$\mathcal{PT}$-symmetric waveguides and the lack of variational techniques

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On the Hilbert space $L^2(\mathbb{R}\times(-1,1))$ let us consider the m-sectorial operator H_α defined as the Laplacian on $H^2(\mathbb{R}\times(-1,1))$ subjected to the following complex Robin-type boundary conditions:

$$\frac{\partial\psi}{\partial y}+i\alpha(x)\psi=0 \qquad \text{for} \qquad (x,y)\in\mathbb{R}\times\{\pm1\}\,,$$

where $\psi\in\mathfrak{D}(H_\alpha)$ and $\alpha:\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous. H_α is not self-adjoint unless $\alpha=0$, however, it is $\mathcal{T}$-self-adjoint and it commutes with the antilinear product operator $\mathcal{PT}$ where $(\mathcal{P}\psi)(x):=\psi(-x)$ and $(\mathcal{T}\psi)(x):=\overline{\psi(x)}$.

In a joint paper with D. Borisov [1], we performed a detailed spectral analysis of H_α. It was established that the residual spectrum is always empty and that the essential spectrum (due to Wolf) is real provided that α is a compactly supported perturbation of a constant function α_0:

$$\sigma_{\rm ess}(H_\alpha)=[\mu_0^2,\infty) \qquad \text{where} \qquad \mu_0:=\min\{|\alpha_0|,\pi/2\}\,.$$

Moreover, assuming that the perturbation $\alpha-\alpha_0$ is small in the supremum norm and using a perturbative method, we derived sufficient conditions for the existence of discrete real weakly-coupled eigenvalues, converging to the threshold μ_0^2 of the essential spectrum when the perturbation vanishes. For instance, one of the sufficient conditions reads

$$|\alpha_0|<\pi/2 \qquad \text{and} \qquad \alpha_0\int_{\mathbb{R}}(\alpha(x)-\alpha_0)\,dx<0\,,$$

while an opposite sign in the latter inequality ensures that there are no such weakly-coupled eigenvalues.

An open problem is to *show the existence of discrete spectra by some qualitative methods*, regardless of the strength of the perturbation. It is particularly frustrating that the variational techniques powerfully used in self-adjoint waveguides are not available here.

Finally, let us mention a joint paper with M. Tater [2], where we analysed the spectrum of H_α by numerical methods. It turns out that there might be complex conjugate pairs of discrete eigenvalues if $\alpha-\alpha_0$ is big in the supremum norm (see a related animation [3]). *Prove it.*

References

[1] D. Borisov and D. Krejčiřík, *$\mathcal{PT}$-symmetric waveguides*, Integral Equations Operator Theory **62** (2008), 489–515.

[2] D. Krejčiřík and M. Tater, *Non-Hermitian spectral effects in a $\mathcal{PT}$-symmetric waveguide*, J. Phys. A: Math. Theor. **41** (2008) 244013.

[3] `http://gemma.ujf.cas.cz/~david/KT.html`