$\mathcal{PT}$-symmetric Laplace-Beltrami operator in the strip on a sphere

Consider the Laplace-Beltrami operator in a tubular neighborhood Ω of the equator of a sphere, subject to complex Robin parity and time preserving boundary conditions,

$$\frac{\partial \psi}{\partial n} + \mathrm{i}\alpha\psi = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where n is a unit normal vector and $\alpha \in \mathbb{R}$, see [1] for details.

The eigenvalue problem for this two-dimensional differential operator can be reduced by separation of variables to the eigenvalue problem of m-sectorial operators in $L^2(-a,a)$ with $a < \pi/2$:

$$\begin{gathered} H_{\alpha,\beta,m} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_m(x), \quad V_m(x) = \frac{8m^2 - 3 - \cos 2x}{8\cos^2 x}, \\ \mathrm{Dom}\,(H_{\alpha,\beta,m}) = \left\{\psi \in W^{2,2}(-a,a) : \psi'(\pm a) + (\mathrm{i}\alpha \pm \beta)\psi(\pm a) = 0\right\}. \end{gathered} \tag{2}$$

where $m \in \mathbb{Z}$, $\alpha \in \mathbb{R}$, $\beta = (\tan a)/2$.

$H_{\alpha,\beta,m}$ is not self-adjoint unless $\alpha = 0$, nonetheless it is $\mathcal{PT}$-symmetric, *i.e.* it commutes with the antilinear operator $\mathcal{PT}$, where $(\mathcal{P}\psi)(x) := \psi(-x)$ and $(\mathcal{T}\psi)(x) := \overline{\psi(x)}$.

It is known that the spectrum of $(H_{\alpha,\beta,m} - V_m)$ is real if $\beta > 0$, see [1, Prop. 4.3.]. The eigenvalues of $H_{\alpha,\beta,m}$ with sufficiently large real part are necessarily real by a standard perturbation argument [1, Prop. 4.4.]. Nonetheless, the numerical analysis of eigenvalues of $H_{\alpha,(\tan a)/2,m}$ suggests that all eigenvalues are real. *Open problem is to prove it.*

The details can be found in [1, Section 4.4.2.].

References

[1] David Krejčiřík and Petr Siegl. $\mathcal{PT}$-symmetric models in curved manifolds. *Journal of Physics A: Mathematical and Theoretical*, 43(48):485204, 2010.