The spectrum of second order multi-point problems

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Multi-point problems

Consider the eigenvalue problem

\[- u'' = \lambda ru \quad \text{on } (-1, 1),\]

where \( r \in C^1[-1, 1], \ r > 0 \) and \( \lambda \in \mathbb{R} \), together with the multi-point boundary conditions

\[ u(\pm 1) = \sum_{i=1}^{m^\pm} \alpha_i^\pm u(\eta_i^\pm), \]

where \( m^\pm \geq 1 \) are integers, and, for \( i = 1, \ldots, m^\pm \): \( \alpha_i^\pm \in \mathbb{R}, \ \eta_i^\pm \in (-1, 1) \).

An eigenvalue is a (real) number \( \lambda \) for which (1)-(2) has a non-trivial solution \( u \), called an eigenfunction.
Functional setting, nodal sets

More precisely, we consider solutions as elements of

\[ X := \{ u \in C^2[-1,1] : u \text{ satisfies } (2) \}. \]

Similarly to Sturm-Liouville theory with separated boundary conditions, we get eigenfunctions with prescribed nodal properties.

For \( \nu = \pm \) and \( k = 1, 2, \ldots \), we define \( T^\nu_k \subset X \) by:

\[ u \in T^\nu_k \iff (a) \quad u'(\pm1) \neq 0 \text{ and } \nu u'(-1) > 0; \]
\[ (b) \quad u' \text{ has only simple zeros, and exactly } k \text{ zeros in } (-1,1); \]
\[ (c) \quad u \text{ has a zero strictly between each consecutive zero of } u'. \]

We also define \( T_k = T^+_k \cup T^-_k, \quad k = 1, 2, \ldots \)

NB A function \( u \in T_k \) has at least \( k - 1 \) and at most \( k \) zeros in \((-1,1)\).
Main result

It is convenient to use the notation

\[ \alpha^\pm := (\alpha_1^\pm, \ldots, \alpha_{m^\pm}^\pm) \in \mathbb{R}^{m^\pm}, \quad |\alpha^\pm| := \sum_{i=1}^{m^\pm} |\alpha_i^\pm|. \]

Theorem 1 (Rynne-G. NA 2011)

For any \( r \in C^1[-1, 1], \ r > 0, \) there exists \( \gamma = \gamma(r) \in (0, 1] \) such that if \( |\alpha^\pm| < \gamma \) then all the eigenvalues are real and simple, and they form a strictly increasing sequence \( \lambda_k = \lambda_k(r) > 0, \ k \geq 1. \) Furthermore, each eigenvalue \( \lambda_k \) has an eigenfunction \( u_k \in T_k^+ \), and we have \( \lim_{k \to \infty} \lambda_k = \infty. \)
Proof

Let \( w(\lambda, \theta) \) be the solution of \(-u'' = \lambda ru\) satisfying

\[
w(\lambda, \theta)(0) = \sin \theta, \quad w(\lambda, \theta)'(0) = \lambda^{1/2} \cos \theta,\]

[case \( r \equiv 1: \ w(\lambda, \theta)(x) = \sin(\lambda^{1/2}x + \theta) \)]

and consider the \( C^1 \) functions \( \Gamma^\pm : (0, \infty) \times \mathbb{R} \times \mathbb{R}^{m^\pm} \to \mathbb{R} \)

defined by

\[
\Gamma^\pm(\lambda, \theta, \alpha^\pm) := w(\lambda, \theta)(\pm 1) - \sum_{i=1}^{m^\pm} \alpha_i^\pm w(\lambda, \theta)(\eta_i^\pm).
\]

Now, \( \lambda \) eigenvalue of (1)-(2) \( \iff \ \Gamma^\pm(\lambda, \theta, \alpha^\pm) = 0 \)

and then \( u = w(\lambda, \theta) \) is a corresponding eigenfunction.
\[ \Gamma^\pm(\lambda, \theta, \alpha^\pm) := w(\lambda, \theta)(\pm 1) - \sum_{i=1}^{m^\pm} \alpha_i^\pm w(\lambda, \theta)(\eta_i^\pm) = 0 \quad (3) \]

For \( \alpha^\pm = 0 \) (Dirichlet b.c.) Sturm-Liouville theory yields
\[ \lambda_0^k, \ u_0^k = w(\lambda_0^k, \theta_0^k), \quad k = 1, 2, \ldots, \]
with the usual properties.

