\textbf{\(PT\)-symmetric Laplace-Beltrami operator in the strip on a sphere}

Consider the Laplace-Beltrami operator in a tubular neighborhood \(\Omega\) of the equator of a sphere, subject to complex Robin parity and time preserving boundary conditions,
\[
\frac{\partial \psi}{\partial n} + i\alpha \psi = 0 \quad \text{on} \quad \partial \Omega,
\]
where \(n\) is a unit normal vector and \(\alpha \in \mathbb{R}\), see [1] for details.

The eigenvalue problem for this two-dimensional differential operator can be reduced by separation of variables to the eigenvalue problem of \(m\)-sectorial operators in \(L^2(-a,a)\) with \(a < \pi/2\):
\[
H_{\alpha,\beta,m} = -\frac{d^2}{dx^2} + V_m(x), \quad V_m(x) = \frac{8m^2 - 3 - \cos 2x}{8\cos^2 x},
\]
\[
\text{Dom}(H_{\alpha,\beta,m}) = \{ \psi \in W^{2,2}(-a,a) : \psi'(\pm a) + (i\alpha \pm \beta)\psi(\pm a) = 0 \}.
\]
where \(m \in \mathbb{Z}, \alpha \in \mathbb{R}, \beta = (\tan a)/2\).

\(H_{\alpha,\beta,m}\) is not self-adjoint unless \(\alpha = 0\), nonetheless it is \(\mathcal{PT}\)-symmetric, i.e. it commutes with the antilinear operator \(\mathcal{PT}\), where \((\mathcal{P}\psi)(x) := \psi(-x)\) and \((\mathcal{T}\psi)(x) := \overline{\psi}(x)\).

It is known that the spectrum of \((H_{\alpha,\beta,m} - V_m)\) is real if \(\beta > 0\), see [1, Prop. 4.3.]. The eigenvalues of \(H_{\alpha,\beta,m}\) with sufficiently large real part are necessarily real by a standard perturbation argument [1, Prop. 4.4.]. Nonetheless, the numerical analysis of eigenvalues of \(H_{\alpha,(\tan a)/2,m}\) suggests that all eigenvalues are real. \textit{Open problem is to prove it.}

The details can be found in [1, Section 4.4.2.].

\textbf{References}