UPPER BOUNDS ON THE NORM OF THE RESOLVENT

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It is well known that the spectrum of a non-self-adjoint operator does not control its resolvent and that the latter may become very large far from the spectrum. Some general upper bounds on resolvents are provided by the abstract operator theory, and restricting the attention to the setting of semiclassical operators on $\mathbf{R}^n$, let us give a rough statement of such bounds. Assume that $P = p^w(x, hD_x)$ is the semiclassical Weyl quantization on $\mathbf{R}^n$ of a nice symbol p with $\operatorname{Re} p \geq 0$, say. Then the norm of the resolvent of P is bounded from above by a quantity of the form $\mathcal{O}(1) \exp(\mathcal{O}(1)h^{-n})$, provided that $z \in \operatorname{neigh}(0, \mathbf{C})$ is not too close to the spectrum of P. On the other hand, the available lower bounds on the resolvent of P, in the interior of the range of the symbol, coming from the pseudospectral considerations, are typically of the form $C_N^{-1} h^{-N}$, $N \in \mathbf{N}$, or $(1/C)e^{1/(Ch)}$, provided that p enjoys some analyticity properties, [DSZ04]. There appears to be therefore a substantial gap between the available upper and lower bounds on the resolvent, especially when $n \geq 2$, which, to the best of my knowledge, has so far only been bridged in the very special case of elliptic quadratic differential operators, see [HSV13].

Open problem: Is the upper bound sharp (especially for dimension $n \geq 2$)?

References

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