THE SPECTRUM OF THE HILBERT MATRIX AS AN OPERATOR ON ℓ_p

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Given a complex number $\lambda \in \mathbb{C}\backslash\{-1,-2,-3,\ldots\}$, we consider the matrix H_λ with coefficients $a_{mn} = (m+n+1-\lambda)^{-1}$, $m, n = 0, 1, 2, \ldots$ Note that

$$H_0 = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

is the classical Hilbert matrix, which defines a selfadjoint operator on ℓ_2 whose spectrum is purely continuous and coincides with $[0, \pi]$.

We say that $\mu \in \mathbb{C}$ is a *latent root* of H_λ if there exists a non-null sequence $(x_n)_{n=0}^\infty$ such that $\sum_{n=0}^\infty a_{mn}x_n$ converges to μx_m for every $m \geq 0$.

It was proved by Rosenblum [3] that for a real $\lambda < 1$, every complex number with positive real part is a *latent root* of H_λ. Moreover, he observed that in this case H_λ defines a bounded operator on ℓ_p for $2 < p < \infty$ and that $\pi \sec \pi u$ is an eigenvalue of $H_\lambda : \ell_p \to \ell_p$ for $|\mathbb{R}\mathrm{e}\, u| < 1/2 - 1/p$.

Note that $\{\pi \sec \pi u : |\mathbb{R}\mathrm{e}\, u| < 1/2\} = \{\mu : \mathbb{R}\mathrm{e}\, \mu > 0\}$, and that the latent roots of H_λ for λ complex have been recently studied in [1].

Problem. *Determine the spectrum, the point spectrum and the essential spectrum of H_λ as an operator on ℓ_p for $2 < p < \infty$.*

We refer to [2] for the following related result: The spectrum of the Cesàro matrix C as an operator on ℓ_p $(1 < p < \infty)$ is $D_p := \{\mu : |\mu - q/2| \leq q/2\}$, where $1/p + 1/q = 1$, its essential spectrum is the boundary of D_p, and the point spectrum of the conjugate operator C^* is the interior of D_p.

References

[1] A. Aleman, A. Montes Rodríguez and A. Sarafoleanu. *The eigenfunctions of the Hilbert matrix.* Preprint, 2010.

[2] M. González. *The fine spectrum of the Cesàro operator in ℓ_p $(1 < p < \infty)$.* Arch. Math. 44 (1985), 355–358.

[3] M. Rosenblum. *On the Hilbert matrix I.* Proc. Amer. Math. Soc. 9 (1958) 137–140.

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