Self-adjoint Jacobi operators possessing generalized eigenvectors with the strongly increasing phase sequence: do they exist?

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Let us consider the right-side infinite Jacobi matrix

$$\begin{pmatrix} q_1 & w_1 & & & \\ w_1 & q_2 & w_2 & & \\ & w_2 & q_3 & w_3 & \\ & & w_3 & q_4 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

determined by real sequences $\{q_n\}_{n\geq 1}$, $\{w_n\}_{n\geq 1}$, $w_n \neq 0$, and the Jacobi operator J in the Hilbert space $\ell^2(\mathbb{N})$ of square-summable complex sequences on $\mathbb{N} := \{1, 2, \ldots\}$. J is the restriction of *the formal Jacobi operator* $\mathcal{J}$ to the domain

$$D(J) := \{u \in \ell^2(\mathbb{N}) : \ \mathcal{J}u \in \ell^2(\mathbb{N})\},$$

where $\mathcal{J}$ acts in the vector space $\ell(\mathbb{N})$ of all complex sequences on $\mathbb{N}$, and it is given by

$$(\mathcal{J}u)(n) := w_{n-1}u(n-1) + q_n u(n) + w_n u(n+1), \qquad n \in \mathbb{N}, \tag{0.1}$$

for any $u = \{u(n)\}_{n\geq 1} \in \ell(\mathbb{N})$.

For $\lambda \in \mathbb{C}$ we consider *generalized eigenvectors* of J for λ, i. e., such $u = \{u(n)\}_{n\geq 1} \in \ell(\mathbb{N})$ that

$$((\mathcal{J} - \lambda)u)(n) = 0, \qquad n \geq \mathbf{2}. \tag{0.2}$$

By $\mathrm{Sol}(\lambda)$ we denote the linear space of all the generalized eigenvectors of J for λ, so $\dim \mathrm{Sol}(\lambda) = 2$.

Various forms of asymptotics for two linearly independent generalized eigenvectors have been found through some asymptotic tricks for difference equations, see e. g.[1, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15]. For many of the described Jacobi operators, for λ-s from some non-empty open intervals the authors proved the existence of two linearly independent vectors u_+, u_- from $\mathrm{Sol}(\lambda)$, having the general form

$$u_\pm(n) = r(n) \exp(\pm i a_n) \cdot s_\pm(n), \quad n \in \mathbb{N}. \tag{0.3}$$

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In this formula r is the positive *explicit modulus*, a — the real *explicit phase*, $s_\pm$ are the complex *implicit terms* with some explicit "convergence properties", for instance: $s_\pm(n) \longrightarrow 1$ or $s_\pm(n) = p_\pm(n) + o(1)$, with $p_\pm$ being explicitly computable non-zero periodic sequences. In particular, such situation seems to be typical for the so-called *Jordan box case* (or *critical case*) (— see e. g. [3] for the definition). In many examples $r(n) := n^{-b}$ with $b \leq \frac{1}{2}$.

It is proved in [11] that such a form of a base of $\mathrm{Sol}(\lambda)$ guarantees non-existence of subordinated solutions of (0.2), and consequently, by subordination theory ([2, 10]), this gives the absolute continuity of J on the appropriate interval of $\mathbb{R}$.

According to my knowledge, in the existing literature dealing with asymptotics of some base vectors u_+, u_- of $\mathrm{Sol}(\lambda)$ with the general form 0.3 (in the self-adjoint case), we can find only the explicit phase sequences a being "weakly increasing", i.e., $(\Delta a)(n) \longrightarrow 0$ or $a(n) = n^\alpha(c + o(1))$ with $c > 0$ and $0 < \alpha < 1.$, if we limit ourselves to all λ-s from some non-empty open intervals of $\mathbb{R}$.

Open problem

The problem is to construct such a self-adjoint Jacobi operator (by finding explicit formulae on its weights $\{w_n\}_{n\geq 1}$ and diagonals $\{q_n\}_{n\geq 1}$) for which the asymptotic formula (0.3) of some base vectors u_+, u_- of $\mathrm{Sol}(\lambda)$ holds, with the **"strongly increasing"** phase sequences a for all λ-s from a non-empty open interval $A \subset \mathbb{R}$. More precisely, for any $\lambda \in A$ we would like (0.3) with the following conditions to be satisfied:

1. r — positive sequence, a — real sequence
2. $(\Delta a)(n) \longrightarrow +\infty$ or $a(n) = n^\alpha(c + o(1))$ with $c > 0$ and $\alpha > 1$
3. $s_\pm(n) = p_\pm(n) + o(1)$, with $p_\pm$ being explicitly computable non-zero periodic sequences (e. g., $p_\pm \equiv 1$, if possible),

where r, a, c, $p_\pm$ can depend on $\lambda \in A$.

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