Two open problems from

linear stability analysis in hydrodynamics

Christiane Tretter

University of Bern, Switzerland

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1. Orr-Sommerfeld problem [Orr 1907, Sommerfeld 1909]

Flow of a viscous incompressible fluid between two parallel plates (linearization of Navier-Stokes):

$$\Big((-\mathrm{D}_x^2+\alpha^2)^2+\mathrm{i}\alpha R(V(-\mathrm{D}_x^2+\alpha^2)+V'')\Big)y = \underbrace{\mathrm{i}\alpha Rc}_{=:\lambda}(-\mathrm{D}_x^2+\alpha^2)y$$

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Poiseuille flow $V(x) = 1 - x^2$: R_{crit} unknown!!!!

Heisenberg's PhD thesis

Über Stabilität und Turbulenz von Flüssigkeitsströmen.

Inaugural-Dissertation

zur Erlangung der Doktorwürde der

Hohen philosophischen Fakultät II Section

der Ludwig-Maximilians-Universität München

vorgelegt am 10. Juli 1923

von

Werner Heisenberg.

Über Stabilität und Turbulenz von Flüssigkeitsströmen. 605

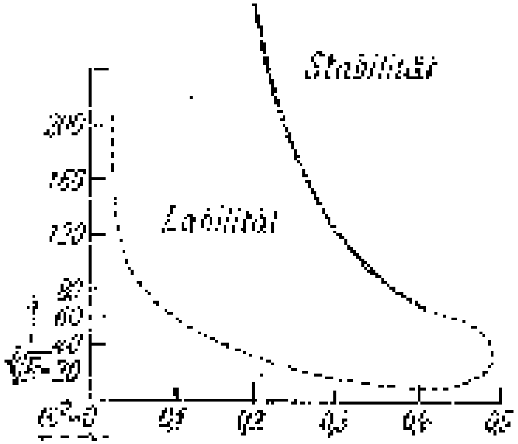
1. Es gibt sowohl einen Maximalwert von α, wie einen Minimalwert von R, nach deren Über- bzw. Unterschreitung die Labilität aufhört.

2. Für einen *bestimmten* Wert von R existiert sowohl ein *Maximal-* wie auch ein *Minimal*wert von α; innerhalb dieser Werte herrscht *Labilität*, außerhalb *Stabilität*.

3. Der Maximalwert von α liegt etwa bei $\alpha = 0,7$ ($\alpha^2 = \frac{1}{2}$). Der Minimalwert von R bei Größen der Ordnung 10^3. Eine einigermaßen genaue Berechnung dieses Minimalwertes aus der Figur ist nicht möglich.

published in Annalen der Physik 74 (1924)

"Heisenberg tongue" (1924 and [Lin, 1955])



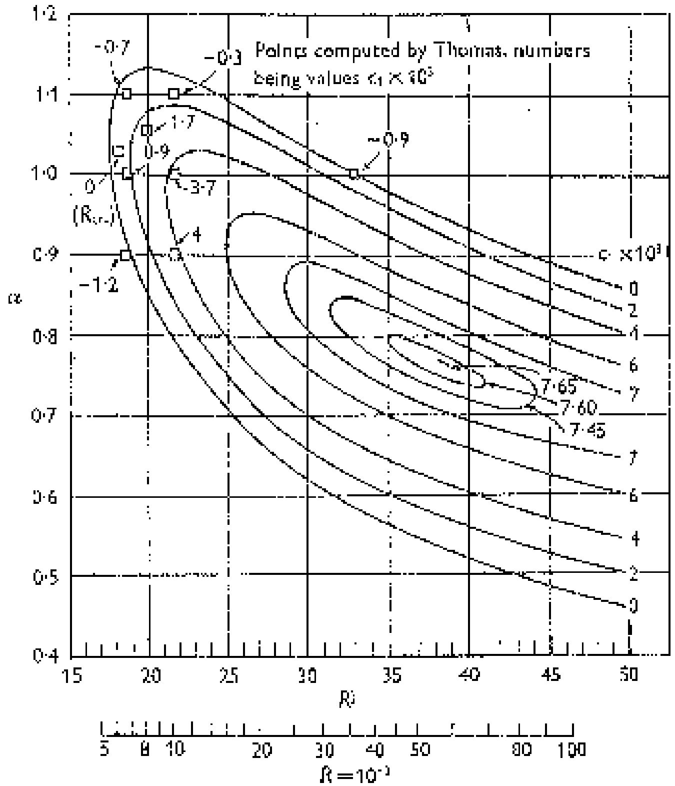


Fig. 3.1. Stability characteristics of plane Poiseuille motion (after Shen, 1954).

