Let $A$ be a bounded linear operator on a Banach space $X$. Its spectral radius is defined by

$$spr(A) := \max \{|z| : z \in \sigma(A)\}.$$ 

Gelfand proved the classical formula

$$spr(A) = \lim_{n \to \infty} \|A^n\|^{\frac{1}{n}}.$$ 

Obviously, $0 \leq spr(A) \leq \|A\|$. The question arises: What is the gap between $\|A\|$ and $spr(A)$? Introduce the denotation

$$\text{gap}(A) := \|A\| - spr(A).$$

**Open problems:**

i) For which class of operators holds

$$\text{gap}(A) > 0 \text{ or } \text{gap}(A) = 0,$$

respectively.

ii) What is the smallest $m \in (0, 1]$, such that

$$spr(A) \leq m\|A\|$$

or

$$\text{gap}(A) \geq (1 - m)\|A\| ?$$

**Example 1.** Let $X = \ell^1(\mathbb{N})$ and $A$ be the weighted shift-operator defined according to the canonical standard basis by the infinite matrix

$$
\begin{pmatrix}
0 & 0 & & \\
\ b_1 & 0 & & \\
\ b_2 & 0 & & \\
\ b_1 & 0 & & \\
\ b_2 & 0 & & \\
\ \vdots & \vdots & \ddots & \\
\end{pmatrix}
$$

where $b_1, b_2 > 0$ and $b_1b_2 = 1$. In this case $\|A\| = \max\{b_1, b_2\}$ and $\sigma(A) = \{z \in \mathbb{C} : |z| \leq 1\}$ and therefore $spr(A) = 1$. Thus

- $\text{gap}(A) = 0$: If $b_1 = b_2 = 1$ then $\|A\| = spr(A)$.
- $\text{gap}(A) > 0$: If $b_1 \neq b_2$ then $\|A\| > spr(A)$. 

*Date: 2015 AIM.*
This kind of estimates are useful in the following situation. Let $K$ be a compact perturbation of $A$. Study the discrete spectrum of $B := A + K$. We are able to analyze the moments and the number of eigenvalues of $B$ outside a ball of radius $\|A\|$. It is more interesting and also natural to enlarge this region up to the complement of a ball with radius $\text{spr}(A)$.