Geometrically induced spectral properties of physical systems

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DSc Thesis

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Résumé

The thesis is aiming at mathematical studies of spectral problems, coming both from modern as well as classical physics, where the significant features of geometry as regards physical properties play a crucial role. We focus on the properties of the nodal set of eigenfunctions and low-lying eigenvalues of vibrating systems, on the influence of curvature and torsion on spectral properties of curved quantum-waveguide nanostructures, on the heat flow in twisted tubes and on the influence of intrinsic curvature on quantum transport on manifolds.

Mathematically, we deal with a spectral-geometric analysis on bounded and quasi-cylindrical domains or, more generally, on non-compact non-complete Riemannian manifolds. The main achievements are represented by the proof of the nodal-line conjecture for a large class of non-convex and possibly multiply connected domains and the establishment of Hardy-type inequalities in twisted tubes and negatively curved surfaces.

To my wife, Veronika

"And what is this God?" I asked the earth and it answered: "I am not He," and all the things that are on the earth confessed the same answer. I asked the sea and the deeps and the creeping things with living souls, and they replied, "We are not your God. Look above us." I asked the blowing breezes, and the universal air with all its inhabitants answered: "I am not God." I asked the heaven, the sun, the moon, the stars, and "No," they said, "we are not the God for whom you are looking." And I said to all those things which stand about the gates of my senses: "Tell me something about my God, you who are not He. Tell me something about Him." And they cried out in a loud voice: "He made us."

St. Augustine, *Confessions*

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Preface

The study of relations between geometry and spectral analysis constitutes an important domain of mathematical physics. Already from a purely mathematical point of view, it is interesting to understand the influence of the shape or metric of a manifold and the type of boundary conditions to the spectra of associated differential operators; and vice versa, one can try to characterize intrinsic properties of a manifold from given spectral data. The physical interest comes from the fact that many systems in Nature are described by partial differential equations, and the latter are often studied by means of a spectral analysis of the corresponding differential operators.

Moreover, eigenvalues and eigenfunctions of the spectral problems usually have direct physical interpretations, and estimating these less accessible quantities on the basis of more accessible geometrical data may already be of a practical interest for the engineer or the physicist.

A typical example is the spectral problem for the Laplace operator in a Euclidean domain. This is a usual model for stationary states of a vibrating object in acoustics and of certain waves in electromagnetism (the Helmholtz equation), or for bound states of a quantum particle constrained to a nanostructure in quantum physics (the stationary Schrödinger equation). It is also related to the stochastic motion of a Brownian particle (the simplest version of the Fokker-Planck equation) and to other diffusive processes. However, apart from very simple symmetric situations where one can employ a separation of variables, no explicit formulae for solutions are available. But the geometry of real systems can be rather complicated, and it is necessary to develop alternative methods of spectral theory in order to provide rigorous information about the spectrum.

The goal of the present thesis is to contribute to this vast area of mathematical physics by analysing the interplay between the geometry and spectrum in the following problems, coming both from classical as well as modern physics:

I. vibrating systems and the nodal-line conjecture,

II. twisting versus bending in waveguide-like structures,

III. quantum mechanics on curved manifolds.

***ad* I.** Historically, probably the first study of spectral-geometric relationship can be associated with the work of Lord Rayleigh on vibrating systems from the second half of the 19th century. His textbook, *The Theory of Sound* [58], is still referred to by acoustic engineers today and has led to a number of interesting mathematical conjectures, some of them being solved much later or still open. An example of the latter is the famous conjecture of L. E. Payne's from 1967 [56], which states that the nodal line of any second eigenfunction of the Dirichlet Laplacian in an arbitrary planar domain touches the boundary.

Our contribution to this problem is both positive and negative. First, in [25] (Chapter 3), we show that the nodal-line conjecture actually does not hold for unbounded domains. Second, in [26] (Chapter 4), we establish the validity of the conjecture for thin curved tubes (of arbitrary cross-section and in any dimension). It is for the first time when the conjecture has been proved for non-convex domains without any symmetry conditions.

In addition to the aforementioned analysis of nodal set of eigenfunctions, we obtain new isoperimetric-type estimates for low-lying eigenvalues of the Dirichlet Laplacian in star-shaped domains [27] (Chapter 5) and study the instability of solutions to the damped wave equation in possibly unbounded domains [23] (Chapter 6).

***ad* II.** Waveguide is a physical device (usually of tubular shape) that exhibits propagating states: electromagnetic or acoustic waves in the context of classical physics, or scattering states in quantum mechanics. Mathematically, one deals with the so-called quasi-cylindrical domains, for which the spectral analysis is most difficult because of the presence of both eigenvalues and continuous spectrum. Here the interest in the interplay between the geometry and spectrum for such systems is mainly due to the advent of nanotechnology in the second half of the 20th century. Modern experimental techniques make it possible to fabricate tiny semiconductor structures (often called nanostructures) of various shapes devised and reproducible in the laboratory and yet small enough to exhibit quantum effects, some of them being of purely geometric origin.

Probably the most beautiful phenomenon is the existence of curvature-induced bound states in quantum waveguides, mathematically first described by P. Exner and P. Šeba in 1989 [19]. This paper initiated extensive theoretical studies of waveguide-like objects in quantum mechanics and the research field is still active today, partly because of the advent of new structures such as carbon nanotubes and graphenes. Our contribution to the study of the effect of *bending* in waveguides consists mainly in generalizing the results to higher dimensions [7] (Chapter 7) and to more general boundary conditions [50, 48, 24] (Chapters 10–12) and in applying new mathematical methods in the analysis of scattering states [49] (Chapter 13).

However, a genuine breakthrough in the theory is represented by our paper [17] (Chapter 8), in which we observe that the geometric deformation of *twisting* leads to a completely opposite effect in quantum waveguides, mathematically described by the existence of Hardy-type inequalities. Surprisingly, the effect of twisting has been overlooked for almost two decades. We also prove a variant of our original result in a different geometrical setting [44] (Chapter 9) and apply the Hardy-type inequalities to diffusive processes [51, 52] (Chapters 14–15).

***ad* III.** The ambient manifold of a quantum waveguide is usually identified with the flat Euclidean space. This restriction is obviously due to the semiconductor-physics motivation, however, at least from the mathematical point of view, one may be interested equally in the situations when it is a general Riemannian manifold. Moreover, this more general setting leads to an interesting conceptual question: *Which geometry is better to travel in?* Or, more precisely, what is the effect of ambient curvature on quantum transport?

We have analysed the problem in the simplest non-trivial case when the ambient space of the quantum traveller is a tubular neighbourhood of an infinite curve in a two-dimensional (abstract) Riemannian manifold. Our principal results can be roughly summarized as follows: positive curvature hurts the transport [45] (Chapter 16), while negative curvature improves it [46] (Chapter 17). Mathematically, the former follows as a consequence of the existence of discrete eigenvalues for the Laplace-Beltrami operator, while the latter is based on Hardy-type inequalities. The results have important consequences for the large time behaviour of the Brownian motion [43] (Chapter 18).

This thesis may be considered as a research report mostly based on the aforementioned papers of the author obtained in the last few years. On the other hand, in the following introductory Chapter 1, we provide a unifying mathematical framework to study the problems coming from the various physical motivations and review the basic material which is needed. Furthermore, in Chapter 2, we give a brief and intentionally informal summary of the main results obtained in the papers. In this sense we believe that the two chapters represent a self-contained treatment of the recent research, accessible to non-specialists and, in particular, to students interested in the topics where functional analysis (especially spectral theory) meets geometry.

The thesis is thus divided into four parts. Part 0 consists of the two introductory Chapters 1–2, while Parts I–III (Chapters 3–18) contain the published material as described above. For the convenience of the reader, we present here the articles on which the thesis is based:

Chapter 3 P. Freitas and D. Krejčiřík, *Location of the nodal set for thin curved tubes*, Indiana Univ. Math. J. **57** (2008), 343–376.

Chapter 4 P. Freitas and D. Krejčiřík, *Unbounded planar domains whose second nodal line does not touch the boundary*, Math. Res. Lett. **14** (2007), 107–111.

Chapter 5 P. Freitas and D. Krejčiřík, *A sharp upper bound for the first Dirichlet eigenvalue and the growth of the isoperimetric constant of convex domains*, Proc. Amer. Math. Soc. **136** (2008), 2997–3006.

Chapter 6 P. Freitas and D. Krejčiřík, *Instability results for the damped wave equation in unbounded domains*, J. Differential Equations **211** (2005), 168–186.

Chapter 7 B. Chenaud, P. Duclos, P. Freitas, and D. Krejčiřík, *Geometrically induced discrete spectrum in curved tubes*, Differential Geom. Appl. **23** (2005), 95–105.

Chapter 8 T. Ekholm, H. Kovařík, and D. Krejčiřík, *A Hardy inequality in twisted waveguides*, Arch. Ration. Mech. Anal. **188** (2008), 245–264.

Chapter 9 H. Kovařík and D. Krejčiřík, *A Hardy inequality in a twisted Dirichlet-Neumann waveguide*, Math. Nachr. **281** (2008), 1159–1168.

Chapter 10 D. Krejčiřík and J. Kříž, *On the spectrum of curved quantum waveguides*, Publ. RIMS, Kyoto University **41** (2005), 757–791.

Chapter 11 D. Krejčiřík, *Spectrum of the Laplacian in a narrow curved strip with combined Dirichlet and Neumann boundary conditions*, ESAIM: Control, Optimisation and Calculus of Variations **15** (2009), 555–568.

Chapter 12 P. Freitas and D. Krejčiřík, *Waveguides with combined Dirichlet and Robin boundary conditions*, Math. Phys. Anal. Geom. **9** (2006), 335–352.

Chapter 13 D. Krejčiřík and R. Tiedra de Aldecoa, *The nature of the essential spectrum in curved quantum waveguides*, J. Phys. A: Math. Gen. **37** (2004), 5449–5466.

Chapter 14 D. Krejčiřík and E. Zuazua, *The Hardy inequality and the heat equation in twisted tubes*, J. Math. Pures Appl. **94** (2010), 277–303.

Chapter 15 D. Krejčiřík and E. Zuazua, *The asymptotic behaviour of the heat equation in a twisted Dirichlet-Neumann waveguide*, J. Differential Equations **250** (2011), 2334–2346.

Chapter 16 D. Krejčiřík, *Quantum strips on surfaces*, J. Geom. Phys. **45** (2003), 203–217.

Chapter 17 D. Krejčiřík, *Hardy inequalities in strips on ruled surfaces*, J. Inequal. Appl. **2006** (2006), Article ID 46409, 10 pages.

Chapter 18 M. Kolb and D. Krejčiřík, *The Brownian traveller on manifolds*, preprint on arXiv:1108.3191 [math.AP] (2011).

Except for unifying cosmetical amendments, the contents of Chapters 3–18 coincide with the published versions of the building papers. This decision leads to two counter effects. First, the notation introduced in Part 0 (Chapters 1–2) may occasionally differ from that used in the individual articles presented in Parts I–III (Chapters 3–18). This is balanced by the fact that each of the Chapters 3–18 can be read as an independent reasearch work, in its original version. Second, more importantly, we decided not to correct misprints and possible mistakes we have encountered after the publication of some of the papers. *Errare humanum est.* In fact, we are aware of just a few cases, which are treated in this thesis by adding a short errata section after the list of references of the corresponding chapter.

I conclude by thanking the large number of people who have stimulated my interest in spectral geometry over the last ten years, particularly in relation to the contents of this thesis. The most important of these have been my former teachers, P. Duclos and P. Exner, and, more recently, P. Freitas. I am very grateful to my co-authors from the above papers and to many other good friends and colleagues. I am especially indebted to P. Siegl, who read the whole thesis and offered invaluable comments. Finally I want to record my thanks to my wife Veronika; I would never have been able to write this thesis without her support.

