Perturbation determinants and Evans function

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Table of contents

1. Introduction
2. Schatten–von Neumann class
3. Evans function
4. Fredholm determinant for the travelling wave problems
    - Reformulation
    - Connection

Introduction

Consider the following eigenvalue problem

$$L(\lambda)Y = \big(\mathrm{d}/\mathrm{d}x - A(\cdot,\lambda)\big)\,Y = 0, \tag{1}$$

where $\mathrm{dom}(L(\cdot)) = H^1(\mathbb{R},\mathbb{C}^n) \subset L^2(\mathbb{R},\mathbb{C}^n)$ and $A(\cdot,\lambda)$ is analytic in λ. Assume

$$A(x,\lambda) = A_0(x,\lambda) + V(x) \tag{2}$$

where $A_0(x,\cdot)$ is bounded and continuous and

$$\|V\|_{\mathbb{C}^{n\times n}} \in L^1(\mathbb{R},\mathbb{C}) \tag{3}$$

Our objective is to find values of λ for which $\dim\ker\big(L(\lambda)\big) \neq 0$.

For such a purpose, the following methods are usually preferred:

1. computing eigenvalues associated with the finite dimensional operator approximating $L(\cdot)$. This can be achieved, e.g., by a finite difference/element methods and so on; or
2. computing the zeros of the so called *Evans function* $E(\lambda)$.

Our strategy to locate λ is to compute the zeros of an analytic function. Such a function is defined as the Fredholm determinant.

Let $\mathcal{H}$ $\mathcal{J}_\infty$ denote a separable Hilbert space and the set of compact operators in $\mathcal{H}$, respectively. Then, the Schatten–von Neumann classes of compact operators are defined by

$$\mathcal{J}_p = \{A \in \mathcal{J}_\infty : \mathrm{tr}(|A|^p) < \infty\} \quad (1 \leqslant p < \infty)$$

with norm

$$\|A\|_p^p = \mathrm{tr}(|A|^p).$$

The operator A is of trace-class if $\|A\|_1 < \infty$ and is of Hilbert–Schmidt class if $\|A\|_2 < \infty$. Given any operator $A \in \mathcal{J}_p$, the p-modified Fredholm determinants are given by

$$\det_p(\mathrm{id} - A) = \prod_{n=1}^{\infty} \left[(1 - \mu_n) \exp\left(\sum_{j=1}^{p-1} \mu_n^j / j \right) \right],$$

where $\{\mu_n\}_{n \geqslant 1}$ are the set of eigenvalues of A. If $A \in \mathcal{J}_1$ then the Fredholm determinant is defined by

$$\begin{aligned} d(\lambda) &:= \det_1(\mathrm{id} - A) \\ &= \prod_{n=1}^{\infty} (1 - \mu_n). \end{aligned}$$

Properties

Let $A\colon \mathcal{H} \to \mathcal{H}$ be defined by

$$Au(x) = \int_{\mathbb{R}} k(x,y)u(y)\mathrm{d}y.$$

If $k \in L^2(\mathbb{R}^2)$ then $A \in \mathcal{J}_2$. Indeed, $\|A\|_2 = \|k\|_{L^2(\mathbb{R}^2)}$. Suppose that $A \in \mathcal{J}_1$ with $k \in C(\mathbb{R}^2)$ then

$$\mathrm{tr}(A) = \int_{\mathbb{R}} k(x,x)\mathrm{d}x.$$

Moreover, we have

$$d(\lambda) = \det_2(\mathrm{id} - A)e^{-\mathrm{tr}(A)}.$$

The Evans function $E(\lambda)$ is an analytic function whose zeros coincide in location and multiplicity to the eigenvalues associated with the operator $L(\cdot)$. Explicitly, the Evans function $E(\lambda)$ is defined by

$$E(\lambda) = \mathrm{e}^{\int_0^x \operatorname{tr}\left(A(x,\lambda)\right)\mathrm{d}s} Y^{-}(x,\lambda) \wedge Y^{+}(x,\lambda), \qquad (4)$$

where $Y^{\pm}(x,\lambda)$ are the subspaces decaying at $\pm\infty$.
If $\mu_j \in \sigma\left(A_0(\lambda)\right)$ are simple then the Evans function is reduced to a simple Wronskian, i.e.,

$$E(\lambda) = \mathrm{e}^{\int_0^x \operatorname{tr}\left(A(x,\lambda)\right)\mathrm{d}s} \det_{\mathbb{C}^{n\times n}}(u_1^{-},\cdots,u_k^{+},u_{k+1}^{+},\cdots,u_n^{+})(x,\lambda)$$

where $u_j^{\pm}$ are the solution of (1) decaying at $\pm\infty$.

