This *open problem* concerns the asymptotics of solution of a certain ODE. Consider the differential expression

$$\ell = -\frac{d^2}{dx^2} + \frac{1}{x^4}$$

and the corresponding spectral equation

$$\ell(y) = \lambda y, \qquad \text{for } x > 0 \text{ and } \lambda \in \mathbb{C}. \tag{1}$$

The potential $\frac{1}{x^4}$ is *stongly singular* in the sense of [1], and hence (see [1, 2]) we know that

- There exists a (regular) solution $\phi(\lambda, x)$ of the differential equation (1) that is square integrable at $x = 0$ and entire in λ.
- There exists a second linear independent (singular) solution $\theta(\lambda, x)$ of (1), which is entire in λ, such that the Wronskian $W(\theta, \phi) \equiv 1$.

Since ℓ is in limit point case at ∞, for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there is (up to a scalar multiple) only one solution that is square integrable at ∞, and it can be written as

$$\theta(\lambda, \cdot) + m(\lambda)\phi(\lambda, \cdot),$$

where m is the so-called generalized Titchmarsh-Weyl coefficient.

My question is: Calculate m!

It is obvious that m is not unique and depends on the choice of ϕ and θ. In particular, two different generalized Titschmarsh-Weyl coefficients m_1 and m_2 are related via

$$m_2(\lambda) = e^{-2g(\lambda)} m_1(\lambda) + e^{-g(\lambda)} f(\lambda),$$

where $g(z)$ and $f(z)$ are real entire functions, see [2, Remark 2.7].

In the litterature I found asymptotics of two basic solutions at 0 and at ∞, BUT for different pais of solutions at 0 and $infty$ and without an obvious connection between them.

References

[1] F. Gesztesy and M. Zinchenko, On spectral theory for Schrödinger operators with strongly singular potentials, Math. Nachr. **279** (2006), no.9–10, 1041–1082.

[2] A. Kostenko, A. Sakhnovich, and G. Teschl, Weyl-Titchmarsh Theory for Schrödinger Operators with Strongly Singular Potentials, Int.Math.Res.Not. 2011, Art.ID rnr065, 49pp (2011).

If somebody is familiar with this problem, please let me know!

Annemarie Luger
luger@math.su.se