Prague, Czech Republic
February 2012 *David Krejčiřík*

Part 0

Introductory part

Chapter 1

Introduction

1.1 General physical motivations

Most processes in Nature can be under first approximation described by one of the following linear differential equations:

$$\text{the wave equation} \qquad \partial_t^2 u - \Delta u = 0\,, \tag{1.1}$$

$$\text{the heat equation} \qquad \partial_t u - \Delta u = 0\,, \tag{1.2}$$

$$\text{the Schrödinger equation} \qquad i\partial_t u + \Delta u = 0\,. \tag{1.3}$$

Qualitative properties of the respective solutions are very different, which of course reflects the variety of the physical systems. The wave equation is a classical model for a vibrating string, membrane or elastic solid, but it also models propagation of electromagnetic waves, moreover it arises in relativistic quantum mechanics and cosmology. The heat equation, also known as the diffusion equation, describes in typical applications the evolution in time of the density of some quantity such as the heat, chemical concentration, *etc*, and it also represents the simplest version of the Fokker-Planck equation describing the stochastic motion of a Brownian particle. Finally, the Schrödinger equation is the fundamental equation of quantum theory, which is probably the best physical theory mankind has ever had (at least from the point of view of the technological impact and the number of experiments confirming it).

The common denominator of the above equations is

$$\text{the Helmholtz equation} \qquad -\Delta\psi = \lambda\psi\,, \tag{1.4}$$

which is obtained from (1.1)–(1.3) after a separation of space x and time t variables. (1.4) can be understood as a *spectral problem for the Laplacian* $-\Delta$ in the space variables x, with eigenvalues λ and eigenfunctions ψ usually having direct physical interpretations (*e.g.*, λ have the meaning of squares of frequences for vibrating systems, decay rates for dissipative systems, bound state energies for quantum systems). More importantly, the solutions of the evolution equations (1.1)–(1.3) can be obtained on the basis of a complete spectral analysis of the Laplacian.

As usual for evolution equations, (1.1)–(1.3) are subject to initial conditions at time $t = 0$. If the space variables x are restricted to a subdomain $\Omega \subset \mathbb{R}^d$, with $d \geq 1$, it is also necessary to equip (1.1)–(1.4) with boundary conditions. The easiest situation is represented by *Dirichlet boundary conditions*

$$\psi = 0 \qquad \text{on} \qquad \partial\Omega\,. \tag{1.5}$$

As well as being simple to treat, these boundary conditions are directly relevant to a number of physical problems (*e.g.*, vibrations of an elastic membrane whose boundary is fixed, heat flow in a medium whose boundary is kept at zero temperature, killing boundary conditions for the Brownian motion, the motion of a quantum particle which is confined to a region by the barrier associated with a large chemical potential, *etc*). Intrinsically harder situation is represented by *Neumann boundary conditions*

$$\frac{\partial\psi}{\partial n} = 0 \qquad \text{on} \qquad \partial\Omega\,, \tag{1.6}$$

where n denotes the unit normal vector field of $\partial\Omega$, however, these are also important in physical applications (*e.g.*, the vibration of a membrane at those parts of the boundary which are free to move, the flow of a fluid through a channel or past an obstacle, the flow of heat in a medium with an insulated boundary, reflecting

boundary conditions for the Brownian motion, *etc*). Representing an interpolation between the Dirichlet and Neumann boundary conditions, it might be also sometimes relevant to employ *Robin boundary conditions*

$$\frac{\partial\psi}{\partial n} + \alpha\,\psi = 0 \qquad \text{on} \qquad \partial\Omega\,, \tag{1.7}$$

where $\alpha : \partial\Omega \to \mathbb{R}$ ($\alpha = 0$ and $\alpha = \pm\infty$ correspond to Neumann and Dirichlet boundary conditions, respectively). Finally, it is also possible to consider the case of combined boundary conditions, where different kinds of boundary conditions are imposed on distinct parts of $\partial\Omega$.

The objective of the present thesis is to study the interplay between the *spectrum* of the Laplacian (1.4), subject to various kinds of boundary conditions (1.5)–(1.7), and the *geometry* of the domain Ω.

1.2 A unifying mathematical framework

A correct interpretation of the Helmholtz equation (1.4) in a domain Ω (*i.e.* open connected set) is to view it as the spectral problem for the self-adjoint operator on $L^2(\Omega)$ generated by the Laplacian $-\Delta$. A standard procedure to construct the self-adjoint operator consists in starting with the Laplacian defined initially on smooth functions satisfying the required boundary conditions and then taking the Friedrichs extension [10].

1.2.1 Dirichlet boundary conditions

We start with Laplacian defined on the minimal domain $C_0^\infty(\Omega)$. The corresponding Friedrichs extension, denoted by the *Dirichlet Laplacian* $-\Delta_D^\Omega$, is the (self-adjoint) operator associated (in the sense of the first representation theorem [40, Thm. VI.2.1]) with the (closed) quadratic form

$$Q_D^\Omega[\psi] := \|\nabla\psi\|^2\,, \qquad \psi \in \mathfrak{D}(Q_D^\Omega) := W_0^{1,2}(\Omega)\,. \tag{1.8}$$

The simplicity of Dirichlet boundary conditions consists in that we indeed have

$$-\Delta_D^\Omega\psi = -\Delta\psi\,, \qquad \psi \in \mathfrak{D}(-\Delta_D^\Omega) = \left\{\psi \in W_0^{1,2}(\Omega)\middle|\ \Delta\psi \in L^2(\Omega)\right\}\,,$$

where $\phi := \Delta\psi$ means the distributional Laplacian of ψ, *i.e.* $(\nabla\varphi, \nabla\psi) = -(\varphi, \phi)$ for every $\varphi \in C_0^\infty(\Omega)$.

We would like to stress that $-\Delta_D^\Omega$ is well defined in this way for an *arbitrary* domain Ω (no regularity assumptions are needed). But one has to give up the "usual" characterization of $\mathfrak{D}(-\Delta_D^\Omega)$ as a subset of the Sobolev space $W^{2,2}(\Omega)$. If, however, $\partial\Omega$ is sufficiently regular, say Ω bounded and the boundary $\partial\Omega$ is of class C^2, then we indeed have

$$\mathfrak{D}(-\Delta_D^\Omega) = W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega) = \left\{\psi \in W^{2,2}(\Omega)\ \middle|\ \psi \restriction \partial\Omega = 0\right\}\,,$$

where $\psi \restriction \partial\Omega$ means the boundary trace of ψ. Indeed, the first inequality follows from standard elliptic regularity theory [29, Thm. 8.12], while the second one is a consequence of the trivial traces theorem [1, Thm. 5.37].

1.2.2 Neumann boundary conditions

The problem with Neumann boundary conditions is the need for a regularity of Ω. Indeed, if we try to take for the initial domain of the Laplacian the functions from $C_0^\infty(\mathbb{R}^d)$ (by which we mean here the space of functions which are restrictions to $\overline{\Omega}$ of functions in $C_0^\infty(\mathbb{R}^d)$) satisfying the Neumann boundary conditions (1.6) it is clear that we need to assume certain regularity of $\partial\Omega$ in order to ensure that the normal vector field n exists. Let us therefore assume for a moment that Ω is a bounded domain with smooth boundary. Then the Friedrichs extension of such a defined Laplacian, denoted by the *Neumann Laplacian* $-\Delta_N^\Omega$, is the (self-adjoint) operator associated with the (closed) quadratic form

$$Q_N^\Omega[\psi] := \|\nabla\psi\|^2\,, \qquad \psi \in \mathfrak{D}(Q_N^\Omega) := W^{1,2}(\Omega)\,. \tag{1.9}$$

It is remarkable that the Neumann boundary conditions disappear as soon as one passes from the initial operator to the closure of its quadratic form.

The definition of $-\Delta_N^\Omega$ as an operator associated with (1.9) makes sense for *arbitrary* domains Ω (even though the normal direction may not be definable at any point of the boundary $\partial\Omega$) and we take it as the definition of the Neumann Laplacian in the general case (*i.e.* any Ω, possibly unbounded with fractal boundary).

If $\partial\Omega$ is smooth enough, say of class C^0, then (*cf* [16, Thm. V.4.7 & Thm. VII.1.13])

$$-\Delta_N^\Omega\psi = -\Delta\psi\,, \qquad \psi\in\mathfrak{D}(-\Delta_N^\Omega) = \left\{\psi\in W^{1,2}(\Omega)\middle|\ \Delta\psi\in L^2(\Omega)\right\}.$$

Here $\phi := \Delta\psi$ means the distribution satisfying $(\nabla\varphi,\nabla\psi) = -(\varphi,\phi)$ for every $\varphi\in C_0^\infty(\mathbb{R}^d)$. Moreover, if additional regularity is imposed, say Ω is bounded and its boundary is of class C^2, then we indeed have

$$\mathfrak{D}(-\Delta_N^\Omega) = \left\{\psi\in W^{2,2}(\Omega)\ \middle|\ \frac{\partial\psi}{\partial n}\restriction\partial\Omega = 0\right\}.$$

1.2.3 Robin boundary conditions

As for the Neumann boundary conditions, let us assume for a moment that Ω is a bounded domain with smooth boundary. Then the Friedrichs extension of the Laplacian defined initially on the functions from $C_0^\infty(\mathbb{R}^d)$ satisfying the Robin boundary conditions (1.7) with $\alpha\in C_B^1(\partial\Omega)$, denoted by the *Robin Laplacian* $-\Delta_{R_\alpha}^\Omega$, is the (self-adjoint) operator associated with the (closed) quadratic form

$$Q_{R_\alpha}^\Omega[\psi] := \|\nabla\psi\|^2 + \int_{\partial\Omega}\alpha\,|\psi|^2\,, \qquad \psi\in\mathfrak{D}(Q_{R_\alpha}^\Omega) := W^{1,2}(\Omega)\,. \tag{1.10}$$

The definition of $-\Delta_{R_\alpha}^\Omega$ as an operator associated with (1.10) makes sense for all those domains Ω for which the boundary integral is well defined (which, for $\alpha\in L^\infty(\partial\Omega)$, requires the existence of boundary traces $W^{1,2}(\Omega)\hookrightarrow L^2(\partial\Omega)$) and, more strongly, can be treated as a relatively bounded perturbation of the Neumann Laplacian:

$$\forall\psi\in W^{1,2}(\Omega)\,, \qquad \left|\int_{\partial\Omega}\alpha\,|\psi|^2\right| \le a\,\|\nabla\psi\|^2 + b\,\|\psi\|^2$$

with some $a<1$ and $b\in\mathbb{R}$. Under this assumption, we may extend the definition of the Robin Laplacian to more general Ω and α.

If $\alpha\in W^{1,\infty}(\Omega)$ and $\partial\Omega$ is sufficiently regular, say Ω is bounded and its boundary is of class C^2, then we indeed have

$$\mathfrak{D}(-\Delta_{R_\alpha}^\Omega) = \left\{\psi\in W^{2,2}(\Omega)\ \middle|\ \left(\frac{\partial\psi}{\partial n}+\alpha\,\psi\right)\restriction\partial\Omega = 0\right\}.$$

1.2.4 The spectrum

Let H be a self-adjoint operator in a Hilbert space $\mathcal{H}$ (*e.g.*, think about H being one of the Laplacians $-\Delta_D^\Omega$, $-\Delta_N^\Omega$ or $-\Delta_{R_\alpha}^\Omega$ and $\mathcal{H}=L^2(\Omega)$). The *spectrum* of H can be defined by

$$\sigma(H) := \big\{\lambda\in\mathbb{C}\ \big|\quad H-\lambda\quad\text{is not bijective}\big\}.$$

It is easy to see that $\sigma(H)\subset\mathbb{R}$. The set $\rho(H) := \mathbb{C}\setminus\sigma(H)$ is called the *resolvent set* of H.

