Essential spectrum of dissipative Maxwell systems

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Mathematical aspects of the physics with non-self-adjoint operators

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Literature

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Anisotropic Maxwell's equations

Macroscopic electromagnetic properties of a medium are described by Maxwell's equations

$$\partial_t D = \operatorname{curl} H - J, \quad \partial_t B = -\operatorname{curl} E, \quad \operatorname{div} D = \rho, \quad \operatorname{div} B = 0.$$

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Here ϵ, μ are matrix-valued bounded functions representing the electric permittivity and the magnetic permeability of the medium. Assume ϵ, μ are L^∞, symmetric-matrix-valued functions, $\epsilon, \mu \geq c\mathbb{I}$, $c > 0$.

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$$\begin{cases} -i\sigma E + i \operatorname{curl} H - \omega\epsilon E = F_1, & \text{in } \Omega, \\ -i \operatorname{curl} E - \omega\mu H = F_2, & \text{in } \Omega, \\ \nu \times E = 0, & \text{on } \partial\Omega. \end{cases}$$

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(II) Weber's compactness result If Ω is bounded and Lipschitz, $H_0(\operatorname{curl}, \Omega) \cap H(\operatorname{div}, \Omega)$ is compactly embedded in $L^2(\Omega)^3$.

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Assumption. $\epsilon, \mu, \sigma \in L^\infty(\Omega, \mathrm{Sym}_3(\mathbb{R}))$, satisfying

$$\begin{aligned} 0 &< \epsilon_{\min} \leq \eta\cdot\epsilon\eta \leq \epsilon_{\max}, \\ 0 &< \mu_{\min} \leq \eta\cdot\mu\eta \leq \mu_{\max}, \qquad \eta\in\mathbb{R}^3,\ |\eta| = 1. \\ 0 &\leq \sigma_{\min} \leq \eta\cdot\sigma\eta \leq \sigma_{\max}, \end{aligned}$$

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$\omega = 0$ and $\omega = -\mathrm{i}\sigma/\epsilon$ "special points" (essential spectrum).

Some spectral theory

Essential spectrum of a linear operator A in the Hilbert space $\mathcal{H}$:

$$\sigma_e(A) := \Big\{\omega \in \mathbb{C} : \exists u_n \in \mathrm{dom}(A), \|u_n\| = 1, u_n \rightharpoonup 0, \big\|(A-\omega)u_n\big\| \to 0\Big\}.$$

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$$\sigma_e(A) = \left\{\omega \in \mathbb{C} : 0 \in \sigma_e(A(\omega))\right\}$$

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$$\lim_{R\to\infty} \left\{ \sup_{\|x\|>R} \max \left(\|\epsilon(x) - \epsilon_\infty \,\mathrm{id}\,\|, \|\mu(x) - \mu_\infty \,\mathrm{id}\,\|, \|\sigma(x)\| \right) \right\} = 0.$$

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Define $\mathcal{L}(\omega) = \operatorname{curl} \mu^{-1} \operatorname{curl}_0 - \omega(\omega\epsilon + \mathrm{i}\sigma)$ and

$$L_\infty(\omega) := \mu_\infty^{-1} \operatorname{curl}\operatorname{curl}_0 - \omega^2 \epsilon_\infty, \quad \operatorname{dom}(L_\infty) \subset H_0(\operatorname{curl}, \Omega) \cap H(\operatorname{div} 0, \Omega).$$

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$$\mathcal{W}_\nabla(\omega) := P_\nabla \mathcal{L}(\omega) P_\nabla = -\omega P_\nabla(\omega\epsilon + \mathrm{i}\sigma) P_\nabla, \quad \operatorname{dom}(\mathcal{W}_\nabla) = \nabla \dot{H}_0^1(\Omega).$$

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Then: [Lassas ’98], [ABMW ’19], [BFMT ’23]

$$\sigma_e(V) = \sigma_e(\mathcal{L}) = \sigma_e(L_\infty) \cup \sigma_e(\mathcal{W}_\nabla) \subset \mathbb{R} \cup \mathrm{i}\mathbb{R}_{\le 0}$$

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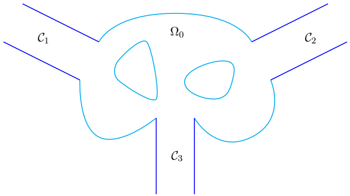
Setting in [FM'24]: domain with multiple cylindrical ends C_i, ϵ, μ asymptotically constant in each cylinder.



