OPEN PROBLEM FOR COMPLEX-VALUED PERIODIC POTENTIALS

R. WEIKARD

If q is a complex-valued periodic potential with period a consider the differential expression $-y'' + qy$. For x_0 varying in $[0, a]$ introduce the solutions $c(\cdot, x_0, \lambda)$ and $s(\cdot, x_0, \lambda)$ of $-y'' + qy = \lambda y$ satisfying the initial conditions $c(x_0, x_0, \lambda) = s'(x_0, x_0, \lambda) = 1$ and $c'(x_0, x_0, \lambda) = s(x_0, x_0, \lambda) = 0$.

The periodic and semi-periodic eigenvalues are given as the zeros of the entire function $(\operatorname{tr} \mathcal{M})^2 - 4$ where $\mathcal{M}$ is the monodromy operator associated to q. In fact $\operatorname{tr} M(\lambda) = c(x_0 + a, x_0, \lambda) + s'(x_0 + a, x_0, \lambda)$ which is, in fact, independent of x_0.

The Dirichlet and Neumann eigenvalues with respect to the interval $[x_0, x_0 + a]$ are given as the zeros of the entire functions $s(x_0 + a, x_0, \cdot)$ and $c'(x_0 + a, x_0, \cdot)$, respectively, and depend, in general, on x_0.

Let $d(\lambda)$, $p(x_0, \lambda)$, and $r(x_0, \lambda)$ denote the multiplicities of λ as zeros of $(\operatorname{tr} \mathcal{M})^2 - 4$, $s(x_0 + a, x_0, \cdot)$ and $c'(x_0 + a, x_0, \cdot)$, respectively (these are also the algebraic multiplicities of the corresponding eigenvalues). Moreover, let $p_i(\lambda) = \min\{p(x_0, \lambda) : x_0 \in [0, a]\}$ and $r_i(\lambda) = \min\{r(x_0, \lambda) : x_0 \in [0, a]\}$.

One has then the following facts (see [GW96]):

(1) $p_i(\lambda) = r_i(\lambda)$.
(2) If $d(\lambda) > 0$, $p(x_0, \lambda) > 0$, and $r(x_0, \lambda) > 0$, then $p_i(\lambda) = r_i(\lambda) > 0$.
(3) $d(\lambda) - p_i(\lambda) - r_i(\lambda) \geq 0$.

Note that a (semi-)periodic eigenvalue λ has geometric multiplicity 2 if and only if it is both a Dirichlet and a Neumann eigenvalue. If λ is a point where two linearly independent Floquet solutions do not exist, i.e., a (semi-)periodic eigenvalue with geometric multiplicity 1, then $d(\lambda) > 0$ and $p_i(\lambda) = r_i(\lambda) = 0$ so that $d(\lambda) - p_i(\lambda) - r_i(\lambda) > 0$. In the self-adjoint case this is the only way to make $d(\lambda) - p_i(\lambda) - r_i(\lambda) > 0$.

Open problem: Now the question arises whether in the non-self-adjoint case it is possible that $d(\lambda) - p_i(\lambda) - r_i(\lambda) > 0$ when λ is a (semi-)periodic eigenvalue of geometric multiplicity 2. Thus, either prove that $d(\lambda) - p_i(\lambda) - r_i(\lambda) > 0$ implies $p_i(\lambda) + r_i(\lambda) = 0$ or give an example to the contrary.

References

[GW96] F. Gesztesy and R. Weikard. Picard potentials and Hill's equation on a torus. *Acta Math.*, 176(1):73–107, 1996.

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