problems.

## Problem 2

*Extend Theorem 1 to problem* (5) *with* $r \not\equiv 1$.

We haven't really tried to do this, but there seem to be considerable technical difficulties, for instance related to integration by parts arguments, a priori estimates, etc.