The set of all eigenvalues of H, *i.e.*,

$$\begin{aligned}\sigma_{\mathrm{p}}(H) &:= \big\{\lambda\in\mathbb{R}\ \big|\quad H-\lambda\quad\text{is not injective}\big\}\\ &= \big\{\lambda\in\mathbb{R}\ \big|\quad \exists\psi\subset\mathfrak{D}(H),\quad \|\psi\|=1,\quad H\psi=\lambda\psi\big\}\end{aligned}$$

is called the *point spectrum* of H. If λ is an eigenvalue of H then the dimension of the kernel of $H-\lambda$ is called the (geometric) *multiplicity* of λ. It is important to stress that, in general, the spectrum is not exhausted by eigenvalues (which is typically the case if Ω is unbounded). The rest in the disjoint partition of the spectrum $\sigma(H)=\sigma_{\mathrm{p}}(H)\cup\sigma_{\mathrm{c}}(H)$ is called the *continuous spectrum* of H:

$$\sigma_{\mathrm{c}}(H) := \big\{\lambda\in\mathbb{R}\ \big|\quad H-\lambda\quad\text{is injective but not surjective}\big\}.$$

We rather use the following, alternative disjoint partition of the spectrum of a self-adjoint operator H:

$$\sigma(H) = \sigma_{\mathrm{disc}}(H)\cup\sigma_{\mathrm{ess}}(H)\,.$$

Here the *discrete spectrum* $\sigma_{\mathrm{disc}}(H)\subset\sigma_{\mathrm{p}}(H)$ consists of those eigenvalues of H which are isolated points of $\sigma(H)$ and have finite multiplicity. The complement

$$\sigma_{\mathrm{ess}}(H) := \sigma(H)\setminus\sigma_{\mathrm{disc}}(H)$$

is called the *essential spectrum* of H and, by definition, it contains either accumulation points of $\sigma(H)$ or isolated eigenvalues of infinite multiplicity.

We say that H has a *compact resolvent* if $(H-\lambda)^{-1}:\mathcal{H}\to\mathcal{H}$ is compact for some, and hence all, $\lambda\in\rho(H)$. An operator H with compact resolvent has a *purely discrete spectrum*, *i.e.* $\sigma_{\mathrm{ess}}(H)=\varnothing$. Conversely, the resolvent of a self-adjoint operator with purely discrete spectrum is necessarily compact.

1.2.5 Basic tools

The *spectral theorem* is by far the most fundamental tool in the spectral theory of self-adjoint operators. It is essentially a generalization of a well known result in finite-dimensional Hilbert spaces that any self-adjoint matrix can be diagonalized.

Theorem 1.1 (Spectral theorem)**.** *For every self-adjoint operator H on $\mathcal{H}$ there exists exactly one spectral family E_H for which*

$$H = \int_{\sigma(H)} \lambda \, dE_H(\lambda) \,.$$

For the notion of spectral family and integration with respect to it, we refer to [63, Sec. 7.2]. We shall not use much the spectral theorem itself in this thesis. But we shall widely use some of its important consequences.

The following characterization tells us that the points in the (essential) spectrum can be considered as approximate eigenvalues.

Theorem 1.2 (Weyl's criterion)**.** *Let H be a self-adjoint operator on $\mathcal{H}$. A point λ belongs to $\sigma(H)$ if, and only if, there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}(H)$ such that*

1. $\forall n \in \mathbb{N}, \quad \|\psi_n\| = 1 \,,$
2. $H\psi_n - \lambda\psi_n \xrightarrow[n\to\infty]{} 0 \quad$ *in* $\quad \mathcal{H}.$

Moreover, λ belongs to $\sigma_{\mathrm{ess}}(H)$ if, and only if, in addition to the above properties,

3. $\psi_n \xrightarrow[n\to\infty]{w} 0 \quad$ *in* $\quad \mathcal{H}.$

See [63, Sec. 7.4] for a proof of this theorem. The sequence satisfying the items 1–3 is called a *singular sequence*.

The following theorem, known also as the *Rayleigh-Ritz variational principle*, provides a variational characterization of eigenvalues below the essential spectrum (*cf* [10, Sec. 4.5]).

Theorem 1.3 (Minimax principle)**.** *Let H be a self-adjoint operator on $\mathcal{H}$ that is bounded from below (*i.e. $\inf\sigma(H) > -\infty$*), and let Q be the associated sesquilinear form. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of numbers defined by*

$$\lambda_n := \inf_{\mathcal{L}^n \subset \mathfrak{D}(H)} \sup_{\psi\in\mathcal{L}^n} \frac{(\psi, H\psi)}{\|\psi\|^2} = \inf_{\mathcal{L}^n \subset \mathfrak{D}(Q)} \sup_{\psi\in\mathcal{L}^n} \frac{Q[\psi]}{\|\psi\|^2} \,, \tag{1.11}$$

where $\mathcal{L}^n$ is any n-dimensional subspace of the corresponding domain. Then

1. $\lambda_\infty := \lim_{n\to\infty} \lambda_n = \inf\sigma_{\mathrm{ess}}(H)$ *, with the convention that the essential spectrum is empty if $\lambda_\infty = +\infty$.*
2. $\{\lambda_n\}_{n=1}^{\infty} \cap (-\infty, \lambda_\infty) = \sigma_{\mathrm{disc}}(H) \cap (-\infty, \lambda_\infty)$ *, each $\lambda_n \in (-\infty, \lambda_\infty)$ being an eigenvalue of H repeated a number of times equal to its multiplicity.*

Summing up, each λ_n represents either a discrete eigenvalue or the threshold of the essential spectrum of H. Obviously, if the spectrum of H is purely discrete (*i.e.* H is an operator with compact resolvent), all the eigenvalues may be characterized by this variational principle. In any case, the spectral threshold of H always coincides with λ_1:

Corollary 1.1. *Under the hypotheses of Theorem 1.3,*

$$\inf\sigma(H) = \inf_{\psi\in\mathfrak{D}(H)\setminus\{0\}} \frac{(\psi, H\psi)}{\|\psi\|^2} = \inf_{\psi\in\mathfrak{D}(Q)\setminus\{0\}} \frac{Q[\psi]}{\|\psi\|^2} \,. \tag{1.12}$$

Theorem 1.3 is an extremely useful tool in practical problems in quantum mechanics, (*e.g.*, for computation of eigenvalues of many-body Hamiltonians in quantum chemistry). In this thesis, however, we shall mainly use the variational method to obtain qualitative properties of the spectrum of the Laplacians on $L^2(\Omega)$ as regards the geometry of Ω.

Another important consequence of Theorem 1.3 is the method of *Dirichlet-Neumann bracketing*, as explained in [59, Sec. XIII.15]. As a very special example, we have the operator inequalities

$$-\Delta^{\Omega}_{R_{\alpha_-}} \leq -\Delta^{\Omega}_{N} \leq -\Delta^{\Omega}_{R_{\alpha_+}} \leq -\Delta^{\Omega}_{D} \,,$$

which appropriately transfer to inequalities for eigenvalues below the essential spectrum.

1.3 Glazman's classification of domains

Since the present thesis deals mainly (but not exclusively) with Dirichlet boundary conditions, this section is devoted to a summary of the significant features of a domain Ω as regards the spectrum of the associated Dirichlet Laplacian $-\Delta_D^\Omega$. We base the analysis on the useful geometric classification of Euclidean domains due to I. M. Glazman [30] (see also [16, Sec. X.6.1]).

Definition 1.1 (Glazman's classification of Euclidean domains)**.** A domain $\Omega \subset \mathbb{R}^d$ is

- *quasi-conical* if it contains arbitrarily large balls;
- *quasi-cylindrical* if it is not quasi-conical but it contains infinitely many (pairwise) disjoint identical (*i.e.* of the same radius, congruent) balls;
- *quasi-bounded* if it is neither quasi-conical nor quasi-cylindrical.

Obviously, each domain $\Omega \subset \mathbb{R}^d$ belongs to one of the classes. Bounded domains represent a subset of quasi-bounded domains, but the latter class is much larger as we shall see below. The whole Euclidean space $\mathbb{R}^d$ or its conical sector are examples of quasi-conical domains. Finally, an infinite (solid) cylinder $\mathbb{R} \times B$, where B is a $(d-1)$-dimensional ball, is a quasi-cylindrical domain. See Figure 1.1 for typical examples in $\mathbb{R}^2$.

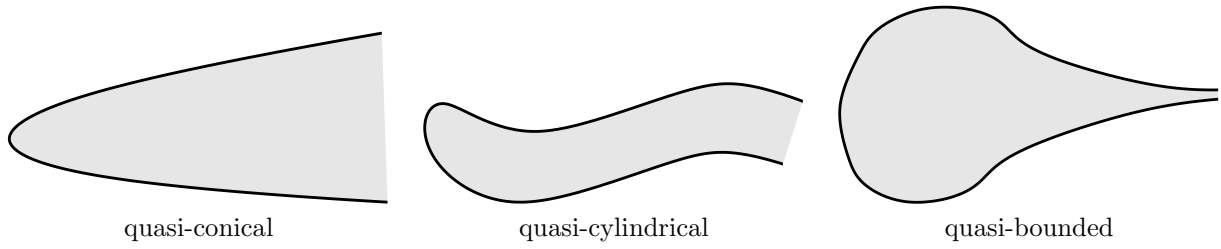


Figure 1.1: Examples of planar domains as regards the Glazman classification

1.3.1 Quasi-conical domains

The Dirichlet spectrum is easiest to locate for quasi-conical domains:

Theorem 1.4 (Spectrum of quasi-conical domains)**.** *If Ω is quasi-conical, then*

$$\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{ess}}(-\Delta_D^\Omega) = [0, \infty)\,.$$

In particular, $\sigma_{\mathrm{disc}}(-\Delta_D^\Omega) = \varnothing$.

This theorem can be established quite easily by constructing a suitable singular sequence of Theorem 1.2 which is supported on the growing balls (*cf* [16, Thm. X.6.5]). From this proof, it is also clear that the same spectral result holds also for $-\Delta_N^\Omega$ and $-\Delta_{R_\alpha}^\Omega$ with $\alpha \geq 0$ (for $\alpha \leq 0$ we only have the inclusion $\sigma_{\mathrm{ess}}(-\Delta_{R_\alpha}^\Omega) \supset [0, \infty)$ in general).

It follows that the spectrum of the Dirichlet Laplacian in quasi-cylindrical domains $\Omega \subset \mathbb{R}^d$ coincides *as a set* with the spectrum of $-\Delta_D^{\mathbb{R}^d}$ (the free quantum Hamiltonian in $\mathbb{R}^d$). Of course, Theorem 1.4 does not say anything about the *nature* of the essential spectrum.

1.3.2 Quasi-bounded domains

Theorem 1.4 says that the spectrum of quasi-conical domains is purely essential, the discrete spectrum is empty. The other extreme case is represented by quasi-bounded domains, for which (as for bounded domains) the spectrum is purely discrete, at least under some regularity assumptions.

More precisely, a sufficient and necessary condition for $-\Delta_D^\Omega$ to have a purely discrete spectrum is that the embedding

$$W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega) \qquad \text{is compact.} \tag{1.13}$$

Shortly below we shall see that the quasi-boundedness is *necessary* for (1.13) to hold (*i.e.*, there is always an essential spectrum for quasi-conical and quasi-cylindrical domains, *cf* Theorem 1.6). However, it is not sufficient [1, Sec. 6.14]. The validity of (1.13) is well studied in the theory of Sobolev spaces (see, *e.g.*, [1, 16]). Here we provide one example of sufficient condition (*cf* [16, Thm. 5.17]):

Theorem 1.5 (Discreteness of spectra for quasi-bounded domains)**.** *One has*

$$\limsup_{|x|\to\infty,\ x\in\Omega} \big|\Omega\cap B_1(x)\big| = 0 \qquad \Longrightarrow \qquad (1.13)\ \textit{holds.}$$

Consequently, $\sigma(-\Delta_D^\Omega) = \sigma_{\rm disc}(-\Delta_D^\Omega)$, $\sigma_{\rm ess}(-\Delta_D^\Omega) = \varnothing$.

It is interesting to compare the sufficient condition of Theorem 1.5 with the following equivalent characterization of quasi-bounded domains (*cf* Definition 1.1):

$$\Omega \text{ is quasi-bounded} \qquad \Longleftrightarrow \qquad \limsup_{|x|\to\infty,\ x\in\Omega} \operatorname{dist}(x,\partial\Omega) = 0\,.$$

Roughly speaking, quasi-bounded domains are "narrow at infinity", but the narrowness must be "inessential in an integral sense" to have (1.13).

