NON-SELF-ADJOINT INVERSE PROBLEMS

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We are interested in identifying a non-self-adjoint operator associated with an evolution equation (parabolic or hyperbolic) through "observations" of the solution as time evolves. Thus for example in a certain Hilbert space we have

\[ u'(t) = Au(t) \quad \text{and} \quad u(0) = f \]

(1)

where, for simplicity, we assume that

\[ A = L + B \]

with \( L \) is a given (known) self-adjoint operator with "nice properties" while \( B \) is an unknown non-self-adjoint perturbation. For example \( Ay(x) = y''(x) - q(x)y(x) \) or \( Au = \Delta u - q(x)u \) with \( \text{Im} q(x) \neq 0 \). We assume that we can observe the solution through a functional \( \langle \cdot, g \rangle \) say

\[ \omega(t) = \langle u(t), g \rangle. \]

For example if \( u(x,t) \) is the solution of a heat equation, where \( x \in \Omega \subset \mathbb{R}^n \), and \( p \in \partial \Omega \), then \( \omega(t) = u(p,t) \) (temperature) or \( \omega(t) = \partial_n u(p,t) \) (heat transfer) are usual observations/readings of the solution on the boundary. Thus we want to recover \( A \) or at least its spectrum \( \sigma_A = \{ \lambda_n \} \subset \mathbb{C} \) from the observation mapping

\[ u(0) \rightarrow \omega(t). \]

To do so, although we do NOT know \( A \), we assume that it has a discrete spectrum \( \{ \lambda_n \} \subset \mathbb{C} \), and in general \( \text{Im} \lambda_n \to 0 \) as \( n \to \infty \), while \( \text{Re} \lambda_n \to -\infty \). If we denote its eigenfunctions by \( \varphi_{n,0} \) and its associated eigenfunctions (roots) by \( \varphi_{n,\nu} \) for \( \nu = 1, \ldots, m_n - 1 \), where \( m_n \) is the multiplicity of the eigenvalue \( \lambda_n \), then we can write a formal solution to the evolution equation

\[ u(t) = \sum_{n \geq 1} \sum_{\nu = 0}^{m_n - 1} e^{\lambda_n t} c_{n\nu}(f) p_{n\nu}(t) \varphi_{n\nu} \]

(2)

where the Fourier coefficients are \( c_{n\nu}(f) = \langle f, \psi_{n\nu} \rangle \) and \( \{ \psi_{n\nu} \} \) is the biorthogonal system to \( \{ \varphi_{n,\nu} \} \). Here \( p_{n\nu} \) are polynomials generated by the multiplicity of the eigenvalue \( \lambda_n \). The observation then is given by

\[ \omega(t) = \sum_{n \geq 1} \sum_{\nu = 0}^{m_n - 1} e^{\lambda_n t} c_{n\nu}(f) p_{n\nu}(t) \langle \varphi_{n\nu}, g \rangle. \]

(3)

In the best case, when all \( c_{n\nu}(f) \neq 0 \) and \( \langle \varphi_{n\nu}, g \rangle \neq 0 \) then it is possible to evaluate/extract all the \( \lambda_n \) from the observation 2.

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Date: 2015 AIM.
Open problems:

i) How do you choose the initial condition $f$, so we can observe all $e^{\lambda_n t}$, that is all $c_{n\nu}(f) \neq 0$? We need to know something about the biorthogonal system $\{\psi_{n\nu}\}$.

ii) How do you choose the observation $g$ so all $\langle \varphi_{n\nu}, g \rangle \neq 0$? We need to know something about the root functions $\{\varphi_{n,\nu}\}$.

iii) How smooth is the sum (2), so we can choose $g$? We need some information on the type of convergence in (2) so (3) holds.

iv) How do we extract the $\lambda_n$ and their multiplicity from a given signal given by (4) in finite time? When $\lambda_n$ are complex values and the sum contains polynomials in $t$, it is much harder than the real case.

v) Find the best $f$ and $g$ that allow the identification of $A$ by using the smallest number of observations. Evolution equations are often found in control theory, and for that purpose, we need finite number of observations done in finite time.