Basis properties of the eigenvectors of a $\mathcal{PT}$-symmetric Sturm-Liouville operator

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Let $f(x) = \sin(x)$ and $0 < \varepsilon < 2$. The singular Sturm-Liouville differential operator

$$\ell[u](x) = \varepsilon(fu')'(x) + u'(x) \tag{1}$$

with periodic boundary conditions at $\pm\pi$ originated in applications from hydrodynamics [1], and it has recently attracted some interest due to various unusual stability and symmetry properties. The spectrum of ℓ was examined simultaneously in various works by Chugunova, Karabash, Pelinovsky and Pyatkov [4, 5], and Davies and Weir [6, 7, 8, 9, 10]. Remarkably it was noted that the associated closed operator, defined on a suitable domain reproducing the singularities and boundary conditions, has a purely discrete spectrum comprising conjugate pairs lying on the imaginary axis and accumulating only at $\pm i\infty$ and a set of eigenfunctions which is complete, but does not form a Riesz basis of $L^2(-\pi, \pi)$.

In [2, 3] it was shown that the spectrum has a similar structure, if we replace $\sin(x)$ by a more general function f, assuming that it is

1. absolutely continuous and 2π-periodic,
2. differentiable except possibly at a finite number of points excluding integer multiples of π,
3. $1 = f'(0) \neq 0$ exists and $f(x) = x + O(x^2)$ near x, and
4. $f(x + \pi) = -f(x)$, $f(-x) = -f(x)$ and $f(x) > 0$ for all $x \in (0, \pi)$.

In [3] it was also shown that the set of eigenfunctions does not form a Riesz basis of the L^2 space. Whether the set of eigenfunctions is also complete in this general case remains an open question.

References

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