Let us consider the Convolution operator $C(a)$ on $L^2(\mathbb{R}^2)$ defined by,

$$\begin{array}{llcl} C(a): & L^2(\mathbb{R}^2) & \rightarrow & L^2(\mathbb{R}^2) \\ & f(x) & \mapsto & \lambda f(x) + \int_{\mathbb{R}^2} u(x-t) f(t) dt \end{array}$$

where $u \in L^1(\mathbb{R}^2)$ and $\lambda \in \mathbb{C}$. The function $a := \lambda + \mathcal{F}(u)$, where $\mathcal{F}$ is the Fourier transform, is usually called the symbol of the operator.

Given a closed connected subset Γ of the unit circle, we define the cone K, to be the following angular sector :

$$K := \{t(x, y) : t \in [0, +\infty[\text{ and } (x, y) \in \Gamma\}$$

and the convolution operator on the cone K, the operator:

$$C_K(a) = \chi_K C(a) \chi_K I + (1 - \chi_K) I,$$

where $\chi_K I$ is the characteristic function of K.

Theorem. *([1])*
Let $a = \lambda + \mathcal{F}(u)$ *, with* $u \in L^1(\mathbb{R}^2)$ and $\lambda \in \mathbb{C}$. The operator $C_K(a)$ is Fredholm if and only if a is invertible, (i.e a does not vanish in $\mathbb{R}^2$ and $\lambda \neq 0$). Furthemore if $C_K(a)$ is Fredholm then its index is zero.

A generalization of this result for a larger class of operators is obtained in ([2], Cor.4.17) .

Define the bidimensional Wiener algebra as

$$W(\mathbb{R}^2) := \{\lambda + \mathcal{F}(u) : u \in L^1(\mathbb{R}^2), \lambda \in \mathbb{C}\}$$

Problem. For any $a \in W(\mathbb{R}^2)$ and any cone K, does the invertibility of a implies the invertibility of the convolution operator on the cone $C_K(a)$?

Remark. The result is true if the cone is a half-space or if the symbol is of a particular type.

References

[1] Simonenko, I. B. Convolution type operators in cones. (Russian) Mat. Sb. (N.S.) 74 (116) 1967 298–313.

[2] Mascarenhas, H.; Silbermann, B. Spectral approximation and index for convolution type operators on cones on Lp(R2). Integral Equations Operator Theory 65 (2009), no. 3, 415–448.