Example 1.1 (Spiny urchin)**.** To see that Theorem 1.5 represents just a sufficient condition, we recall the nice example of *spiny urchin* [1, Sec. 6.17] (see the left part of Figure 1.2):

$$\Omega := \mathbb{R}^2 \setminus \bigcup_{k=1}^{\infty} S_k\,, \tag{1.14}$$

where the sets S_k are specified in polar coordinates $(r,\vartheta)\in\mathbb{R}_+\times[0,2\pi)$ by

$$S_k := \big\{(r,\vartheta)\ |\quad r\geq k \quad\wedge\quad \vartheta = n\pi/2^k \quad \text{for} \quad n = 1,2,\dots,2^{k+1}\big\}\,.$$

Note that this domain, though quasi-bounded, is simply connected and has empty exterior. Clearly, it does not satisfy the hypothesis of Theorem 1.5. However, (1.13) holds for it. In fact, it turns out that the compactness of (1.13) depends in an essential way on the dimension of $\partial\Omega$. Any quasi-bounded domain whose boundary consists of reasonably regular $(d-1)$-dimensional surfaces satisfies (1.13).

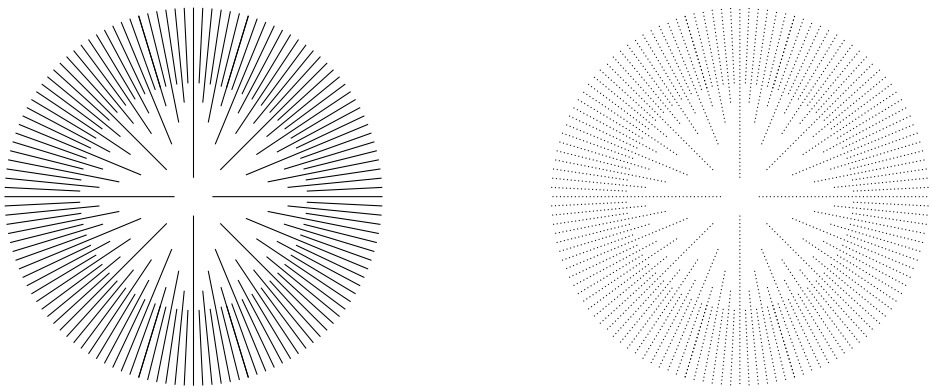
On the other hand, if we replace the lines in (1.14) by "dots accumulating at infinity" (see the right part of Figure 1.2), *i.e.*, we define $\dot\Omega$ as the domain in $\mathbb{R}^2$ obtained by deleting from the plane the union of the sets

$$\dot S_k := \big\{(r,\vartheta)\ |\quad r = k+\sqrt{m} \quad \text{for} \quad m\in\mathbb{N} \quad\wedge\quad \vartheta = n\pi/2^k \quad \text{for} \quad n = 1,2,\dots,2^{k+1}\big\}\,,$$

then exactly the same proof as that of Theorem 1.4 for quasi-conical domains imply that

$$\sigma(-\Delta_D^{\dot\Omega}) = \sigma_{\rm ess}(-\Delta_D^{\dot\Omega}) = [0,\infty)\,.$$

This is obvious since a finite number of points in an open planar set (*e.g.*, an arbitrarily large disc) form a polar set. □



(removing infinite rays)
purely discrete spectrum

(removing dots accumulating at infinity)
purely essential spectrum

Figure 1.2: *Spiny urchin* as an example of a highly irregular quasi-bounded domain

Finally, let us remark that in $d=1$ one knows that quasi-boundedness is necessary and sufficient for an arbitrary (not necessary connected) open subset $\Omega\subset\mathbb{R}$ to satisfy (1.13). In higher dimensions, the necessary and sufficient conditions for the validity of (1.13) can be obtained in terms of capacity (*cf* [16, Sec. X.6.1]).

1.3.3 Quasi-cylindrical domains

Finally, we consider the class of quasi-cylindrical domains. They are most interesting because they generally contain both the essential and discrete spectra. But this make them also most difficult to study. The following result is probably the only one which can be stated in general.

Theorem 1.6 (Spectrum of quasi-cylindrical domains)**.** *Let Ω be quasi-cylindrical. Set*

$$R_{\max} := \sup\left\{R \mid \Omega \text{ contains a sequence of disjoint balls of radius } R\right\}.$$

Then

$$\inf \sigma_{\mathrm{ess}}(-\Delta_D^\Omega) \leq \frac{\mu_1}{R_{\max}^2},$$

where μ_1 denotes the lowest eigenvalue of the Dirichlet Laplacian in the unit ball in $\mathbb{R}^d$ (μ_1 depends exclusively on the dimension d). Consequently, $\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{ess}}(-\Delta_D^\Omega) \cup \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)$ in general.

Proof. The idea is to construct a non-compact sequence supported on the disjoint balls. Let $\{x_n\}_{n\in\mathbb{N}^*} \subset \Omega$ be a set of points such that $\{B_R(x_n)\}_{n\in\mathbb{N}^*} \subset \Omega$ is the set of mutually disjoint balls for all $R \in (0, R_{\max})$. Let ψ be the first eigenfunction of $-\Delta_D^{B_R(0)}$, normalized to 1 in $L^2(B_R(0))$. For all $n \in \mathbb{N}^*$, we set

$$\psi_n(x) := \psi(x - x_n)$$

and extend it by zero to the whole Ω. Then ψ_n's are mutually orthonormal in $L^2(\Omega)$ and satisfy $\|\nabla\psi_n\|^2_{L^2(\Omega)} = \mu_1/R^2$. Hence, choosing the n-dimensional subspace $\mathcal{L}^n = \mathrm{span}\{\psi_1, \dots, \psi_n\}$ in the minimax principle (Theorem 1.3), we get

$$\lambda_n \leq \mu_1/R^2$$

for *all* $n \in \mathbb{N}^*$. Consequently, $\inf \sigma_{\mathrm{ess}}(-\Delta_D^\Omega) = \lambda_\infty \equiv \lim_{n\to\infty} \lambda_n \leq \mu_1/R^2$. Since the argument held for all $R \in (0, R_{\max})$, we conclude with the stated inequality. □

Remark 1.1. It is not necessary to assume that Ω is quasi-cylindrical in Theorem 1.6. The result applies to quasi-conical domains as well (with $R_{\max} = \infty$), in agreement with Theorem 1.4.

For quasi-cylindrical domains the precise location of the essential spectrum and the question of the existence of eigenvalues are difficult and challenging problems. A detailed study of the spectral properties for a special class of quasi-cylindrical domains, *tubes*, constitutes a major part of the present thesis.

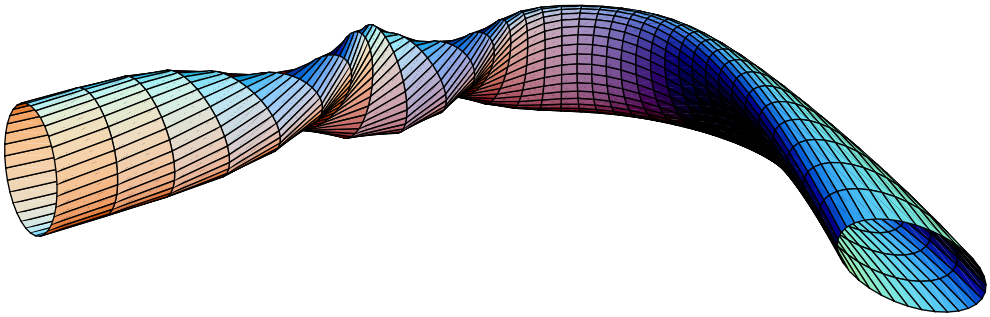


Figure 1.3: An example of a tube of elliptical cross-section. Twisting and bending are demonstrated on the left and right part of the picture, respectively.

1.4 Tubes

The tubes we consider are obtained as a suitable deformation of the *straight tube* $\Omega_0 := I \times \omega$. Here I is an open interval (bounded or unbounded) and $\omega \subset \mathbb{R}^{d-1}$ is a bounded domain that plays the role of cross-section of the tube. We do not assume any regularity conditions about the boundary $\partial\omega$. For simplicity of the presentation, in this introductory section we consider only the three-dimensional case (*i.e.* $d = 3$), which already contains all the interesting features. The deformed tube, denoted by Ω, is then introduced as follows.

1.4.1 The reference curve

We begin with a C^3-smooth curve $\Gamma : I \to \mathbb{R}^3$, which is assumed to be parameterized by its arc-length. Regarding Γ as a path of a unit-speed traveller in the space, it is convenient to describe its motion in a (non-inertial) reference frame moving along the curve. One usually adopts the so-called *distinguished Frenet frame* $\{e_1, e_2, e_3\}$ which satisfies the Serret-Frenet formulae (*cf* [42, Sec. 1.2])

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}^{\cdot} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{1.15}$$

Here κ and τ are the *curvature* and *torsion* of Γ, respectively. The elements of the triad $\{e_1, e_2, e_3\}$ are called the *tangent*, *normal* and *binormal* vectors, respectively, and are obtained from $\{\dot\Gamma, \ddot\Gamma, \dddot\Gamma\}$ by the Gramm-Schmidt orthonormalization process. One has to keep in mind that not every curve in $\mathbb{R}^3$ possesses the distinguished Frenet frame, but it always exists if the curvature is never vanishing, *i.e.* $\kappa > 0$.

1.4.2 The general moving frame

A general moving frame along Γ can be introduced by rotating normal components of the distinguished Frenet frame. More specifically, given a C^1-smooth function $\theta : I \to \mathbb{R}$, we define a rotation matrix-valued function $\mathcal{R}^\theta : I \to \mathrm{SO}(3)$ by

$$\mathcal{R}^\theta := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

This give rise to a new moving frame $\{e_1^\theta, e_2^\theta, e_3^\theta\}$ by setting

$$e_i^\theta := \sum_{j=1}^{3} \mathcal{R}_{ij}^\theta \, e_j \,, \qquad i \in \{1,2,3\}. \tag{1.16}$$

Note that $e_1^\theta(s) = e_1(s)$ for all $s \in \mathbb{R}$, while $e_2^\theta(s)$ (respectively $e_3^\theta(s)$) is rotated with respect to $e_2(s)$ (respectively $e_3(s)$) by the angle $\theta(s)$. Using (1.15), it is easy to check that the new frame evolves along the curve via

$$\begin{pmatrix} e_1^\theta \\ e_2^\theta \\ e_3^\theta \end{pmatrix}^{\cdot} = \begin{pmatrix} 0 & \kappa\cos\theta & \kappa\sin\theta \\ -\kappa\cos\theta & 0 & \tau - \dot\theta \\ -\kappa\sin\theta & -(\tau - \dot\theta) & 0 \end{pmatrix} \begin{pmatrix} e_1^\theta \\ e_2^\theta \\ e_3^\theta \end{pmatrix}. \tag{1.17}$$

1.4.3 The curved tube

The *curved tube* Ω is defined by moving the cross-section ω along the reference curve Γ together with a generally rotated frame (1.16). More precisely, we set

$$\Omega := \mathcal{L}(\Omega_0), \tag{1.18}$$

where Ω_0 is the straight tube and

$$\mathcal{L} : \Omega_0 \to \mathbb{R}^3 : \left\{ (s,t) \mapsto \Gamma(s) + \sum_{j=2}^{3} t_j \, e_j^\theta(s) \right\}. \tag{1.19}$$

See Figure 1.3.

The image of Ω_0 by $\mathcal{L}$ can be quite complex, in particular Ω can have self-intersections we would like to avoid. Furthermore, our strategy to deal with the curved geometry of the tube will be to use the identification

$$\Omega \simeq (\Omega_0, G),$$

where the latter is the Riemannian manifold Ω_0 equipped with the metric $G = (G_{ij})$ induced by the mapping $\mathcal{L}$, *i.e.*,

$$G_{ij} := (\partial_i \mathcal{L}) \cdot (\partial_j \mathcal{L}), \qquad i,j \in \{1,2,3\}.$$

Here the dot denotes the scalar product in $\mathbb{R}^3$. In other words, we parameterize Ω globally by means of the "coordinates" (s,t) of (1.19). To this aim, we need to impose natural restrictions in order to ensure that $\mathcal{L}$ induces a *diffeomorphism* between Ω_0 and Ω.

