Self-adjoint Jacobi operators possessing generalized eigenvectors with the strongly increasing phase sequence: do they exist?

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Let us consider the right-side infinite Jacobi matrix

\[
\begin{pmatrix}
q_1 & w_1 \\
w_1 & q_2 & w_2 \\
w_2 & q_3 & w_3 \\
w_3 & q_4 & \ddots \\
& \ddots & \ddots
\end{pmatrix}
\]

determined by real sequences \(\{q_n\}_{n \geq 1}\), \(\{w_n\}_{n \geq 1}\), \(w_n \neq 0\), and the Jacobi operator \(J\) in the Hilbert space \(\ell^2(\mathbb{N})\) of square-summable complex sequences on \(\mathbb{N} := \{1, 2, \ldots\}\). \(J\) is the restriction of the formal Jacobi operator \(J\) to the domain

\[
D(J) := \{ u \in \ell^2(\mathbb{N}) : Ju \in \ell^2(\mathbb{N}) \},
\]

where \(J\) acts in the vector space \(\ell(\mathbb{N})\) of all complex sequences on \(\mathbb{N}\), and it is given by

\[
(Ju)(n) := w_{n-1}u(n-1) + q_n u(n) + w_n u(n+1), \quad n \in \mathbb{N},
\]

for any \(u = \{u(n)\}_{n \geq 1} \in \ell(\mathbb{N})\).

For \(\lambda \in \mathbb{C}\) we consider generalized eigenvectors of \(J\) for \(\lambda\), i.e., such \(u = \{u(n)\}_{n \geq 1} \in \ell(\mathbb{N})\) that

\[
((J - \lambda)u)(n) = 0, \quad n \geq 2.
\]

By \(\text{Sol}(\lambda)\) we denote the linear space of all the generalized eigenvectors of \(J\) for \(\lambda\), so \(\dim \text{Sol}(\lambda) = 2\).

Various forms of asymptotics for two linearly independent generalized eigenvectors have been found through some asymptotic tricks for difference equations, see e.g., [1, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15]. For many of the described Jacobi operators, for \(\lambda\)-s from some non-empty open intervals the authors proved the existence of two linearly independent vectors \(u_+, u_-\) from \(\text{Sol}(\lambda)\), having the general form

\[
u_\pm(n) = r(n) \exp(\pm ia_n) \cdot s_\pm(n), \quad n \in \mathbb{N}.
\]

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In this formula $r$ is the positive explicit modulus, $a$ — the real explicit phase, $s_\pm$ are the complex implicit terms with some explicit “convergence properties”, for instance: $s_\pm(n) \to 1$ or $s_\pm(n) = p_\pm(n) + o(1)$, with $p_\pm$ being explicitly computable non-zero periodic sequences. In particular, such situation seems to be typical for the so-called Jordan box case (or critical case) (— see e. g. [3] for the definition).

In many examples $r(n) := n^{-b}$ with $b \leq \frac{1}{2}$.

It is proved in [11] that such a form of a base of Sol($\lambda$) guarantees non-existence of subordinated solutions of (0.2), and consequently, by subordination theory ([2, 10]), this gives the absolute continuity of $J$ on the appropriate interval of $\mathbb{R}$.

According to my knowledge, in the existing literature dealing with asymptotics of some base vectors $u_+, u_-$ of Sol($\lambda$) with the general form 0.3 (in the self-adjoint case), we can find only the explicit phase sequences $a$ being “weakly increasing”, i.e., $(\Delta a)(n) \to 0$ or $a(n) = n^{\alpha}(c + o(1))$ with $c > 0$ and $0 < \alpha < 1$, if we limit ourselves to all $\lambda$-s from some non-empty open intervals of $\mathbb{R}$.

Open problem

The problem is to construct such a self-adjoint Jacobi operator (by finding explicit formulae on its weights $\{w_n\}_{n\geq 1}$ and diagonals $\{q_n\}_{n\geq 1}$) for which the asymptotic formula (0.3) of some base vectors $u_+, u_-$ of Sol($\lambda$) holds, with the “strongly increasing” phase sequences $a$ for all $\lambda$-s from a non-empty open interval $A \subset \mathbb{R}$. More precisely, for any $\lambda \in A$ we would like (0.3) with the following conditions to be satisfied:

1. $r$ — positive sequence, $a$ — real sequence
2. $(\Delta a)(n) \to +\infty$ or $a(n) = n^{\alpha}(c + o(1))$ with $c > 0$ and $\alpha > 1$,
3. $s_\pm(n) = p_\pm(n) + o(1)$, with $p_\pm$ being explicitly computable non-zero periodic sequences (e. g., $p_\pm \equiv 1$, if possible),

where $r, a, c, p_\pm$ can depend on $\lambda \in A$.

References


