On a connected non-compact Riemannian manifold $M$ of dimension $d$, let us consider the second-order operator

$$P := -\partial_i G^{ij} \partial_j + \partial_i E^i - F^i \partial_i + V$$

with real-valued coefficients satisfying some mild local-integrability conditions so that $P$ is a locally uniformly elliptic operator on $M$ obeying the (weak) maximum principle. We say that $P$ is symmetric if $E^i = 0 = F^i$; then the Friedrichs extension of $P$ defined initially on $C_0^\infty(M)$ gives rise to a self-adjoint operator on $L^2(M)$ satisfying Dirichlet boundary conditions on $\partial M$ in a generalized sense.

In any case, let $k_P(x, x', t)$ denote the positive minimal (Dirichlet) heat kernel of $P$. We say that $P$ is subcritical (respectively, critical) if for any fixed $x, x' \in M$, $x \neq x'$, we have that $k_P(x, x', \cdot) \in L^1(\mathbb{R}^+)$ (respectively, $k_P(x, x', \cdot) \not\in L^1(\mathbb{R}^+)$). In a joint paper with M. Fraas and Y. Pinchover [2], we made the following conjecture:

**Conjecture 1 ([2])** Let $P_+$ and $P_0$ be respectively subcritical and critical operators on $M$. Then

$$\lim_{t \to \infty} \frac{k_{P_+}(x, x', t)}{k_{P_0}(x, x', t)} = 0$$

locally uniformly for $(x, x') \in M \times M$.

The relevance of this conjecture becomes clearer if we recall the relationship of the subcriticality/criticality to properties of positive solutions of the elliptic equation $Pu = 0$. The generalized principal eigenvalue $\lambda_0$ of $P$ is defined as the supremum over all $\lambda \in \mathbb{R}$ such that there exists a positive (weak) solution $u$ of $Pu = \lambda u$. The solution is (up to a normalization) unique for critical operators. If $P$ is symmetric, then $\lambda_0$ coincides with the bottom of the spectrum of the Friedrichs extension.

Let us assume that $\lambda_0 \geq 0$. Then $\lambda_0 = 0$ for any critical operator, while $\lambda_0 \geq 0$ for any subcritical operator. If the generalized principle eigenvalue of $P_+$ is positive, then it is easy to see that Conjecture 1 holds, so the only interesting situation is when it is equal to zero. Moreover, Conjecture 1 holds if $P_0$ is positive-critical, i.e., $\varphi^* \varphi \in L^1(M)$ where $\varphi$ and $\varphi^*$ are the unique solutions of $P_0 u = 0$ and $P_0^* u = 0$, respectively. Finally, it follows from [4] that Conjecture 1 holds for Schrödinger operators with short-range potentials.

An open question is to prove (or disprove) Conjecture 1 under the general hypotheses.

In [2], we established, *inter alia*, the following result for potential-type perturbations:

**Theorem 1 ([2])** Let $P_0$ be critical in $M$ and let $P_+ = P_0 + V$ where $V$ is a non-zero non-negative potential. Then Conjecture 1 holds true if any of the two following conditions is satisfied:

1. $P_0$ is symmetric.
2. Davies’ conjecture holds for both $P_0$ and $P_+$. 
By Davies’ conjecture we mean the following conjecture, which was raised in [1] by E. B. Davies in the self-adjoint case.

**Conjecture 2 (Davies’ conjecture)** Fix reference points \(x_0, x'_0 \in M\). Then

\[
\lim_{t \to \infty} \frac{k_P(x, x', t)}{k_P(x_0, x'_0, t)} = a(x, y)
\]

exists and is positive for all \(x, x' \in M\).

Obviously, Conjecture 2 holds if \(P\) is positive-critical. Moreover, it holds true in the symmetric case if the solution of \(Pu\) is unique. In particular, it holds true for a critical symmetric operator.

Theorem 1 suggests that Conjectures 1 and 2 are closely related. However, is it necessary to suppose the validity of Conjecture 2 in Theorem 1 for the non-symmetric case (2)?

Conjecture 1 can be regarded as a point-wise version of another conjecture, made in the self-adjoint case in a joint paper with E. Zuazua [3]:

**Conjecture 3 ([3])** Let \(P_+\) and \(P_0\) be respectively subcritical and critical operators on \(M\). Then there exists a positive weight \(w : M \to \mathbb{R}\) such that

\[
\lim_{t \to \infty} \frac{\|e^{-P_+ t}\|_{L^2(M,w) \to L^2(M)}}{\|e^{-P_0 t}\|_{L^2(M,w) \to L^2(M)}} = 0.
\]

This conjecture is proved in [3] for the Dirichlet Laplacian on a special class of quasi-cylindrical domains. There does not seem to be a direct relationship between Conjectures 1 and 3. Moreover, it is not clear whether the sufficient conditions established in [2, Thm. 3.1] for the validity of Conjecture 4 are satisfied for the domains considered in [3].

References