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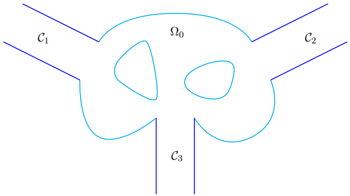

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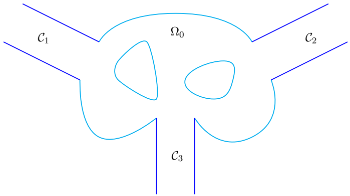

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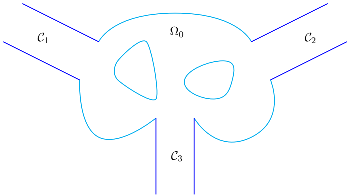

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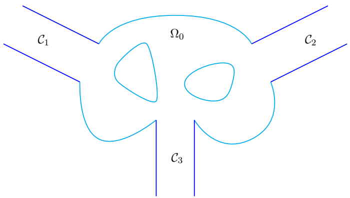

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(!) Immediate Glazman decomposition on V fails.

Rational dependence on the frequency

- Systems in the form

$$V(\omega) = \begin{pmatrix} -\mathrm{i}\sigma & \mathrm{i}\,\mathrm{curl} \\ -\mathrm{i}\,\mathrm{curl}_0 & 0 \end{pmatrix} - \omega \mathbb{I} + \begin{pmatrix} \frac{\theta_e^2}{(\omega+\mathrm{i}\gamma_e)} & 0 \\ 0 & \frac{\theta_m^2}{(\omega+\mathrm{i}\gamma_m)} \end{pmatrix}, \quad \omega \in \mathbb{C} \setminus \{-\mathrm{i}\gamma_e, -\mathrm{i}\gamma_m\},$$

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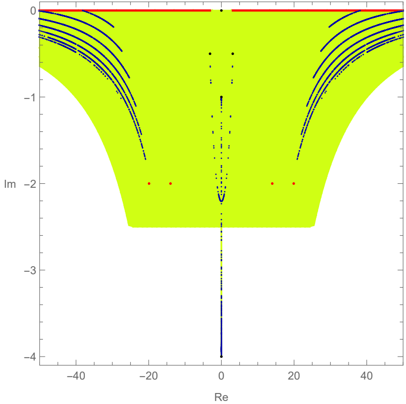

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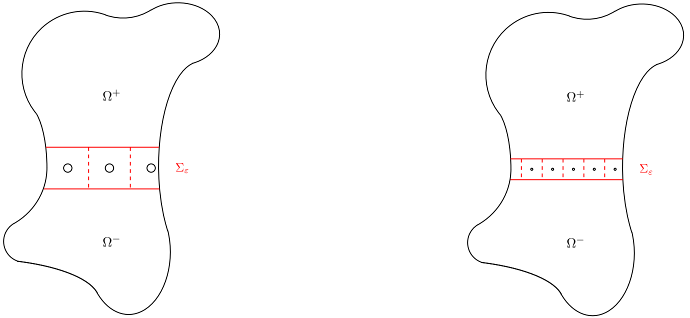

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For which frequencies $\omega \in \mathbb{C}$ can we solve this transmission problem?

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Basic intuition:

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If the open problem holds, i.e., no disks of eigenvalues of V_Ω, then

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$\tilde{\sigma}_e(\mathcal{C}(\cdot, \alpha))$ is the *extended* essential spectrum of the operator family $\mathcal{C}$.

