SPECTRAL RADIUS AND OPERATOR NORM

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Let A be a bounded linear operator on a Banach space X. Its spectral radius is defined by

$$\operatorname{spr}(A) := \max\{|z| : z \in \sigma(A)\}.$$

Gelfand proved the classical formula

$$\operatorname{spr}(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}.$$

Obviously, $0 \leq \operatorname{spr}(A) \leq \|A\|$. The question arises: What is the gap between $\|A\|$ and $\operatorname{spr}(A)$? Introduce the denotation

$$\operatorname{gap}(A) := \|A\| - \operatorname{spr}(A).$$

Open problems:

i) For which class of operators holds

$$\operatorname{gap}(A) > 0 \text{ or } \operatorname{gap}(A) = 0,$$

respectively.

ii) What is the smallest $m \in (0, 1]$, such that

$$\operatorname{spr}(A) \leq m\|A\|$$

or

$$\operatorname{gap}(A) \geq (1 - m)\|A\| \quad ?$$

Example 1. Let $X = \ell^1(\mathbb{N})$ and A be the weighted shift-operator defined according to the canonical standard basis by the infinite matrix

$$\begin{pmatrix} 0 & & & & & \\ b_1 & 0 & & & & \\ & b_2 & 0 & & & \\ & & b_1 & 0 & & \\ & & & b_2 & 0 & \\ & & & & \ddots & \ddots \end{pmatrix}$$

where $b_1, b_2 > 0$ and $b_1 b_2 = 1$. In this case $\|A\| = \max\{b_1, b_2\}$ and $\sigma(A) = \{z \in \mathbb{C} : |z| \leq 1\}$ and therefore $\operatorname{spr}(A) = 1$. Thus

- $\operatorname{gap}(A) = 0$: If $b_1 = b_2 = 1$ then $\|A\| = \operatorname{spr}(A)$.
- $\operatorname{gap}(A) > 0$: If $b_1 \neq b_2$ then $\|A\| > \operatorname{spr}(A)$.

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This kind of estimates are useful in the following situation. Let K be a compact perturbation of A. Study the discrete spectrum of $B := A + K$. We are able to analyze the moments and the number of eigenvalues of B outside a ball of radius $\|A\|$. It is more interesting and also natural to enlarge this region up to the complement of a ball with radius $\operatorname{spr}(A)$.