Using (1.17), we find

$$G = \begin{pmatrix} h^2 + h_2^2 + h_3^2 & h_2 & h_3 \\ h_2 & 1 & 0 \\ h_3 & 0 & 1 \end{pmatrix}, \quad \begin{aligned} h(s,t) &:= 1 - [t_2 \cos\theta(s) + t_3 \sin\theta(s)]\,\kappa(s)\,, \\ h_2(s,t) &:= -t_3\,[\tau(s) - \dot\theta(s)]\,, \\ h_3(s,t) &:= t_2\,[\tau(s) - \dot\theta(s)]\,. \end{aligned} \tag{1.20}$$

Consequently,

$$|G| := \det(G) = h^2\,.$$

By virtue of the inverse function theorem, the mapping $\mathcal{L}$ induces a *local* diffeomorphism provided that the Jacobian h does not vanish on Ω_0. In view of the uniform bounds

$$0 < 1 - a\,\|\kappa\|_{L^\infty(I)} \;\le\; h \;\le\; 1 + a\,\|\kappa\|_{L^\infty(I)} < \infty\,, \tag{1.21}$$

where the cross-section quantity

$$a := \sup_{t\in\omega} |t|$$

measures the distance of the farthest point of $\overline{\omega}$ to the origin of $\mathbb{R}^2$, the positivity of h is guaranteed by the hypothesis

$$\kappa \in L^\infty(\mathbb{R}) \qquad \text{and} \qquad a\,\|\kappa\|_{L^\infty(I)} < 1\,. \tag{1.22}$$

The mapping then becomes a *global* diffeomorphism if, in addition to (1.22), we assume that

$$\mathcal{L} \quad \text{is injective}\,. \tag{1.23}$$

We remark that the tube Ω represents an example of quasi-cylindrical domains if the interval I is unbounded, otherwise it is bounded.

1.4.4 The natural hypotheses

For the convenience of the reader, we summarize here characteristic conditions needed for the construction of the tube Ω:

1. the reference curve Γ is C^3-smooth and possesses the distinguished Frenet frame;

2. the cross-section ω is bounded;

3. the angle function θ is C^1-smooth;

4. (1.22) *and* (1.23) *hold.*

These hypotheses will be assumed henceforth, without any further repetitions.

Remark 1.2. Relaxing the geometrical interpretation of Ω being a non-self-intersecting tube in $\mathbb{R}^3$, it is possible to consider (Ω_0, G) as an abstract Riemannian manifold where only the reference curve Γ is embedded in $\mathbb{R}^3$. Then one does not need to assume (1.23), and the spectral results below hold in this more general situation, too.

1.4.5 The definitions of bending and twisting

It is clear from the equations of motion of the general moving frame (1.17) that there are two independent geometric effects in curved tubes (*cf* Figure 1.3).

Definition 1.2 (bending)**.** The tube Ω is said to be *bent* if, and only if, the reference curve Γ is not a straight line, *i.e.*, $\kappa \neq 0$. Otherwise it is said *unbent*.

Definition 1.3 (twisting)**.** The tube Ω is said to be *twisted* if, and only if, the cross-section ω is not rotationally invariant with respect to the origin and $\tau - \dot\theta \neq 0$. Otherwise it is said *untwisted*.

Of course, in the second definition it is necessary to assume that the cross-section is not rotationally invariant with respect to the origin, since the shape of the tube Ω is not influenced by a special choice of θ if the cross-section is a disc or an annulus centered at the reference line.

The message of the first definition is clear: the reference curve must be non-trivially curved to give rise to a bending of the tube. On the other hand, the requirement $\tau - \dot\theta \neq 0$ is less intuitive in the definition of twisting, unless Γ is straight. We point out that the special choice of rotations $\dot\theta = \tau$ corresponds to a parallel transport of ω along Γ (we refer to [47] for more details and other equivalent definitions of twisting).

1.4.6 The Laplacian

The Laplacians $-\Delta_D^\Omega$, $-\Delta_N^\Omega$ or $-\Delta_{R_\alpha}^\Omega$ are introduced as the self-adjoint operators associated with the closed quadratic forms (1.8), (1.9) or (1.10), respectively. For simplicity, in this introduction, let us consider the Dirichlet Laplacian only.

Our strategy to investigate $-\Delta_D^\Omega$ is to express it in the coordinates determined by (1.19). More specifically, recalling the diffeomorphism between Ω_0 and Ω given by $\mathcal{L}$, we can identify the Hilbert space $L^2(\Omega)$ with

$$\mathcal{H} := L^2\big(\Omega_0, h(s,t)\,ds\,dt\big)$$

and the Dirichlet Laplacian can be identified with the Laplace-Beltrami operator

$$H\psi := -\Delta_G\psi\,, \qquad \psi\in\mathfrak{D}(H) := \Big\{\psi\in W_0^{1,2}(\Omega_0,G)\;\Big|\;\Delta_G\psi\in\mathcal{H}\Big\}\,. \tag{1.24}$$

Here

$$-\Delta_G := -|G|^{-1/2}\partial_i|G|^{1/2}G^{ij}\partial_j \tag{1.25}$$

is a general expression for the Laplacian expressed in curvilinear coordinates (the Einstein summation convention is adopted, with the range of indices being $1,2,3$) and $W_0^{1,2}(\Omega_0,G)$ denotes the completion of $C_0^\infty(\Omega_0)$ with respect to the norm

$$\|\psi\|_{\mathcal{H}_1} := \sqrt{(\partial_i\psi, G^{ij}\,\partial_j\psi)_{\mathcal{H}} + \|\psi\|_{\mathcal{H}}^2}\,.$$

In our case, when the metric is given by (1.20), we have $|G|^{1/2}=h$ and

$$G^{-1} = \frac{1}{h^2}\begin{pmatrix} 1 & -h_2 & -h_3\\ -h_2 & h^2+h_2^2 & h_2h_3\\ -h_3 & h_3h_2 & h^2+h_3^2\end{pmatrix}.$$

The quadratic form associated with H is given by

$$Q[\psi] = \big(\partial_i\psi, G^{ij}\,\partial_j\psi\big)_{\mathcal{H}} = \big\|h^{-1}\big[\partial_1\psi + (\tau-\dot\theta)\partial_u\psi\big]\big\|_{\mathcal{H}}^2 + \|\nabla'\psi\|_{\mathcal{H}}^2\,, \qquad \psi\in\mathfrak{D}(Q)=W^{1,2}(\Omega_0,G)\,,$$

where $\nabla' := (\partial_2,\partial_3)$ is the transverse gradient and $\partial_u := t_3\partial_2 - t_2\partial_3$ is the transverse angular derivative.

If the functions κ and $\tau-\dot\theta$ are bounded, then the $\mathcal{H}_1$-norm is equivalent to the usual norm in $W^{1,2}(\Omega_0)$ and $W_0^{1,2}(\Omega_0,G)=W_0^{1,2}(\Omega_0)$.

As usual, the idea of expressing the Laplacian in curvilinear coordinates is that a simple operator $(-\Delta)$ acting on a complicated space (Ω) is transformed to a more complicated operator $(-\Delta_G)$ acting on a simple space (Ω_0).

1.4.7 A unitarily equivalent operator

Finally, we transform H on $\mathcal{H}$ into a unitarily equivalent operator on a fully straighten Hilbert space $L^2(\Omega_0)$. This is enabled by the unitary transform

$$\hat U : \mathcal{H}\to L^2(\Omega_0) : \Big\{\psi\mapsto h^{1/2}\psi\Big\}\,,$$

which leads to $\hat H := \hat U H\hat U^{-1}$. To apply $\hat U$, however, it is necessary to strengthen the regularity assumptions to $\Gamma\in C^4(I;\mathbb{R}^3)$ and $\theta\in C^2(I;\mathbb{R})$. To find an explicit form of $\hat H$, we note that the differential expression (1.25) is transformed to

$$|G|^{1/4}(-\Delta_G)|G|^{-1/4} = -|G|^{-1/4}\partial_i|G|^{1/2}G^{ij}\partial_j|G|^{-1/4} = -\partial_iG^{ij}\partial_j + V\,,$$

where

$$V := \partial_i(G^{ij}F_j) + F_iG^{ij}F_j \qquad \text{with} \qquad F_i := \partial_i(\log|G|^{1/4})\,.$$

(This is a general formula valid for any smooth metric G.) Hence,

$$\hat H\psi = -\partial_iG^{ij}\partial_j\psi + V\psi\,, \qquad \psi\in\mathfrak{D}(\hat H) = \Big\{\psi\in\mathfrak{D}(\hat Q)\;\Big|\;-\partial_iG^{ij}\partial_j\psi + V\psi\in L^2(\Omega_0)\Big\}\,, \tag{1.26}$$

in the distributional sense, where

$$\hat Q[\psi] := \big(\partial_i\psi, G^{ij}\partial_j\psi\big) - 2\,\Re\big(\partial_i\psi, G^{ij}F_j\psi\big) + \big(\psi, F_iG^{ij}F_j\psi\big)\,, \qquad \psi\in\mathfrak{D}(\hat Q) := \overline{C_0^\infty(\Omega_0)}^{\sqrt{\hat Q[\cdot]+\|\cdot\|^2}}.$$

Here we denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and inner product in $L^2(\Omega_0)$, respectively.

Again, if κ, $\dot\kappa$, τ and $\dot\theta$ are bounded functions, then the topology of $\mathfrak{D}(\hat h)$ is equivalent to that of $W^{1,2}(\Omega_0)$ and we have $\mathfrak{D}(\hat h) = W_0^{1,2}(\Omega_0)$. Moreover, if in addition $\ddot\kappa$, $\dot\tau$ and $\ddot\theta$ are bounded, then V is a bounded function and

$$\hat Q[\psi] = \big\|h^{-1}\big[\partial_1\psi + (\tau-\dot\theta)\partial_u\psi\big]\big\|^2 + \|\nabla'\psi\|^2 + (\psi, V\psi)\,.$$

Then the study of $-\Delta_D^\Omega$ is reduced to the study of a Schrödinger-type operator $\hat H$.

Chapter 2

Presentation of results

This chapter is devoted to a brief and intentionally somewhat informal summary of the results presented in the subsequent chapters. The latter represent research articles of the author and are divided into the following three parts:

I. vibrating systems and the nodal-line conjecture,

II. twisting versus bending in waveguide-like structures,

III. quantum mechanics on curved manifolds.

In view of the unifying spectral-theoretic approach presented in the previous chapter, this division may seem a bit artificial and there are indeed intersections. However, the individual papers were initially motivated by various physical models and this is reflected in different classes of domains Ω typically considered in the respective parts.

Part I is motivated by the classical wave equation (1.1), for which the domain Ω is typically (but not exclusively) bounded or quasi-bounded. Part II is motivated by the Schrödinger equation (1.3) in quantum waveguides, for which Ω is a quasi-cylindrical domain (typically an unbounded tube). This study also naturally leads to an analysis of the heat equation (1.2) in such unbounded domains. Finally, part III is again motivated by the Schrödinger equation (1.3), however, Ω is a quasi-cylindrical subdomain of an abstract Riemannian manifold (rather than of a Euclidean space).

2.1 *Ad* Part I: Vibrating systems

The simplest mathematical model for a vibrating membrane with fixed edge is the wave equation (1.1) in a bounded planar domain Ω, subject to the Dirichlet boundary condition $u = 0$ on $\partial\Omega$. The eigenfunctions u_n and eigenvalues λ_n of the associated spectral problem (1.4) or, more precisely, of the Dirichlet Laplacian $-\Delta_D^\Omega$ are modes and squared frequencies of vibrations, respectively. The zero set of an eigenfunction corresponds to stationary points of the membrane vibrated in a resonant frequency; it is a curve known as the *nodal line* that forms peculiar shapes (also known as Chladni's patterns): various crossing curves or closed loops.

2.1.1 The nodal-line conjecture

It turns out that the shape of the nodal line is related to acoustic properties of the membrane. In particular, it is important to know whether the nodal line of the *second* eigenfunction u_2 can form a closed loop or not. In this context, the famous conjecture of L. E. Payne's from 1967 [56] states that the nodal line of any second eigenfunction of the Dirichlet Laplacian in an arbitrary bounded two-dimensional Euclidean domain touches the boundary.

So far, it has been shown that the conjecture holds for convex domains [53, 38, 33] and there exist counterexamples with multiply connected domains [35]. Nevertheless, it is still an open question whether the conjecture holds for simply connected domains.

