LARGE-TIME BEHAVIOR OF THE HEAT EQUATION: SUBCRITICALITY VERSUS CRITICALITY

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This open problem is a repetition of the open problem raised during previous meetings in Prague (2010) and Barcelona (2012)

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but little progress has been made so far. Please visit the links above for more details and references.

Our conjecture is that the solutions of the heat equation "decay faster" for large times provided that the generator is "more positive" in the sense of the validity of a Hardy-type inequality. There exist both semigroup (with Zuazua [KZ10]) and heat-kernel (with Fraas and Pinchover [FKP10]) versions of the conjecture and the latter involves non-self-adjoint operators too. The conjecture has been supported by several particular situations, but there exists no general result yet.

In the self-adjoint case, the conjectures can be stated as follows. Let Ω be an open connected subset of $\mathbb{R}^d$. Let H_0 and H_+ be two self-adjoint operators in $L^2(\Omega)$ such that $\inf \sigma(H_0) = \inf \sigma(H_+) = 0$. Assume that H_+ is *subcritical*, in the sense that there is a smooth positive function $\rho : \Omega \to \mathbb{R}$ such that $H_+ \geq \rho$ (Hardy inequality). On the other hand, H_0 is assumed to be *critical*, in the sense that $\inf \sigma(H_0 - V) < 0$ for any non-negative non-trivial $V \in C_0^\infty(\Omega)$.

Conjecture 1 (Semigroup version, [KZ10])**.** There is a positive function (weight) $w : \Omega \to \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{\left\|e^{-H_+ t}\right\|_{L^2_w(\Omega)\to L^2(\Omega)}}{\left\|e^{-H_0 t}\right\|_{L^2_w(\Omega)\to L^2(\Omega)}} = 0\,.$$

Conjecture 2 (Heat-kernel version, [FKP10])**.** Let H_+ and H_0 be in addition elliptic differential operators whose coefficients satisfy some minimal regularity assumptions so that the heat kernels exist. Then

$$\lim_{t\to\infty} \frac{e^{-H_+ t}(x,x')}{e^{-H_0 t}(x,x')} = 0$$

locally uniformly for $(x,x') \in \Omega \times \Omega$.

Open problem: Prove the conjectures or find a counterexample.

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References

[FKP10] M. Fraas, D. Krejčiřík, and Y. Pinchover. On some strong ratio limit theorems for heat kernels. *Discrete Contin. Dynam. Systems A*, 28:495–509, 2010.

[KZ10] D. Krejčiřík and E. Zuazua. The Hardy inequality and the heat equation in twisted tubes. *J. Math. Pures Appl.*, 94:277–303, 2010.