This open problem concerns the asymptotics of solution of a certain ODE. Consider the differential expression

$$\ell = -\frac{d^2}{dx^2} + \frac{1}{x^4}$$

and the corresponding spectral equation

$$\ell(y) = \lambda y, \quad \text{for } x > 0 \text{ and } \lambda \in \mathbb{C}. \quad (1)$$

The potential $\frac{1}{x^4}$ is strongly singular in the sense of [1], and hence (see [1, 2]) we know that

- There exists a (regular) solution $\phi(\lambda, x)$ of the differential equation (1) that is square integrable at $x = 0$ and entire in $\lambda$.

- There exists a second linear independent (singular) solution $\theta(\lambda, x)$ of (1), which is entire in $\lambda$, such that the Wronskian $W(\theta, \phi) \equiv 1$.

Since $\ell$ is in limit point case at $\infty$, for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$ there is (up to a scalar multiple) only one solution that is square integrable at $\infty$, and it can be written as

$$\theta(\lambda, \cdot) + m(\lambda)\phi(\lambda, \cdot),$$

where $m$ is the so-called generalized Titchmarsh-Weyl coefficient.

My question is: Calculate $m$!

It is obvious that $m$ is not unique and depends on the choice of $\phi$ and $\theta$. In particular, two different generalized Titchmarsh-Weyl coefficients $m_1$ and $m_2$ are related via

$$m_2(\lambda) = e^{-2g(\lambda)}m_1(\lambda) + e^{-g(\lambda)}f(\lambda),$$

where $g(z)$ and $f(z)$ are real entire functions, see [2, Remark 2.7].

In the literature I found asymptotics of two basic solutions at 0 and at $\infty$, BUT for different pairs of solutions at 0 and $\infty$ and without an obvious connection between them.

References


If somebody is familiar with this problem, please let me know!

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