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$$\sigma_e(A_\mu) = \begin{cases} \emptyset, & \text{if } \mu \neq 1, \\ \{0\}, & \text{if } \mu = 1, N = 2, \\ \supset \{0\}, & \text{if } \mu = 1, N \geq 3, + \text{ geom. assumption} \end{cases}$$

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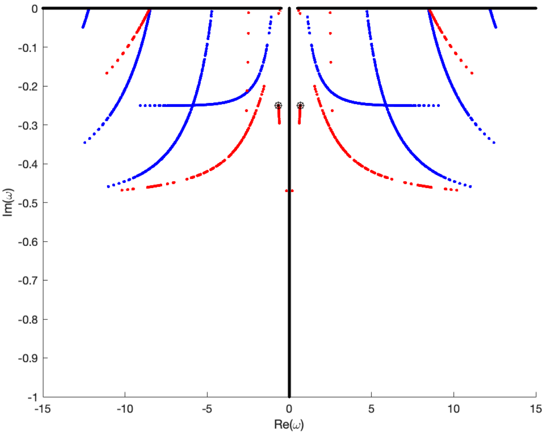

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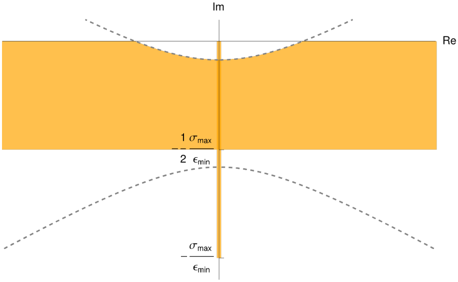
- Get rid of gradient fields to prove 'hole around 0'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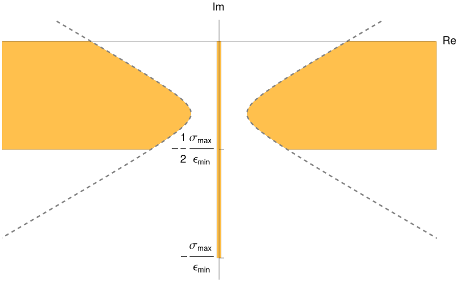
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for $\omega\in\rho(V)$.

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Note that for $\omega = 0$ and $\omega = -\mathrm{i}$, $\omega(\omega + \mathrm{i}\chi)$ vanishes identically in a subdomain of Ω; we get essential spectrum.

In the previous example $\sigma_e(\mathcal{W}_\nabla) = \{0, -\frac{\mathrm{i}}{2}, -\mathrm{i}\}$. Why?

Recall $\quad \mathcal{W}_\nabla(\omega) = -P_\nabla[\omega(\omega + \mathrm{i}\chi)]P_\nabla, \qquad \chi(x) = \chi_{(0,1)}(x_1).$

Note that $P_\nabla = \nabla\Delta^{-1}\,\mathrm{div}$, with Δ Dirichlet Laplacian acting from $\dot{H}^1_0(\Omega)$ to $H^{-1}(\Omega)$. Equivalent to study

$$-\,\mathrm{div}(\omega(\omega + \mathrm{i}\chi))\nabla : \dot{H}^1_0(\Omega) \to H^{-1}(\Omega)$$

Note that for $\omega = 0$ and $\omega = -\mathrm{i}$, $\omega(\omega + \mathrm{i}\chi)$ vanishes identically in a subdomain of Ω; we get essential spectrum.

For $\omega = -\mathrm{i}/2$ we have $\omega^2 = -\omega(\omega + \mathrm{i})$ (relative contrast $= -1$)...can construct black hole modes.

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For $\nu \neq 1/2$, this operator is a finite-rank perturbation of a boundedly invertible matrix, so $-\mathrm{i}\nu$ is not in the essential spectrum. For $\nu = 1/2$,

$$u_n(x_1, x_2, x_3) := \begin{cases} (1-(x_1-1)\kappa_n(\coth(\kappa_n)-1))\psi_n(x_2, x_3)\frac{\sinh(\kappa_n x_1)}{\sinh(\kappa_n)}, & x_1 \in (0, 1), \\ \psi_n(x_2, x_3)\exp(-\kappa_n(x_1-1)), & x_1 > 1, \end{cases}$$

forms a Weyl singular sequence, so $-\mathrm{i}/2 \in \sigma_e(V)$.