The large-time behaviour of the heat kernel: subcriticality versus criticality

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On a connected non-compact Riemannian manifold M of dimension d, let us consider the second-order operator

$$P := -\partial_i G^{ij} \partial_j + \partial_i E^i - F^i \partial_i + V$$

with real-valued coefficients satisfying some mild local-integrability conditions so that P is a locally uniformly elliptic operator on M obeying the (weak) maximum principle. We say that P is symmetric if $E^i = 0 = F^i$; then the Friedrichs extension of P defined initially on $C_0^\infty(M)$ gives rise to a self-adjoint operator on $L^2(M)$ satisfying Dirichlet boundary conditions on ∂M in a generalized sense.

In any case, let $k_P(x, x', t)$ denote the positive minimal (Dirichlet) heat kernel of P. We say that P is *subcritical* (respectively, *critical*) if for any fixed $x, x' \in M$, $x \neq x'$, we have that $k_P(x, x', \cdot) \in L^1(\mathbb{R}_+)$ (respectively, $k_P(x, x', \cdot) \notin L^1(\mathbb{R}_+)$). In a joint paper with M. Fraas and Y. Pinchover [2], we made the following conjecture:

Conjecture 1 ([2]) *Let P_+ and P_0 be respectively subcritical and critical operators on M. Then*

$$\lim_{t \to \infty} \frac{k_{P_+}(x, x', t)}{k_{P_0}(x, x', t)} = 0$$

locally uniformly for $(x, x') \in M \times M$.

The relevance of this conjecture becomes clearer if we recall the relationship of the subcriticality/criticality to properties of *positive solutions* of the elliptic equation $Pu = 0$. The *generalized principal eigenvalue* λ_0 of P is defined as the supremum over all $\lambda \in \mathbb{R}$ such that there exists a positive (weak) solution u of $Pu = \lambda u$. The solution is (up to a normalization) unique for critical operators. If P is symmetric, then λ_0 coincides with the bottom of the spectrum of the Friedrichs extension.

Let us assume that $\lambda_0 \geq 0$. Then $\lambda_0 = 0$ for any critical operator, while $\lambda_0 \geq 0$ for any subcritical operator. If the generalized principle eigenvalue of P_+ is positive, then it is easy to see that Conjecture 1 holds, so the only interesting situation is when it is equal to zero. Moreover, Conjecture 1 holds if P_0 is *positive-critical*, *i.e.*, $\varphi^* \varphi \in L^1(M)$ where φ and φ^* are the unique solutions of $P_0 u = 0$ and $P_0^* u = 0$, respectively. Finally, it follows from [4] that Conjecture 1 holds for Schrödinger operators with *short-range* potentials.

An open question is to *prove (or disprove) Conjecture 1* under the general hypotheses.

In [2], we established, *inter alia*, the following result for potential-type perturbations:

Theorem 1 ([2]) *Let P_0 be critical in M and let $P_+ = P_0 + V$ where V is a non-zero non-negative potential. Them Conjecture 1 holds true if any of the two following conditions is satisfied:*

(1) *P_0 is symmetric.*

(2) *Davies' conjecture holds for both P_0 and P_+.*

By *Davies' conjecture* we mean the following conjecture, which was raised in [1] by E. B. Davies in the self-adjoint case.

Conjecture 2 (Davies' conjecture) *Fix reference points $x_0, x_0' \in M$. Then*

$$\lim_{t\to\infty} \frac{k_P(x,x',t)}{k_P(x_0,x_0',t)} = a(x,y)$$

exists and is positive for all $x, x' \in M$.

Obviously, Conjecture 2 holds if P is positive-critical. Moreover, it holds true in the symmetric case if the solution of Pu is unique. In particular, it holds true for a critical symmetric operator.

Theorem 1 suggests that Conjectures 1 and 2 are closely related. However, *is it necessary to suppose the validity of Conjecture 2 in Theorem 1 for the non-symmetric case (2)?*

Conjecture 1 can be regarded as a point-wise version of another conjecture, made in the self-adjoint case in a joint paper with E. Zuazua [3]:

Conjecture 3 ([3]) *Let P_+ and P_0 be respectively subcritical and critical operators on M. Then there exists a positive weight $w : M \to \mathbb{R}$ such that*

$$\lim_{t\to\infty} \frac{\|e^{-P_+t}\|_{L^2(M,w)\to L^2(M)}}{\|e^{-P_0t}\|_{L^2(M,w)\to L^2(M)}} = 0\,.$$

This conjecture is proved in [3] for the Dirichlet Laplacian on a special class of quasi-cylindrical domains. There does not seem to be a direct relationship between Conjectures 1 and 3. Moreover, *it is not clear whether the sufficient conditions established in [2, Thm. 3.1] for the validity of Conjecture 1 are satisfied for the domains considered in [3].*

References

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[3] D. Krejčiřík and E. Zuazua, *The Hardy inequality and the heat equation in twisted tubes*, J. Math. Pures Appl. **94** (2010), 277–303.

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