“Strange” functions in the Laplace operator domains and their relation to the eigenfunctions

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Let Ω be a region in $\mathbb{R}^2$, with a boundary $\partial\Omega$, and consider the corresponding Laplace operator $-\Delta$ on $L^2(\Omega)$ with mixed Dirichlet-Neumann boundary conditions, defined in the standard way using the one-to-one correspondence between symmetric below bounded quadratic forms and self-adjoint operators, *cf* [1]. It is very often supposed, that the operator domain of such operator is the subset of the Sobolev space $W^{2,2}(\Omega)$, which satisfies the prescribed boundary conditions. However, Dirichlet Laplacians studied in [2], or mixed Dirichlet-Neumann ones studied in [1] are examples of operators, for which this assertion is not true.

Namely, it was shown in [2], that for bounded regions with piece-wise C^3-boundary, which has finite number of angles larger than π, there exists for each such angle one function, which is not in $W^{2,2}(\Omega)$ and which belongs to the operator domain. Let us present here the example of such a function, constructed by O. V. Guseva, see [2]. Let Ω be a sector of the unit ball with the angle $\pi\,\beta^{-1}$, $1/2 < \beta < 1$. The function

$$u(r,\vartheta) := \xi(r)\, r^{\beta} \sin \beta\vartheta,$$

where (r,ϑ) are polar coordinates and $\xi \in C^{\infty}((0,\infty))$ with $\xi(t) = 1$ if $t \le 1/3$ and $\xi(t) = 0$ if $t \ge 2/3$, obviously satisfies Dirichlet boundary conditions, and moreover $-\Delta u \in L^2(\Omega)$. However its second derivatives do not belong to $L^2(\Omega_\beta)$, *i.e.* $u \notin W^{2,2}(\Omega)$.

Similar functions, corresponding to every point of change from one boundary condition to another, belong to operator domain in the case of the combination of Dirichlet and Neumann boundary conditions on straight strips in the plane too [1, 3]. The operator domain is then a span of all these “Guseva-like” functions and of the subset of $W^{2,2}(\Omega)$ space, satisfying prescribed boundary conditions.

The natural question is: **What is a role of such Guseva-like functions?** In particular, **do they contribute to the Laplacian eigenvalues?**

Let us consider, for example, the well known unbounded L-shaped region (waveguide). It is known that in such a case there is a unique discrete eigenvalue of the Laplacian below the essential spectrum, *cf* [4]). Since the L-shaped region possesses on the boundary the angle larger than π, its operator domain contains Guseva-like functions. Is the inner product of this function and the eigenfunction zero or not?

References

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