OPEN PROBLEM FOR COMPLEX-VALUED PERIODIC POTENTIALS

R. WEIKARD

If \( q \) is a complex-valued periodic potential with period \( a \) consider the differential expression \(-y'' + qy\). For \( x_0 \) varying in \([0, a]\) introduce the solutions \( c(\cdot, x_0, \lambda) \) and \( s(\cdot, x_0, \lambda) \) of \(-y'' + qy = \lambda y\) satisfying the initial conditions \( c(x_0, x_0, \lambda) = s'(x_0, x_0, \lambda) = 1 \) and \( c'(x_0, x_0, \lambda) = s(x_0, x_0, \lambda) = 0 \).

The periodic and semi-periodic eigenvalues are given as the zeros of the entire function \((\text{tr } \mathcal{M})^2 - 4\) where \( \mathcal{M} \) is the monodromy operator associated to \( q \). In fact \( \text{tr } M(\lambda) = c(x_0 + a, x_0, \lambda) + s'(x_0 + a, x_0, \lambda) \) which is, in fact, independent of \( x_0 \).

The Dirichlet and Neumann eigenvalues with respect to the interval \([x_0, x_0 + a]\) are given as the zeros of the entire functions \( s(x_0 + a, x_0, \cdot) \) and \( c'(x_0 + a, x_0, \cdot) \), respectively, and depend, in general, on \( x_0 \).

Let \( d(\lambda), p(x_0, \lambda), \) and \( r(x_0, \lambda) \) denote the multiplicities of \( \lambda \) as zeros of \((\text{tr } \mathcal{M})^2 - 4, s(x_0 + a, x_0, \cdot) \) and \( c'(x_0 + a, x_0, \cdot) \), respectively (these are also the algebraic multiplicities of the corresponding eigenvalues). Moreover, let \( p_i(\lambda) = \min \{ p(x_0, \lambda) : x_0 \in [0, a] \} \) and \( r_i(\lambda) = \min \{ r(x_0, \lambda) : x_0 \in [0, a] \} \).

One has then the following facts (see [GW96]):

1. \( p_i(\lambda) = r_i(\lambda) \).
2. If \( d(\lambda) > 0, p(x_0, \lambda) > 0, \) and \( r(x_0, \lambda) > 0 \), then \( p_i(\lambda) = r_i(\lambda) > 0 \).
3. \( d(\lambda) - p_i(\lambda) - r_i(\lambda) \geq 0 \).

Note that a (semi-)periodic eigenvalue \( \lambda \) has geometric multiplicity 2 if and only if it is both a Dirichlet and a Neumann eigenvalue. If \( \lambda \) is a point where two linearly independent Floquet solutions do not exist, i.e., a (semi-)periodic eigenvalue with geometric multiplicity 1, then \( d(\lambda) > 0 \) and \( p_i(\lambda) = r_i(\lambda) = 0 \) so that \( d(\lambda) - p_i(\lambda) - r_i(\lambda) > 0 \). In the self-adjoint case this is the only way to make \( d(\lambda) - p_i(\lambda) - r_i(\lambda) > 0 \).

Open problem: Now the question arises whether in the non-self-adjoint case it is possible that \( d(\lambda) - p_i(\lambda) - r_i(\lambda) > 0 \) when \( \lambda \) is a (semi-)periodic eigenvalue of geometric multiplicity 2. Thus, either prove that \( d(\lambda) - p_i(\lambda) - r_i(\lambda) > 0 \) implies \( p_i(\lambda) + r_i(\lambda) = 0 \) or give an example to the contrary.

References


Date: 2015 AIM.