Let us also mention that the study of nodal sets of eigenfunctions may of course be extended in a natural way to higher dimensions [39, 20, 41] and manifolds [13, 5, 60, 22]. It can be stated in a generalized form as follows:

Conjecture 2.1 (Nodal-line conjecture)**.** *For any simply-connected domain $\Omega \subset \mathbb{R}^d$,*

$$\overline{u_2^{-1}(0)} \cap \partial\Omega \neq \varnothing\,.$$

That is, the nodal set must touch the boundary.

As a matter of fact, it has been shown recently in [41] that the conjecture does not hold for $d \geq 3$, even if the domain Ω is simply-connected. Therefore, the validity of the conjecture constitutes an open problem only for $d = 2$. However, even if the conjecture is violated in general, it is important to identify domains for which it is satisfied and, moreover, provide more information on the location and geometry of the nodal set.

The positive result

In the joint work [26] (Chapter 3) with P. Freitas, we establish the validity of the conjecture for sufficiently *thin* bounded tubes Ω (*i.e.* I bounded in the notation of Section 1.4), in Euclidean spaces of *arbitrary dimension*. The tube is allowed to be bent (and therefore non-convex), but it is assumed that it is not twisted (*i.e.* $\dot{\theta} = \tau$ if $d = 3$). We allow for the tube to have an *arbitrary cross-section*, and thus we do not exclude the case of multiply connected domains either.

This result may be extended to higher eigenfunctions and we actually show that, given a natural number N greater than or equal to two, for any $2 \leq n \leq N$ the nodal set of the n-th eigenfunction u_n divides the tube Ω into precisely n subdomains, and the closure of each of these subdomains has a non-empty intersection with $\partial\Omega$, provided that the diameter of the cross-section of Ω is sufficiently small. Moreover, we locate the nodal set near the zeros of the solution of an ordinary differential equation which is associated to the tube in a natural way, via the geometry of the reference curve (see Figure 2.1).

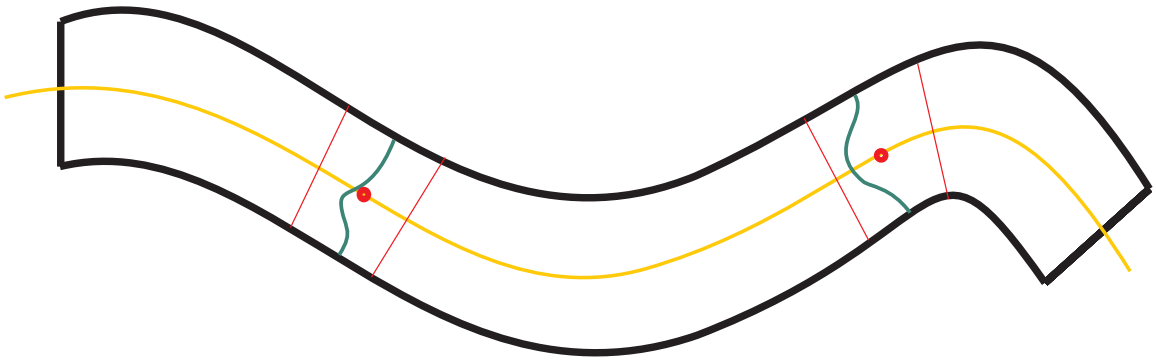


Figure 2.1: A visualization of our result for $d = 2$ and $n = 3$. The nodal lines of u_3 are located close to the points representing the nodal set of ϕ_3 of H_{eff}.

In addition to the fact that we support the validity of the conjecture by a large class of not-previously-considered domains, also our methodological approach to the problem is new. Indeed, our results are based on a resolvent-type convergence of the Dirichlet Laplacian in the tube Ω to a Schrödinger operator on the reference curve that we explain now.

The "thin-width" limit is realized by scaling a fixed cross-section ω by a small positive number ε, *i.e.* we consider the family of tubes Ω_ε obtained after the replacement $\omega \mapsto \varepsilon\omega$. In a first step, we establish the convergence

$$-\Delta_D^{\Omega_\varepsilon} - \varepsilon^{-2} E_1 \;\xrightarrow[\varepsilon\to 0]{}\; H_{\text{eff}} := -\Delta_D^{\Gamma} - \frac{\kappa^2}{4} \tag{2.1}$$

in a norm-resolvent sense, where E_1 is the first eigenvalue of the Dirichlet Laplacian in the fixed cross-section ω. Note that the operators on the left and right parts of the limit act on Hilbert spaces of different dimensions. This conceptual problem is overcome by using the unitarily equivalent operator (1.26), denoted by $\hat{H}_\varepsilon$, instead of $-\Delta_D^{\Omega_\varepsilon}$ and by an appropriate identification of the "straight" Hilbert spaces $L^2(\Omega_0) \simeq L^2(I) \otimes L^2(\omega)$ and $L^2(I)$.

It follows that the eigenfunctions ψ_n^ε of $\hat{H}_\varepsilon$ converge to $\phi_n \otimes \mathcal{J}_1$ in the L^2-topology, where ϕ_n are eigenfunctions of H_{eff} and $\mathcal{J}_1$ is the eigenfunction of $-\Delta_D^\omega$ corresponding to E_1. The idea is that Conjecture 2.1 trivially holds for the limiting eigenfunctions, as a consequence of the separation of variables. Moreover, the nodal set of ψ_n^ε is expected to be located close to the set $\phi_n^{-1}(0) \times \omega$, where the zeros of ϕ_n are governed by Sturm-Liouville oscillation theorems.

However, the L^2-convergence of eigenfunctions is not sufficient to guarantee the corresponding convergence of nodal sets. In fact, even the uniform C^0-convergence is not enough, because of the presence of Dirichlet boundary conditions (ψ_n^ε could have closed nodal surfaces close to the boundary). For these reasons, we also need to apply methods known from elliptic regularity theory in a scale of Sobolev spaces and to use the

maximum principle in a refined way in order to establish a better point-wise convergence, from which the ultimate result about the convergence of nodal sets can be deduced.

In the end of our paper [26], we extend the results to the Laplace-Beltrami operator in tubular neighbourhoods of curves on two-dimensional Riemannian manifolds.

The negative result

In the other joint paper [25] (Chapter 4) with P. Freitas, we show that the restriction to bounded domains in the nodal-line conjecture is crucial. Indeed, if one does not require the domain to be bounded, then the nodal line need not touch the boundary even under the same assumptions that have been previously used in the bounded case to prove the conjecture. More precisely, we prove that there exists a simply-connected unbounded planar domain Ω which is convex and symmetric with respect to two orthogonal directions, and for which the nodal line of a second eigenfunction does not touch the boundary $\partial\Omega$. This domain can be chosen as one of the following two types (see Figure 2.2):

(i) the distance between the nodal line of a second eigenfunction and the boundary $\partial\Omega$ is bounded away from zero, but the spectrum of the Dirichlet Laplacian is not purely discrete;

(ii) the spectrum consists only of discrete eigenvalues, but the infimum of the distance between a point on the nodal line of a second eigenfunction and the boundary $\partial\Omega$ is zero.

Note that case (i) is realized by a quasi-cylindrical domain, while Ω is quasi-bounded in case (ii).

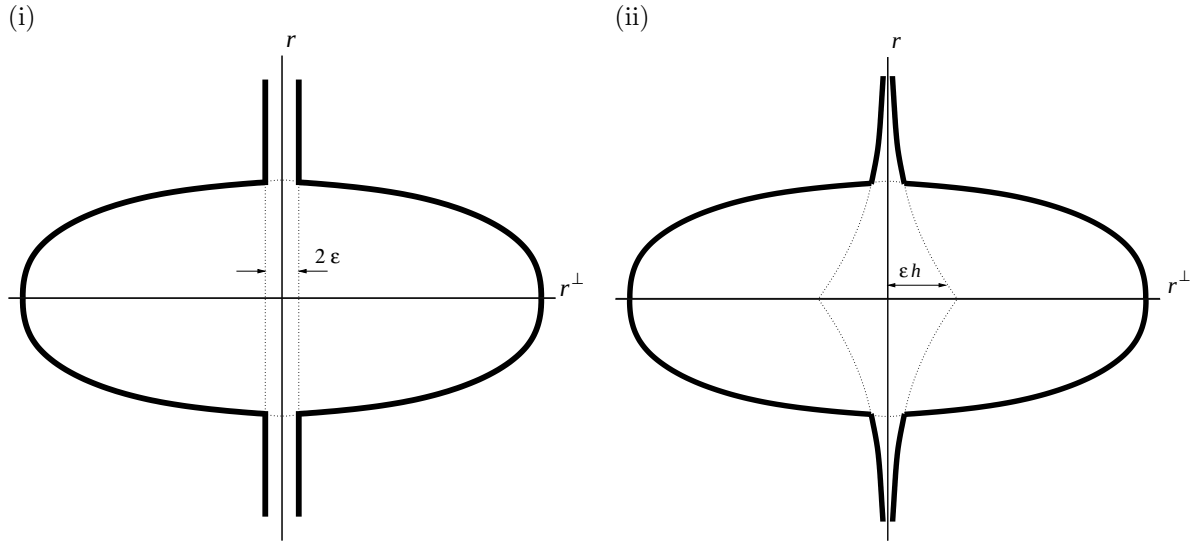


Figure 2.2: Typical domains for which the nodal line of the second eigenfunction does not touch the boundary.

The idea behind both examples is to start from a bounded convex domain Ω_0 which is invariant under reflections through two orthogonal lines r and $r^\perp$ (*cf* Figure 2.2), and which we will assume to be sufficiently long in the direction $r^\perp$, such that its second eigenvalue is simple and any corresponding eigenfunction is antisymmetric with respect to r. In fact, its second nodal line will be given by the closure of $\Omega_0 \cap r$. We then append two sufficiently thin semi-infinite strips to Ω_0 in neighbourhoods of the points where its second nodal line touches the boundary, in such a way that the nodal line coincides with the axis r and thus stays within these strips without touching the boundary. Hence, Conjecture 2.1 is violated.

In order to establish case (i), we consider domains which are asymptotically cylindrical. This means that there also exists essential spectrum, and so it is necessary to prove that the domain does indeed possess a second discrete eigenvalue in this case. In order for condition (ii) to be satisfied, we consider domains which are asymptotically narrow and thus, although the nodal line does not touch the boundary, it does get asymptotically close to it.

It should be stressed that while the nodal line in both our examples does not touch the boundary, it is not closed.

2.1.2 The isoperimetric inequalities

Among all drums of given area, the circular drum is the one which produces the deepest bass note. This is a musical interpretation of the famous Faber-Krahn inequality (conjectured already by Lord Rayleigh in 1877 [58] but proved almost thirty years later, simultaneously and independently by G. Faber and E. Krahn). Mathematically, in any dimension,

$$\lambda_1(\Omega) \geq \lambda_1(B)\,, \tag{2.2}$$

where $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet Laplacian in a bounded domain Ω and B is the ball having the same volume as Ω. Modulus a set of zero capacity, equality in (2.2) is attained if and only if Ω is the ball B and it is thus of interest to understand how strong this connection is. In particular, if the domain Ω is *far away* from B, must its first Dirichlet eigenvalue be much larger than that of the ball?

This particular question was given a positive answer respectively in [34] and [54, 3, 28], where the measure of deviation of the domain from the ball which was used was based respectively on the support function of the domain and the Fraenkel asymmetry (*i.e.* Hausdorff distance if Ω is convex).

In the joint paper [27] (Chapter 5) with P. Freitas, we consider the issue of whether having a large first Dirichlet eigenvalue implies being away from the corresponding ball. The main result of this paper in this direction is the following estimate using the isoperimetric constant as a measure of deviation of convex Ω from B:

$$\frac{|\partial\Omega|}{|\Omega|^{1-1/d}} \;\geq\; \frac{|\partial B|}{|B|^{1-1/d}} \sqrt{\frac{\lambda_1(\Omega)}{\lambda_1(B)}}\, \frac{\pi}{2\sqrt{\lambda_1(B_1)}}\,, \tag{2.3}$$

where B_1 is the d-dimensional ball of unit radius. This bound is, in a sense, optimal for d-dimensional parallelepipeds.

