Large-time behaviour of the heat equation: subcriticality versus criticality

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We expect that the solutions of the heat equation "decay faster" for large times provided that the generator is "more positive" in a weak sense. This rough claim is made precise in terms of the conjectures below.

Let Ω be an open connected subset of $\mathbb{R}^d$. Let H_0 and H_+ be two self-adjoint operators on $L^2(\Omega)$ such that $\inf \sigma(H_0) = \inf \sigma(H_+) = 0$. Assume that H_+ is *subcritical*, in the sense that there is a smooth positive function $\rho : \Omega \to \mathbb{R}$ such that $H_+ \geq \rho$ (Hardy inequality). On the other hand, H_0 is assumed to be *critical*, in the sense that $\inf \sigma(H_0 - V) < 0$ for any non-negative non-trivial $V \in C_0^\infty(\Omega)$.

Conjecture 1 (Semigroup version) *There is a positive function (weight) $w : \Omega \to \mathbb{R}$ such that*

$$\lim_{t\to\infty} \frac{\left\|e^{-H_+ t}\right\|_{L^2_w(\Omega)\to L^2(\Omega)}}{\left\|e^{-H_0 t}\right\|_{L^2_w(\Omega)\to L^2(\Omega)}} = 0\,.$$

This conjecture is raised in a joint paper with Zuazua [6], where it is proved in the special case of H_0 and H_+ being the Dirichlet Laplacians in unbounded three-dimensional straight and twisted tubes, respectively (see also [7] for an analogous two-dimensional model). In [5], the conjecture is established for $\Omega = \mathbb{R}^2$ and H_+ being the magnetic Schrödinger operator (for which the existence of Hardy inequalities is known due to [8]) versus H_0 the "free" Laplacian. The method of [6, 7, 5] (self-similarity variables) seemingly applies to other models (for instance, potential perturbations). The open problem is to establish Conjecture 1 in the general setting.

Conjecture 1 can be regarded as a norm-wise version of another conjecture, raised in a joint paper with Fraas and Pinchover [2]:

Conjecture 2 (Heat-kernel version) *Let H_+ and H_0 be in addition elliptic differential operators whose coefficients satisfy some minimal regularity assumptions so that the heat kernels exist. Then*

$$\lim_{t\to\infty} \frac{e^{-H_+ t}(x,x')}{e^{-H_0 t}(x,x')} = 0$$

locally uniformly for $(x,x') \in \Omega \times \Omega$.

In [2], we stated Conjecture 2 in a more general setting of not necessarily self-adjoint operators and proved it, *inter alia*, in the case of H_+ being a potential perturbation of H_0, provided that H_0 is self-adjoint or Davies' conjecture [1] holds for both H_0 and H_+. The validity of Conjecture 2 in the case of aforementioned three-dimensional tubes and magnetic perturbations in the plane follows from [3] and [4], respectively.

There does not seem to be a direct relationship between Conjectures 1 and 2.

References

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