Real Resonances of Schrödinger Operators with Complex Potentials

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Notation

Consider Schrödinger operators with with a complex-valued potential

$$P = -\Delta + V(x) \quad V(x) = V_1(x) - iV_2(x), \quad x \in \mathbf{R}^n, n \geq 1, \tag{1}$$

with V_j real and

$$|V_j(x)| \leq C\langle x\rangle^{-\rho}, x \in \mathbf{R}^n, \rho > 0. \tag{2}$$

V and P are called dissipative if $V_2 \geq 0$ and $V_2 \neq 0$.
Denote $P_1 = -\Delta + V_1$.

Real resonances

Real resonances $\cong$ spectral singularities of J.T Schwartz (CPAM, 1960).

Definition. Let $\lambda \geq 0$. Assume that $\rho > 2$ if $\lambda = 0$ and $\rho > 1$ if $\lambda > 0$. We call λ a resonance of P if

$$(P - \lambda)u = 0$$

admits a solution $u \in H^2_{\mathrm{loc}}$ verifying one of the Sommerfeld's radiation conditions:

$$u(x) = \frac{e^{\pm i\sqrt{\lambda}|x|}}{|x|^{\frac{n}{2}-1}}(a(\theta) + o(1)), \quad |x| \to \infty,$$

for some $a \in L^2(\mathbf{S}^{n-1})$ and $a \neq 0$. Here $\theta = x/|x|$.

Open Question

Complex eigenvalues of P can only accumulate at real resonances (and eventually, at zero).

If P has only a finite number of real resonances and if the resolvent $R(z) = (P - z)^{-1}$ has only isolated singularity at each resonance, then the number of the complex eigenvalues of P is finite.

Question. What can one say about the singularity of the resolvent $R(z) = (P - z)^{-1}$ at a given real resonance λ?

Radiation conditions

Let $u \in L^{2,-s}$, $s > 1/2$ appropriate, and $Pu = \lambda u$, $\lambda > 0$. Then for $\rho > 1$, one has

$$u(x) = \frac{e^{i\sqrt{\lambda}|x|}}{|x|^{\frac{n}{2}-1}} a_+(\theta) + \frac{e^{-i\sqrt{\lambda}|x|}}{|x|^{\frac{n}{2}-1}} a_-(\theta) + o\left(\frac{1}{|x|^{\frac{n}{2}-1}}\right)$$

and

$$\|a_-\|^2 = \|a_+\|^2 + \frac{1}{\sqrt{\lambda}} \int_{\mathbf{R}^n} V_2(x)|u(x)|^2 dx.$$

If $V_2 = 0$, then u is outgoing if and only if u is incoming and P_1 has no positive resonances. (S. Agmon, 1974).

Question. What are Sommerfeld's radiation conditions when V_2 is long-range?

Dissipative Schrödinger operators

To see the roles played by the radiation conditions of resonant states, let P be a dissipative Schrödinger operator (i.e. $V_2\ ge 0$, $V_2 \neq 0$). Assume that $\rho > 2$ and $n \geq 3$. Then one can show that

- There are no real resonances in $[0, c_0]$ for some $c_0 > 0$.
- For any $\lambda > 0$, $\exists V \in C_0^\infty$, dissipative, such that λ is a real resonance of $-\Delta + V$.
- There are no outgoing resonant states for dissipative Schrödinger operators.

Consequences

As a result, one deduces

- Complex eigenvalues of P can not accumulate at low energies if $\rho > 2$ and $n \geq 3$.
- Time-decay estimates of the semigroup e^{-itP} for positive times. For example, if $n = 3$ and $\rho > 2$, one has the dispersive estimate

$$\|e^{-itP}u\|_{L^\infty} \leq Ct^{-3/2}\|u\|_{L^1}, \quad t > 0, \forall u \in L^1. \tag{3}$$

Rate of energy dispersion of eigenfunctions
(3) implies that for any $z \in \mathbf{C}_-$ and any u with $Pu = zu$, $\|u\|_{L^1} = 1$, one has

$$\|u\|_\infty \leq Ce^{-1}|\Im z|^{3/2}.$$

Singularity of the resolvent

Assume $\rho > 1$ and $n = 3$, V complex potential. Let $\lambda > 0$ be a resonance of P with some incoming resonant states. Denote $K_0 = R_0(\lambda - i0)V$, where $R_0(z) = (-\Delta - z)^{-1}$. Let ϕ_j, $j = 1, \cdots, m$, be a basis of $\ker(1 + K_0)$ in $L^{2,-s}$ for $s > 1/2$ and close to $1/2$. Define the matrix M by

$$M = (\langle B_0 V\phi_i, V^*\phi_j^*\rangle)_{1\le i,j\le m} \tag{4}$$

where $f^* = \overline{f}$ and B_0 is an operator with integral kernel

$$B_0(x, y) = e^{-i\sqrt{\lambda}|x-y|}.$$

Singularity of the resolvent

Theorem 1

Assume $\rho > 5$, E semisimple or nonderogatory, M non-singular. Then one has: $\exists \epsilon_0 > 0$ and an operator T with finite rank such that

$$R(z) - \frac{T}{\lambda - z} = O(1) : L^{2,s} \to L^{2,-s}, s > 5/2,$$

for any $z \in \mathbf{C}_-$ with $|z - \lambda| < \epsilon_0$.

Question. What can one say about the singularity of the resolvent at a given positive resonance in more general situations?