The proof of (2.3) is based on the (sharp) inequality

$$\lambda_1(\Omega) \;\leq\; \lambda_1(B_1)\, \frac{|\partial\Omega|}{d\,\rho_\Omega\,|\Omega|}\,, \tag{2.4}$$

where ρ_Ω is the inradius of Ω. (2.4) is in turn a consequence of a stronger upper bound for $\lambda_1(\Omega)$, holding in the more general case of star-shaped domains, that we establish by using trial functions in Theorem 1.3 with mutually homothetic level sets (see Figure 2.3). The stronger bound depends on the support function of the domain Ω in a non-elementary way (therefore we do not present it here) and it is in fact an extension to arbitrary dimensions of an upper bound for $\lambda_1(\Omega)$ appearing in G. Pólya and G. Szegö's 1951 book [57] in the planar case.

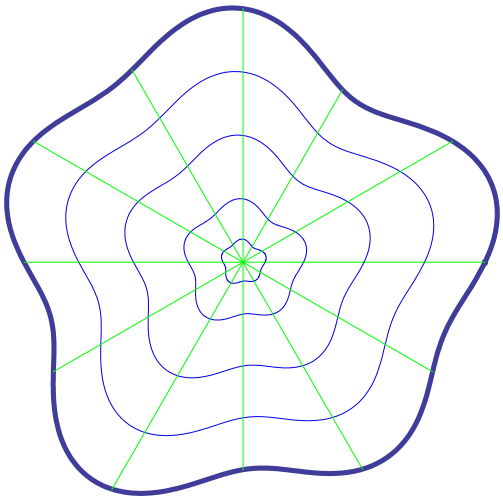


Figure 2.3: A star-shaped domain with curves representing its homothetically shrinking boundary.

As a by-product, we also obtain sharp upper bounds for the second eigenvalue $\lambda_2(\Omega)$ and spectral gap $\lambda_2(\Omega) - \lambda_1(\Omega)$ of convex domains.

2.1.3 The damped wave equation

The wave equation (1.1) is of course an idealization, since it does not take into account dissipation which always exists in real vibrating systems. A more realistic mathematical model is given by the *damped wave equation*

$$\partial_t^2 u + a(x)\,\partial_t u - \Delta u = 0\,. \tag{2.5}$$

Here the positive part of the damping term a corresponds to a dissipation, while negative "damping" models a supply of energy into the system. Apart from viscoelasticity, (2.5) models a variety of evolution processes in other areas of physics: electromagnetism (the telegraph equation), relativistic quantum mechanics, cosmology (the Klein-Gordon equation in a curved space-time), *etc.* The indefinite damping also arises after linearizing semilinear damped wave equations around a stationary solution.

Let us focus on the case of Dirichlet boundary conditions, *i.e.* $u(x,t) = 0$ for every $(x,t) \in \partial\Omega \times (0,\infty)$. In the case where the damping a remains non-negative, the asymptotic behaviour of solutions of (2.5) is well understood [64]. However, the situation is much less clear in the case of the *indefinite damping*, precluding the usage of standard energy methods.

In 1991, G. Chen *et al.* [6] conjectured that for bounded intervals and under certain extra conditions on the damping the trivial solution of (2.5) would remain stable. This was disproved in 1996 by P. Freitas [21], who showed that in the case of bounded domains Ω this sign-changing condition is sufficient to cause the existence of unbounded solutions of (2.5), provided that the supremum norm of the damping is large enough.

Heuristically, this behaviour can be understood from the fact that, when the sign-changing a is replaced by αa and the parameter α increases, equation (2.5) (formally) approaches a backward-forward heat equation. Thus one does expect the appearance of complex eigenvalues μ of the associated (operator pencil) spectral problem

$$\begin{cases} -\Delta u + \mu\, a\, u = -\mu^2 u & \text{in } \Omega\,, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.6}$$

on the positive side of the real axis (which then give rise to unbounded solutions). On the other hand, and still at the heuristic level, note that while for bounded domains the result is not unexpected from the point of view of geometric optic rays either, for unbounded domains this is not as clear.

The lack of results for unbounded domains and/or indefinite damping was a motivation for another joint paper [23] (Chapter 6) with P. Freitas, in which we establish the *instability* of solutions u to the wave equation (2.5) with large (non-homogeneous but time-independent) indefinite damping a in (possibly unbounded) domains. Our approach is to reconsider (2.6) (after the replacement $a \mapsto \alpha a$) as a spectral problem for the linear albeit non-self-adjoint operator

$$A_\alpha := \begin{pmatrix} 0 & 1 \\ \Delta_D^\Omega & -\alpha\, a \end{pmatrix} \qquad \text{on the Hilbert space } \; W_0^{1,2}(\Omega) \times L^2(\Omega) \tag{2.7}$$

and determine the solutions of (2.5) as an application of the corresponding C_0-semigroup on given initial data. We prove that

$$(a_{\min} := \operatorname{ess\,inf} a < 0 \quad \& \quad \alpha > 0 \text{ is large enough}) \qquad \Longrightarrow \qquad \sigma(A_\alpha) \cap (0,\infty) \neq \varnothing\,. \tag{2.8}$$

It follows that the system (2.5) is not stable.

The crucial spectral result (2.8) is a consequence of two lemmata. First, we show how to look for real points in the spectrum of A_α by considering a much simpler (since self-adjoint) spectral problem for the Schrödinger operator $S_\mu := -\Delta_D^\Omega + \mu\, a$.

Lemma 2.1. $\forall \mu \in \mathbb{R}, \qquad -(\mu/\alpha)^2 \in \sigma(S_\mu) \qquad \Longrightarrow \qquad \mu/\alpha \in \sigma(A_\alpha)\,.$

This lemma continues to make sense for complex μ, in such a case, however, one is merely trading one non-self-adjoint problem for another. Second, we establish the following semiclassical-type asymptotics for the Schrödinger operators on arbitrary domains.

Lemma 2.2. $\gamma(\mu) := \inf \sigma(S_\mu) = \mu\,(\operatorname{ess\,inf} a) + o(\mu) \quad$ *as* $\quad \mu \to +\infty.$

It follows from Lemma 2.1 that the conclusion of (2.8) holds provided that there is an intersection for $\mu > 0$ of the ground-state eigencurve $\mu \mapsto \gamma(\mu)$ with the parabola $\mu \mapsto -(\mu/\alpha)^2$. In view of Lemma 2.2 and the fact that $\gamma(0) \geq 0$, this can be always achieved by choosing α sufficiently large. See Figure 2.4 for a possible scenario in the case of a quasi-cylindrical domain Ω.

In fact, the main idea behind the results is the same as that used by P. Freitas in [21], however, the generalization is not straightforward because of the presence of essential spectrum for unbounded domains. Moreover, we work under very mild regularity assumptions about the coefficients of a generalized form of (2.5) and without any restrictions on the geometry of Ω.

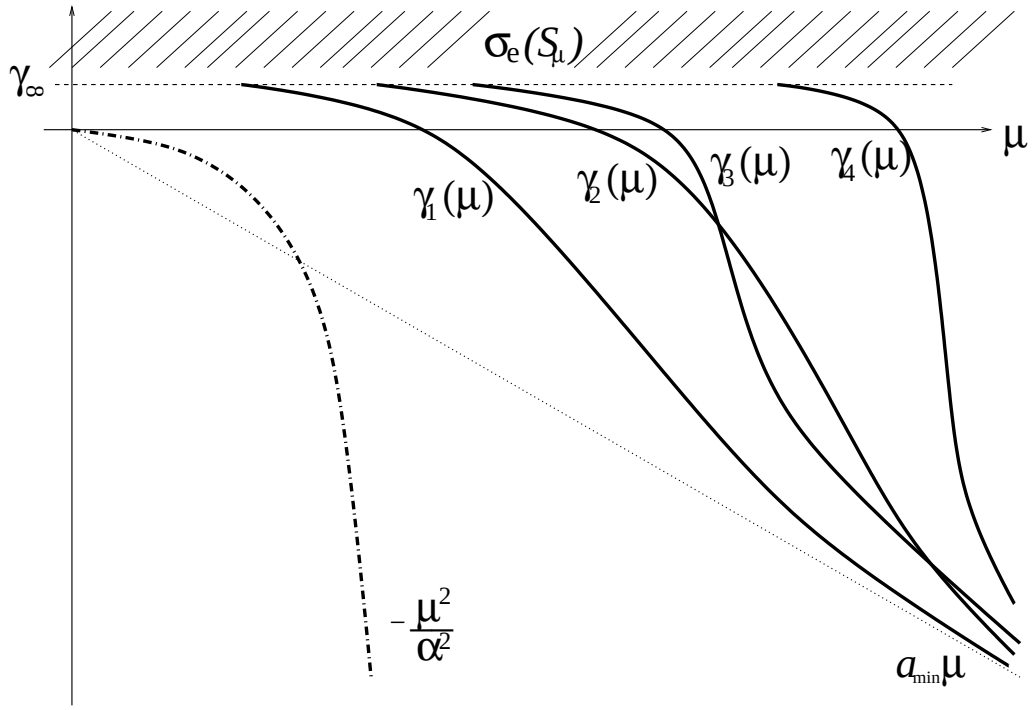


Figure 2.4: The spectrum of S_μ as a function of μ for a quasi-cylindrical domain Ω. The parabola will have intersections with the eigencurves γ_n for all sufficiently large α.

2.2 *Ad* Part II: Quantum waveguides

The evolution of a wavefunction of an effectively free quantum particle confined to a nanostructure with hard walls is described by the Schrödinger equation (1.3) (in suitable units) in a spatial domain Ω, subject to the Dirichlet boundary condition (1.5). The associated spectral problem coincides with (1.4), with eigenfunctions ψ_n representing quantum bound states (stationary solutions) and λ_n being their energies.

For unbounded domains Ω, the spectrum of $-\Delta_D^\Omega$ is not exhausted by eigenvalues, since there is typically also continuous spectrum, which in turn decomposes into absolutely and singularly continuous spectra (the former corresponding to propagating/scattering states and the latter without physical interpretation). This is precisely what happens for *quantum waveguides* modelled by Ω being an unbounded tube of Section 1.4.

Let us remark that the spectral problem for the Laplacian in tubular domains is relevant in other areas of physics as well (electromagnetic and acoustic waveguides, fluid dynamics, *etc*).

2.2.1 The effect of bending

It turns out that the energy spectrum is extremely sensitive to geometric deformations of the quantum waveguide. It is probably best demonstrated by an astonishing 1989 result of P. Exner and P. Šeba [19], who demonstrated the existence of discrete eigenvalues in *bent* two-dimensional strips. A generalization to three-dimensional tubes with circular cross-section was done in [31, 14]. The result is indeed far from being obvious, because the quantum bound states do not have classical counterparts (more precisely, it is a wave effect).

In the joint work with B. Chenaud, P. Duclos and P. Freitas [7] (Chapter 7), we ask the question whether the phenomenon of the existence of bound states in bent quantum waveguides continues to exist, first, for *higher-dimensional tubes* (*i.e.* all $d \geq 2$) and, second, for tubes of general (*i.e.* possibly non-circular) cross-sections ω. After generalizing the geometric concept of curved waveguides to higher dimensions, we successfully extend the standard variational proof of the existence of discrete spectrum to tubes whose (arbitrary) cross-section is appropriately rotated along the reference curve with respect to the Frenet frame, so that it is not twisted (*i.e.* $\dot\theta = \tau$ if $d = 3$); see Figure 2.5. Moreover, we present a new proof of the location of the essential spectrum, which does not require any conditions whatsoever about the derivatives of curvature. Our main results can be stated as follows.

Theorem 2.1 (Bent untwisted tubes)**.** *Let Ω be untwisted. Then*

(i) $\kappa(s) \xrightarrow[|s|\to\infty]{} 0 \quad \Longrightarrow \quad \sigma_{\mathrm{ess}}(-\Delta_D^\Omega) = [E_1, \infty)$,

(ii) $\kappa \neq 0 \quad \Longrightarrow \quad \inf \sigma(-\Delta_D^\Omega) < E_1$.

As a consequence of (i) and (ii), $-\Delta_D^\Omega$ possesses at least one isolated eigenvalue of finite multiplicity in all locally bent tubes.

Recall that E_1 denotes the lowest Dirichlet eigenvalue in the cross-section ω. The corresponding eigenfunction is denoted by $\mathcal{J}_1$.