Given the decomposition (2), we write

$$L(\lambda) = L_0^{-1}(\lambda)\big(\mathrm{id} - L_0^{-1}(\lambda)V\big), \quad \big(\forall\lambda \in \rho\big(L_0(\cdot)\big)\big) \tag{5}$$

where

$$L_0(\lambda) := \mathrm{d}/\mathrm{d}x - A_0(\cdot, \lambda)$$

and $\rho\big(L_0(\cdot)\big)$ is the resolvent set of $L_0(\cdot)$. Assume that $L_0^{-1}(\lambda)V \in \mathcal{J}_\infty$. Then from (5), it follows that

$$\lambda \in \sigma_d\big(L(\cdot)\big) \Leftrightarrow \det_F\big(\mathrm{id} - L_0^{-1}(\lambda)V\big) = 0,$$

where $\det_F$ denote the determinant of a Fredholm operator. Equivalently, we rewrite the above vanishing determinant as

$$\det_F\big(\mathrm{id} - K(\lambda)\big) = 0,$$

where $K(\lambda)$ is the Birman–Schwinger operator given by

$$K(\lambda) = |V|^{1/2} L_0^{-1}(\lambda)\widetilde{V},$$

with $\widetilde{V} = U|V|^{1/2}$ (U a unitary transformation). The integral kernel associated with $K(\lambda)$ is given by

$$k(x,y,\lambda) = \begin{cases} -|V(x)|^{1/2}\Phi(x,\lambda)Q\Phi^{-1}(y,\lambda)\widetilde{V}(y), & x \leqslant y \\ |V(x)|^{1/2}\Phi(x,\lambda)(\mathrm{id}-Q)\Phi^{-1}(y,\lambda)\widetilde{V}(y), & x > y, \end{cases}$$

where Φ is the fundamental matrix solution of $L_0(\lambda)Y = 0$ and Q is a projection operator. Assuming that $K(\lambda) \in \mathcal{J}_\infty$ then

$$\lambda \in \sigma_d\big(L(\cdot)\big) \Leftrightarrow 1 \in \sigma_d\big(K(\lambda)\big).$$

With the assumption that $\rho\big(L_0(\cdot)\big) \neq \emptyset$ and $\|V\|_{\mathbb{C}^{n\times n}} \in L^1$. We have that

$$K(\lambda) \in \mathcal{J}_2, \quad (\text{since } \|k\|_{L^2(\mathbb{R}^2,\mathbb{C}^{n\times n})} < \infty).$$

In general, the integral operator $K(\lambda)$ is of Hilbert–Schmidt class. However, assume that $A_0(x, \lambda) = A_0(\lambda)$ and that $A_0(\lambda)$ is hyperbolic. Moreover, assume that $A_0(\lambda)$ is diagonalisable then we have:

Theorem

For $\lambda \in \rho\big(L_0(\cdot)\big)$, the operator $K(\lambda)$ is of trace class.

Proof.

Write $\hat{g}^{-1}(-i\mathrm{d}/\mathrm{d}x) = \big(\mathrm{d}/\mathrm{d}x - A_0(\lambda)\big)^{-1}$. Then, one can show that $\|\hat{g}^{-1}(\xi)\|^2_{\mathbb{C}^{n\times n}} \leqslant c\frac{1}{1+\xi^2}$. By Corollary 4.8 in (a) it follows that $K(\lambda)$ is of trace class □

Define the *matrix transmission coefficient* $D(\lambda)$ by

$$D(\lambda) := \lim_{x \to \infty} Z_0^-(x, \lambda) Y^+(x, \lambda),$$

where $Z_0^- \in L^2(\mathbb{R}^-, \mathbb{C}^{k \times n})$ and $Y^+ \in L^2(\mathbb{R}^+, \mathbb{C}^{n \times k})$ are the solution of the adjoint problem of $L_0(\lambda)Y = 0$ and the matrix-valued Jost solution decaying at $+\infty$ of $L(\lambda)Y = 0$, respectively. Assume that $A_0(x, \lambda) = A_0(\lambda)$.

Theorem

For $\lambda \in \rho$, we have

$$\det{}_{\mathbb{C}^{k \times k}} D(\lambda) = \frac{E(\lambda)}{c(\lambda)},$$

where $c(\lambda) = \det_{\mathbb{C}^{n \times n}} \Phi(\cdot, \lambda)$

Assume that $K(\lambda)$ is of trace class. Then the following result holds

Theorem

For $\lambda \in \rho\big(L_0(\cdot)\big)$, we have

$$\det_1\big(id - K(\lambda)\big) = \det_{\mathbb{C}^{k\times k}} D(\lambda).$$

Hence

$$\det_1\big(id - K(\lambda)\big) = \frac{E(\lambda)}{c(\lambda)}.$$

That is, the infinite dimensional determinant in the left-hand side is reduced to a finite dimensional determinant!

As a consequence of the above theorem, we have

$$\det_F L(\lambda) = \tilde{c}(\lambda)E(\lambda),$$

where $\tilde{c}(\lambda) = \det_F L_0(\lambda)/\det_{\mathbb{C}^{n\times n}} \Phi(\cdot,\lambda)$.

References

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