Complex symmetric operators

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1. Complex symmetric operators

This section is a brief introduction to complex symmetric operators, a certain class of Hilbert space operators which arise in complex analysis, matrix theory, functional analysis, and even quantum mechanics. The basic definitions and examples are discussed in [8, 10, 11] and a few applications to quantum systems can be found in [15]. We first introduce the notion of a conjugation:

Definition 1.1. A conjugate-linear operator C on a complex Hilbert space $\mathcal{H}$ is called a *conjugation* if $C^2 = I$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all x, y in $\mathcal{H}$.

The standard example of a conjugation is pointwise complex conjugation on a Lebesgue space $L^2(X, \mu)$. It is easy to see that any conjugation is unitarily equivalent to the canonical conjugation on a ℓ^2-space of the appropriate dimension.

Definition 1.2. Let C be a conjugation on $\mathcal{H}$. A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is called *C-symmetric* if $T = CT^*C$. We say that T is a *complex symmetric operator* (CSO) if there exists a C such that T is C-symmetric.

We remark that a slightly more technical definition exists if one wishes to consider unbounded operators [11]. The term *complex symmetric* comes from the fact that an operator is a CSO if and only if it has a symmetric (i.e., self-transpose) matrix representation with respect to some orthonormal basis [10]. In the above it is important to note that C is *conjugate-linear* and thus the study of complex symmetric operators is quite distinct from that of operators on indefinite inner product spaces.

As a simple example, consider the *Volterra integration operator* $T : L^2[0,1] \to L^2[0,1]$ defined by

$$[Tf](x) = \int_0^x f(y)\, dy. \tag{1.1}$$

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It is highly non-normal, being in fact quasinilpotent. However, it is C-symmetric with respect to the conjugation

$$[Cf](x) = \overline{f(1-x)}$$

on $L^2[0,1]$. Now observe that C fixes each element of the orthonormal basis

$$e_n = \exp[2\pi i n(x - \tfrac{1}{2})], \qquad (n \in \mathbb{Z})$$

of $L^2[0,1]$ and that the matrix for T with respect to this basis is simply

$$\left(\begin{array}{cccc|c|cccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & \frac{i}{6\pi} & 0 & 0 & \frac{i}{6\pi} & 0 & 0 & 0 & \cdots \\
\cdots & 0 & \frac{i}{4\pi} & 0 & -\frac{i}{4\pi} & 0 & 0 & 0 & \cdots \\
\cdots & 0 & 0 & \frac{i}{2\pi} & \frac{i}{2\pi} & 0 & 0 & 0 & \cdots \\
\hline
\cdots & \frac{i}{6\pi} & -\frac{i}{4\pi} & \frac{i}{2\pi} & \frac{1}{2} & -\frac{i}{2\pi} & \frac{i}{4\pi} & -\frac{i}{6\pi} & \cdots \\
\hline
\cdots & 0 & 0 & 0 & -\frac{i}{2\pi} & -\frac{i}{2\pi} & 0 & 0 & \cdots \\
\cdots & 0 & 0 & 0 & \frac{i}{4\pi} & 0 & -\frac{i}{4\pi} & 0 & \cdots \\
\cdots & 0 & 0 & 0 & -\frac{i}{6\pi} & 0 & 0 & -\frac{i}{6\pi} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}\right).$$

2. Norms and singular values

Perhaps the most promising development (as far as applications are concerned) in the abstract theory of complex symmetric operators is the notion of *approximate antilinear eigenvalue problems* [5]. This can be viewed as a complex symmetric analogue of the classical Weyl Criterion from the spectral theory of self-adjoint operators [17].

Theorem 2.1. *If T is a bounded C-symmetric operator and $\lambda \in \mathbb{C}$, then $|\lambda|$ belongs to the spectrum of $|T| = \sqrt{T^*T}$ if and only if there exists a sequence of unit vectors x_n that satisfy $\lim_{n\to\infty} \|(T-\lambda C)x_n\| = 0$. Moreover, $|\lambda|$ is a singular value of T if and only if there exists $x \neq 0$ such that $Tx = \lambda Cx$.*

The preceding theorem places in a single framework a number of disparate topics in analysis. For instance, Schmidt vectors and singular numbers in the theory of Hankel operators [13] and the Fredholm eigenvalues of a planar domain [2, 18] both arise from such antilinear eigenvalue problems.

Continuing with the example of the Volterra operator (1.1), a straightforward application of 2.1 reveals that $\|T\| = \frac{2}{\pi}$ (see [11] for details). As another example, a simple application of Theorem 2.1 completely describes the spectral properties of the modulus $|T| = \sqrt{T^*T}$ of a *Foguel operator*

$$T = \begin{pmatrix} S^* & H \\ 0 & S \end{pmatrix} \qquad (2.1)$$

on $l^2(\mathbb{N}) \oplus l^2(\mathbb{N})$ [7] (M. Raghupathi recently obtained similar results using other means [16]). Here H is an infinite Hankel matrix and S is the unilateral shift operator on $\ell^2(\mathbb{N})$. Such operators figure prominently in Pisier's celebrated solution to Halmos' polynomially bounded operator problem [14] (see also the influential papers [1, 3]).

Up to this point, the applications of Theorem 2.1 have been limited mostly to the study of function-related operator theory. It would therefore be of interest to develop this approach further.

Question: Develop numerical methods to compute the singular values of concrete complex symmetric operators (e.g., certain differential and integral operators). For instance, using this method could one compute resolvent norms $\|(T - zI)^{-1}\|$ for certain classes of operators?

3. Decomposition of complex symmetric operators

The class of complex symmetric operators, which contains many highly nonnormal operators, does not yet have a fully developed spectral theory (beyond those basic results which apply to all operators generically). Although the basic linear algebraic principles governing the eigenstructure of complex symmetric operators was developed in [6], there is still much to be done.

It has recently been established [12] that every complex symmetric operator on a *finite-dimensional* space is unitarily equivalent to a direct sum of

1. Irreducible complex symmetric matrices,
2. Matrices of the form $A \oplus A^t$ where A is irreducible and not a complex symmetric operator.[1]

We use the term *irreducible* in the operator-theoretic sense. Namely, $T \in B(\mathcal{H})$ is called irreducible if T is not unitarily equivalent to a direct sum $A \oplus B$. In some sense, the preceding result permits one to decompose complex symmetric operators on finite-dimensional spaces into simpler components. In a similar vein, we consider the following questions:

Question: What is the infinite-dimensional analogue of the preceding result?

Question: Just as multiplication operators $M_z : L^2(X, \mu) \to L^2(X, \mu)$ play a fundamental role in decomposing normal operators, can one develop a comparable *model theory* for complex symmetric operators?[2]

Question: Does every bounded complex symmetric operator (on a separable, infinite-dimensional Hilbert space) have a proper, nontrivial invariant subspace?

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[1]However, it turns out that the direct sum $A \oplus A^t$ *is* a complex symmetric operator.
[2]There is some indication that *truncated Toeplitz operators* may play a role in this endeavor [4, 9, 19].

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