[case \( r \equiv 1 \): \( \lambda_0^k = \left(\frac{k\pi}{2}\right)^2 \) and \( u_0^k(x) = \sin\left(\frac{k\pi}{2}x + \frac{k\pi}{2}\right) \), i.e. \( \theta_0^k = \frac{k\pi}{2} \)]

We solve (3) for \( \alpha^\pm \neq 0 \) by continuation from the case \( \alpha^\pm = 0 \).

By the implicit function theorem, we get solutions
\[ \lambda_k(\alpha^\pm), \ \theta_k(\alpha^\pm), \ u_k(\alpha^\pm) = w(\lambda_k(\alpha^\pm), \theta_k(\alpha^\pm)), \quad k = 1, 2, \ldots, \]
for \( |\alpha^\pm| < \gamma(r) \) with an appropriate \( \gamma(r) \in (0, 1] \).
\[ \Gamma^\pm(\lambda, \theta, \alpha^\pm) := w(\lambda, \theta)(\pm 1) - \sum_{i=1}^{m^\pm} \alpha_i^\pm w(\lambda, \theta)(\eta_i^\pm) = 0 \]

This involves checking that the Jacobian determinant

\[ J(\lambda, \theta, \alpha^\pm) := \begin{vmatrix} \Gamma^-_\lambda(\lambda, \theta, \alpha^-) & \Gamma^-_\theta(\lambda, \theta, \alpha^-) \\ \Gamma^+_\lambda(\lambda, \theta, \alpha^+) & \Gamma^+_\theta(\lambda, \theta, \alpha^+) \end{vmatrix} \neq 0. \quad (4) \]

A key step is to prove that \( w(\lambda, \theta)'(\pm 1) \neq 0 \) if \(|\alpha^\pm| < \gamma(r)\).

A priori estimates for \( w(\lambda, \theta) \), \( w_\lambda(\lambda, \theta) \) and \( w_\theta(\lambda, \theta) \) also contribute to the definition of \( \gamma(r) \).
Open problems

In case $r \equiv 1$, $w(\lambda, \theta)(x) = \sin(\lambda^{1/2}x + \theta)$, and the analysis is easier. In fact, we can take $\gamma(1) = 1$.

Inspecting the simple 3-point problem

$$-u'' = \lambda u \quad \text{on } (0, 1), \quad u(0) = 0, \quad u(1) = \alpha u(\eta),$$

for various values of $\alpha \in \mathbb{R}$ and $\eta \in (0, 1)$, one can observe that:

(i) for $\alpha = 1$, we may have $u'(1) = 0$, so $u \notin T_k$ for any $k \geq 1$;

(ii) for $\alpha > 1$, there may be no eigenfunctions in the sets $T_l, T_{l+1}, \ldots, T_{l+n}$, for arbitrarily large $n$.

We have only been interested in real eigenvalues here.

Problem 1

What happened to the ‘missing eigenvalues’ in (ii), have they become complex?
For $r \equiv 1$, Theorem 1 still holds [Rynne NA 2010] for the ‘$p$-linear’ problem

$$- (|u'|^{p-2}u')' = \lambda r |u|^{p-2}u, \quad u(\pm 1) = \sum_{i=1}^{m^{\pm}} \alpha_i^{\pm} u(\eta_i^{\pm}). \quad (5)$$

There has been a lot work on this, in particular in connection with bifurcation for fully nonlinear problems.

**Problem 2**

Extend Theorem 1 to problem (5) with $r \neq 1$.

We haven’t really tried to do this, but there seem to be considerable technical difficulties, for instance related to integration by parts arguments, a priori estimates, etc.