NON-SELF-ADJOINT INVERSE PROBLEMS

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We are interested in identifying a non-self-adjoint operator associated with an evolution equation (parabolic or hyperbolic) through "observations" of the solution as time evolves. Thus for example in a certain Hilbert space we have

$$u'(t) = Au(t) \quad \text{and } u(0) = f \tag{1}$$

where, for simplicity, we assume that

$$A = L + B$$

with L is a given (known) self-adjoint operator with "nice properties" while B is an unknown non-self-adjoint perturbation. For example $Ay(x) = y''(x) - q(x)y(x)$ or $Au = \Delta u - q(x)u$ with $\operatorname{Im} q(x) \neq 0$. We assume that we can observe the solution through a functional $\langle \cdot, g \rangle$ say

$$\omega(t) = \langle u(t), g \rangle.$$

For example if $u(x,t)$ is the solution of a heat equation, where $x \in \Omega \subset \mathbb{R}^n$, and $p \in \partial\Omega$, then $\omega(t) = u(p,t)$ (temperature) or $\omega(t) = \partial_n u\,(p,t)$ (heat transfer) are usual observations/readings of the solution on the boundary. Thus we want to recover A or at least its spectrum $\sigma_A = \{\lambda_n\} \subset \mathbb{C}$ from the observation mapping

$$u(0) \to \omega(t).$$

To do so, although we do NOT know A, we assume that it has a discrete spectrum $\{\lambda_n\} \subset \mathbb{C}$, and in general $\operatorname{Im} \lambda_n \to 0$ as $n \to \infty$, while $\operatorname{Re} \lambda_n \to -\infty$. If we denote its eigenfunctions by $\varphi_{n,0}$ and its associated eigenfunctions (roots) by $\varphi_{n,\nu}$ for $\nu = 1, ..., m_n - 1$, where m_n is the multiplicity of the eigenvalue λ_n, then we can write a formal solution to the evolution equation

$$u\,(t) = \sum_{n \geq 1} e^{\lambda_n t} \sum_{\nu=0}^{m_n - 1} c_{n\nu}\,(f)\, p_{n\nu}(t) \varphi_{n\nu} \tag{2}$$

where the Fourier coefficients are $c_{n\nu}\,(f) = \langle f, \psi_{n\nu} \rangle$ and $\{\psi_{n\nu}\}$ is the biorthogonal system to $\{\varphi_{n,\nu}\}$. Here $p_{n\nu}$ are polynomials generated by the multiplicity of the eigenvalue λ_n. The observation then is given by

$$\omega(t) = \sum_{n \geq 1} e^{\lambda_n t} \sum_{\nu=0}^{m_n - 1} c_{n\nu}\,(f)\, p_{n\nu}(t) \langle \varphi_{n\nu}, g \rangle. \tag{3}$$

In the best case, when all $c_{n\nu}\,(f) \neq 0$ and $\langle \varphi_{n\nu}, g \rangle \neq 0$ then it is possible to evaluate/extract all the λ_n from the observation (2).

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Open problems:

i) How do you choose the initial condition f, so we can observe all $e^{\lambda_n t}$, that is all $c_{n\nu}(f) \neq 0$? We need to know something about the biorthogonal system $\{\psi_{n\nu}\}$.

ii) How do you choose the observation g so all $\langle \varphi_{n\nu}, g \rangle \neq 0$? We need to know something about the root functions $\{\varphi_{n,\nu}\}$.

iii) How smooth is the sum (2), so we can choose g? We need some information on the type of convergence in (2) so (3) holds.

iv) How do we extract the λ_n and their multiplicity from a given signal given by (3) in finite time? When λ_n are complex values and the sum contains polynomials in t, it is much harder than the real case.

v) Find the best f and g that allow the identification of A by using the smallest number of observations. Evolution equations are often found in control theory, and for that purpose, we need finite number of observations done in finite time.