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Guaranteed numerics: $R_{\mathrm{crit}} \leq 5772.221818$

[Brown, M. Langer, Marletta, C.T., Wagenhofer 2010]

2. Ekman boundary layer problem [Faller 1963, Lilly 1966]

$$\underbrace{\begin{pmatrix} (-\mathrm{D}_x^2+\alpha^2)^2+\mathrm{i}\alpha R(V(-\mathrm{D}_x^2+\alpha^2)+V'') & 2\mathrm{D}_x \\ 2\mathrm{D}_x+\mathrm{i}\alpha RU' & -\mathrm{D}_x^2+\alpha^2+\mathrm{i}\alpha RV \end{pmatrix}}_{=\mathcal{A}} \begin{pmatrix} y \\ z \end{pmatrix} = \lambda \underbrace{\begin{pmatrix} -\mathrm{D}_x^2+\alpha^2 & 0 \\ 0 & I \end{pmatrix}}_{=\mathcal{B}} \begin{pmatrix} y \\ z \end{pmatrix}$$

on $[0,\infty)$, $\lim_{x\to\infty} U', V, V'' = 0$.

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$$\sigma_{\mathrm{ess}}(\mathcal{L}) = \{\lambda\in\mathbb{C} : \mathcal{L}(\lambda) \text{ not Fredholm}\} = \left\{ t \pm \mathrm{i}\, 2\sqrt{1-\frac{\alpha^2}{t}} : t\in[\alpha^2,\infty)\right\}$$

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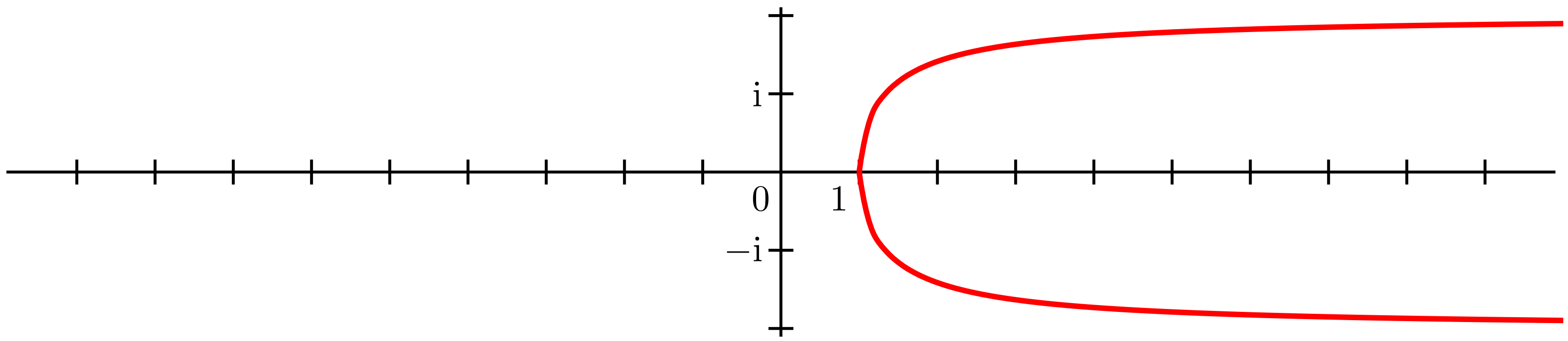

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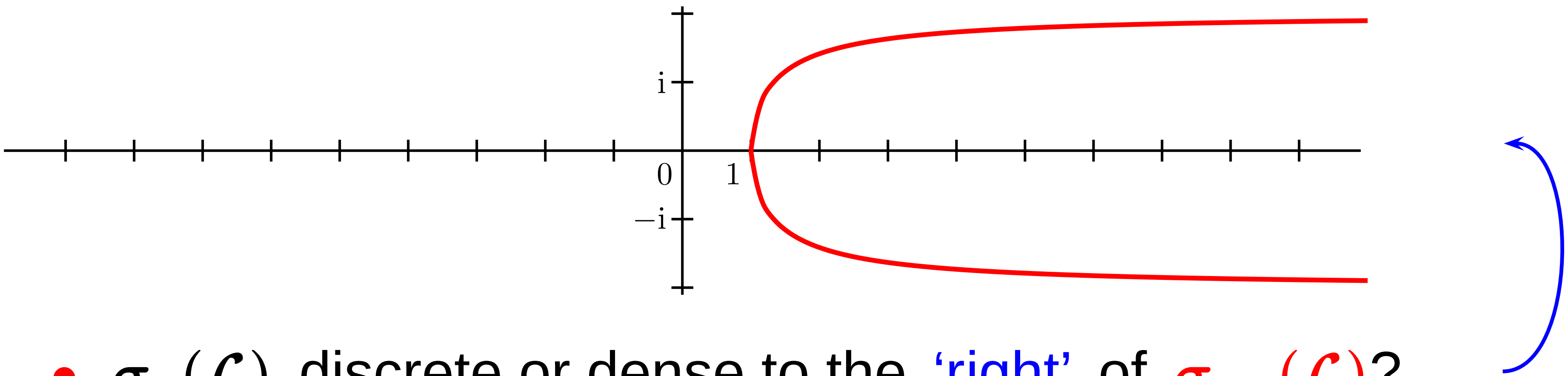

Open:
- $\sigma_{\rm p}(\mathcal{L})$ discrete or dense to the 'right' of $\sigma_{\rm ess}(\mathcal{L})$?
- critical Reynolds number $R_{\rm crit}$?

Prize for solution (for each problem):

+ Bottle of (true) Champagne:

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+ Fast track publication in IEOT . . . . :