UNIFORM TIME-DECAY OF SEMIGROUPS OF CONTRACTIONS

XUE PING WANG

High frequency analysis of propagation of waves in media with variable absorption index leads to the following dissipative Schrödinger equation:

$$\begin{cases} ih\frac{\partial}{\partial t}u^h(x,t) &= P(h)u^h(x,t), \\ u^h(x,0) &= u_0^h(x), \end{cases}$$

where $P(h) = -h^2\Delta + V_1(x) - ihV_2(x)$, $x \in \mathbb{R}^n$, $h \in]0, h_0]$ is a small parameter proportional to wave length and V_j, $j = 1, 2$, are real functions with $V_2 \geq 0$ and $V_2 \neq 0$. Assume that V_j is smooth, satisfying for some $\rho > 0$

$$|\partial_x^\alpha V_j(x)| \leq C_\alpha \langle x \rangle^{-\rho-|\alpha|}, \quad j = 1, 2.$$

Here $\langle x \rangle = (1+|x|^2)^{1/2}$. Let $S_h(t) = e^{-itP(h)/h}$, $t \geq 0$, be the associated semigroup of contractions in $L^2(\mathbb{R}^n)$. Then $\|S_h(t)\| \leq 1$ for all $t \geq 0$ and $h \in]0, h_0]$. The interplay between propagation along the flow of the Hamiltonian $p_1(x,\xi) = \xi^2 + V_1(x)$ and the dissipation governs the long-time behavior of solutions. A global uniforme *a priori* estimate is a first step towards more refined analysis. A natural question in this connexion is the following

Question. Can one establish a uniform time-decay estimate for the semigroup $S_h(t)$ in the form

$$\|\langle x \rangle^{-s} S_h(t) \langle x \rangle^{-s}\| \leq w(t), \quad t > 0, \tag{1}$$

uniformly in $h \in]0, h_0]$? Here $s > 0$ and $w(t)$ is independent of h with $w(t) \to 0$, as $t \to \infty$.

Let $(x(t; y, \eta), \xi(t; y, \eta))$ denote the classical Hamiltonian flow of $p_1(x, \xi)$ with initial data (y, η). Making use of an Egorov's theorem and the argument of [4], one can deduce that a necessary condition for (1) to be true is that

$$|\langle x(t; y, \eta) \rangle^{-s} e^{-2\int_0^t V_2(x(\tau;y,\eta))d\tau} \langle y \rangle^{-s}| \leq w(t), \tag{2}$$

for all $(y, \eta) \in \mathbb{R}^{2d}$. Estimate (2) implies that each bounded classical trajectory should pass through the open set $\{x; V_2(x) > 0\}$ and

$$w(t) \geq C\langle t \rangle^{-s\sigma}$$

with $\sigma = \min\{1, \tau_0\}$, where τ_0 is the divergence rate in t of nontrapping trajectories with energy 0. Is the condition (2) sufficient for a uniform time-decay estimate of the form (1)? If yes, can one take $w(t) = C\langle t \rangle^{-\min\{\frac{n}{2}, s\sigma\}}$? The restriction on the decay rate by $\frac{n}{2}$ comes from the threshold behavior of the semigroup for fixed h.

Recall that in the selfadjoint case ($V_2 = 0$) and with a localization in energies away from $\mathbb{R}_-$, the result is true with $w(t) = C_s\langle t \rangle^{-s}$ for any $s > 0$. More precisely, let $U(t,h) = e^{-itP_1(h)/h}$, $P_1(h) = -h^2\Delta + V_1$, $I =]a,b[$ with $a > 0$, $\chi \in C_0^\infty(I)$. Then the estimate

$$\|\langle x\rangle^{-s}\chi(P_1(h))U(t,h)\langle x\rangle^{-s}\| \le C_s\langle t\rangle^{-\epsilon}, \quad t \in \mathbb{R} \tag{3}$$

holds for some $s, \epsilon > 0$ and uniformly in $h \in]0, h_0]$ *if and only if* every energy E in supp χ is nontrapping. If the latter is satisfied, (3) holds with $\epsilon = s$ and for any $s > 0$. See [3].

For non-selfadjoint operators, one can not use compactly supported cut-off. The main difficulty to prove (1) is the semiclassical analysis near the threshold zero for the dissipative Schrödinger operators. A closely related problem is a global limiting absorption principle on the whole real axis from the the upper half-complex plane and a nice estimate in $h > 0$. For energies away from zero and under the condition (2), this is recently obtained in the PhD thesis of J. Royer (see [2]). Again the question is open near the threshold zero. See [1] for a result in the selfadjoint case for positive potentials.

References

[1] C. Gérard, Semiclassical resolvent estimates for two and three-body Schrödinger operators. Commun. in P. D. E., 15 (1990), no. 8, 1161-1178.

[2] J. Royer, Limiting absorption principle for the dissipative Helmholtz equation, Commun. in P.D.E., 35(2010), no.8, 1458-1489.

[3] X. P. Wang, Time-decay of scattering solutions and classical trajectories, Ann. I.H.P., Sect. A., 47(1987), no.1, 25-37.

[4] X.P. Wang, Semiclassical resolvent estimates for N-body Schrödinger operators. J. Funct. Anal., 97 (1991), no.2, 466-483.

Département de Mathématiques, Université de Nantes, 44322 Nantes Cedex 3 France, E-mail: xue-ping.wang@univ-nantes.fr