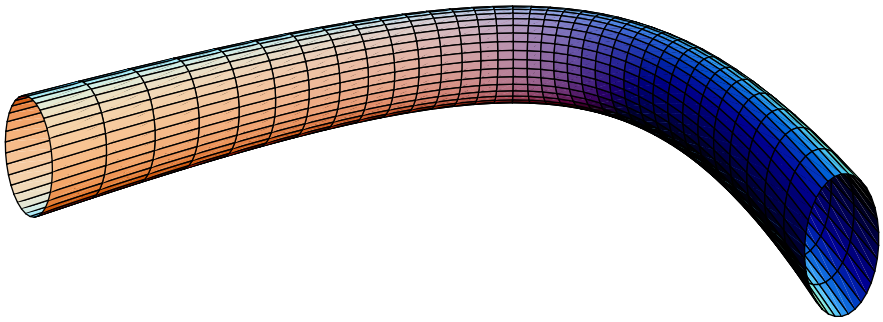


Figure 2.5: An example of bent untwisted waveguide of elliptical cross-section.

Property (i) is easy to understand from the heuristic fact that "the essential spectrum is determined by the behaviour of the geometry at infinity only" (this statement can be made rigorous by the so-called decomposition principle, *cf* [16, Sec. X.1]). In our case, the metric (1.20) is easily seen to converge to the identity I tensor if the curvature vanishes at infinity and the spectrum of (Ω_0, I) coincides with $[E_1, \infty)$. To make these considerations rigorous, without the need to differentiate the metric G in (1.25), we employ a modification of the Weyl criterion of Theorem 1.2 for quadratic forms.

The proof of property (ii) is based on the observation that the generalized eigenfunction $1 \otimes \mathcal{J}_1$ is a critical point of the functional $Q_1[\psi] := Q[\psi] - E_1 \|\psi\|^2_{\mathcal{H}}$ (of course, $1 \otimes \mathcal{J}_1$ does not belong to $\mathcal{H}$, but the statement can be made rigorous by appropriately mollifying the constant function 1 on $\mathbb{R}$). It remains to check that the critical point is not a minimum whenever $\kappa \neq 0$, which we show by an explicit construction.

2.2.2 The effect of twisting

The above paper [7], in which we study the effect of bending in quantum waveguides of arbitrary cross-section, does not answer the question what happens with the spectrum (in particular with the discrete eigenvalues) if the cross-section is *not* appropriately rotated along the reference curve with respect to the Frenet frame, so that the tube is also *twisted* (see Figure 1.3). The effect of twisting is more subtle, because the twist itself (*i.e.* no bending) does not change the spectrum, and more refined functional-analytic techniques are required to describe it. Indeed,

$$\big(\kappa = 0 \quad \& \quad \dot\theta(s) \xrightarrow[|s|\to\infty]{} 0\big) \qquad \Longrightarrow \qquad \sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{ess}}(-\Delta_D^\Omega) = [E_1, \infty)\,,$$

irrespectively of whether the tube is twisted or not. This is probably the reason why the spectral consequences of twisting in quantum waveguides had not been discovered until our breakthrough contribution [17].

In this joint work with T. Ekholm and H. Kovařík [17] (Chapter 8), we actually prove the existence of *Hardy-type inequalities* in twisted unbent waveguides (see Figure 2.6).

Theorem 2.2 (Twisted unbent tubes)**.** *Let Ω be twisted but not bent, $d = 3$. Assume $\dot\theta \in C_0(\mathbb{R})$. Then*

$$\forall \psi \in W_0^{1,2}(\Omega)\,, \qquad \int_\Omega |\nabla\psi|^2 - E_1 \int_\Omega |\psi|^2 \;\geq\; c_H \int_\Omega \frac{|\psi(x)|^2}{1+|x|^2}\,dx\,, \tag{2.9}$$

where c_H is a positive constant independent of ψ.

As a consequence of the functional inequality, twisting acts as a repulsive interaction and the spectrum of a twisted waveguide is stable against small (attractive) perturbations. Furthermore, we employ (2.9) in order to show that the spectrum is stable against geometric deformations as well: a simultaneously twisted and mildly

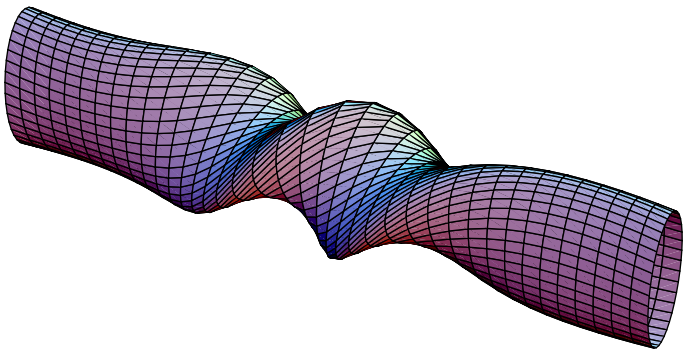


Figure 2.6: An example of twisted unbent waveguide of elliptical cross-section.

bent waveguide does not possess discrete eigenvalues. Putting it somewhat popularly, this result provides a prescription for experimentalists on how to produce bound-state-free waveguide nanostructures.

As a matter of fact, Theorem 2.2 is proved in [17] under the additional hypothesis that $\ddot{\theta}$ is bounded. However, it was later shown in [47, 51], by a different proof, that this technical assumption is not needed. Alternative forms of the Hardy inequality (possibly without the requirement $\dot{\theta} \in C_0(\mathbb{R})$) are presented in [47]. The proof of these references goes as follows. By imposing supplementary Neumann conditions on transverse sections $\pm L \times \omega$, with L such that $(-L, L) \cap \operatorname{supp} \dot{\theta} \neq \varnothing$, one obtains an operator H^N which represents a lower bound to H in the sense of quadratic forms and decomposes into a direct sum of Dirichlet-Neumann Laplacians in the interior and exterior parts of such a divided tube Ω,

$$H \geq H^N = H^N_{\mathrm{int}} \oplus H^N_{\mathrm{ext}} \,.$$

It is easy to see that $\inf \sigma(H^N_{\mathrm{ext}}) \geq E_1$ (with equality if $\dot{\theta}$ vanishes at infinity). But, with a little bit effort, $\inf \sigma(H^N_{\mathrm{int}}) > E_1$ (with strict inequality!) because of compactness. Consequently, $H - E_1 \geq c\, \chi_{(-L,L)}(\cdot)$, where $c := \inf \sigma(H^N_{\mathrm{int}}) - E_1 > 0$ and χ_J denotes the characteristic function of the set $J \times \omega$. To deduce from this "local" (*i.e.* compactly supported) Hardy inequality the "global" (*i.e.* everywhere positive) bound (2.9), one employes the classical one-dimensional Hardy inequality for the Dirichlet Laplacian on the semi-axis $(0, \infty)$. The initial proof of [17] is more technical, based on careful estimates of the form $Q_1[\psi]$ and a ground-state decomposition of ψ.

In the follow-up work with H. Kovařík [44] (Chapter 9), we show that the effect of twisting is robust by establishing an analogous Hardy-type inequality in a two-dimensional waveguide twisted via boundary conditions (see Figure 2.7).

Figure 2.7: Twisting in a two-dimensional strip introduced via switching Dirichlet (thick lines) to Neumann (thin lines) boundary conditions at one point, and *vice versa*.

2.2.3 More general boundary conditions

As we have already mentioned in Introduction, the Dirichlet boundary condition (1.5) represents hard-wall boundaries, which are used to model large chemical barrier on the interface between two semiconductor materials. However, it is not the most general condition modelling impenetrable walls of Ω. A more general boundary condition, which ensures that there is no probability current through the boundary $\partial\Omega$, is given by the Robin boundary conditions (1.7). The values of $\alpha \neq \pm\infty$ may in principle be relevant for different types of interface in a solid and it is thus important to understand the interplay between the boundary conditions, geometry and spectral properties.

In the joint work with J. Kříž [50] (Chapter 10), we make a comparative study of the situation of an unbounded planar strip with Dirichlet, Neumann and a combination of these boundary conditions (see Figure 2.8). It turns out that the existence of bound states is highly sensitive to the nature of boundary conditions imposed: there is always discrete spectrum in locally curved Dirichlet strips, there is no for Neumann strips and, most interestingly, the existence of bound states in strips with Dirichlet condition on one boundary curve and Neumann on the other depends on the direction to which the strip is bent (in the parameterization of Figure 2.8, there are bound states if $\kappa < 0$, no bound states if $\kappa > 0$). The last phenomenon was observed for the first time by J. Dittrich and J. Kříž in 2002 [12]. In addition to an extensive study of the existence and properties of bound states by means of variational methods, we give a new proof of the location of the essential spectrum in quantum waveguides, which does not require any conditions whatsoever about the decay of derivatives of curvature at infinity.

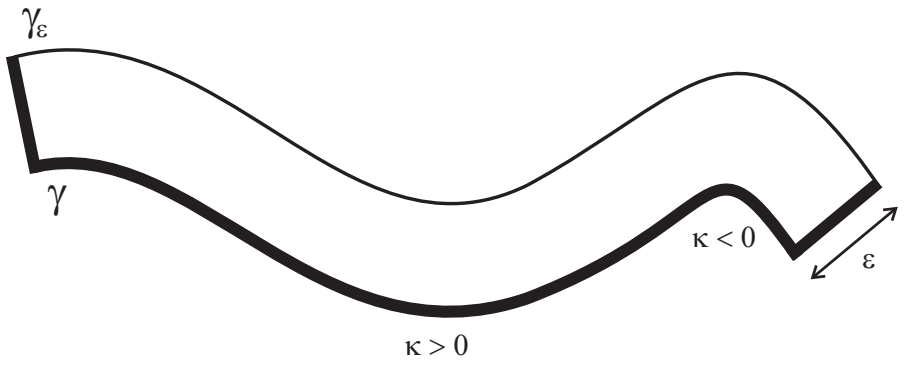


Figure 2.8: A curved strip with combined boundary conditions. The thick and thin lines correspond to Dirichlet and Robin (including possibly Neumann) boundary conditions, respectively (the straight boundary segments are absent for infinite strips). Here the geometry is chosen in such a way that $\kappa > 0$ implies that the parallel curve γ_ε is "locally shorter" than γ, and vice versa.

In order to explain the peculiar spectral properties of the strip with a combination of Dirichlet and Neumann boundary conditions (the model of [12]) established in [50, 12] by the variational proofs, in the follow-up work [48] (Chapter 11), we derive two-term (semiclassical type) asymptotics for the eigenvalues in the limit when the strip width tends to zero. The asymptotics read

$$\lambda_n(-\Delta_{DN}^{\Omega_\varepsilon}) \;=\; \left(\frac{\pi}{2\varepsilon}\right)^2 + \frac{\inf\kappa}{\varepsilon} + o(\varepsilon^{-1}) \qquad \text{as} \qquad \varepsilon \to 0\,, \tag{2.10}$$

where λ_n are the numbers (not necessarily eigenvalues) as defined by the minimax principle (1.11) and $-\Delta_{DN}^{\Omega_\varepsilon}$ denotes the Laplacian in the strip Ω_ε, subject to the corresponding combination of Dirichlet and Neumann boundary conditions. In particular, if $I = \mathbb{R}$ and the curvature κ vanishes at infinity, the leading term $\pi^2/(2\varepsilon)^2$ of (2.10) coincides with the threshold of the essential spectrum of $-\Delta_{DN}^{\Omega_\varepsilon}$. The next term in the expansion then tells us that

- the discrete spectrum exists whenever κ assumes a negative value and ε is sufficiently small;

- the number of the eigenvalues increases to infinity as $\varepsilon \to 0$;

both in coincidence with the results of [50, 12]. Furthermore, we show that the asymptotics (2.10) can be obtained from a form of norm-resolvent convergence which takes into account the width-dependence of the domain of definition of the operators involved.

Finally, in the joint work P. Freitas [24] (Chapter 12), we consider a more general situation when the Neumann boundary condition is replaced by a variable Robin boundary condition. We prove that, for certain α, the spectral threshold of the associated Laplacian is estimated from below by the lowest eigenvalue of the Laplacian in a Dirichlet-Robin annulus determined by the geometry of the strip. Moreover, we show that an appropriate combination of the geometric setting and boundary conditions leads to an (appropriately modified) Hardy-type inequality (2.9) in the infinite strips. As an application, we derive certain stability of the spectrum for the Laplacian in Dirichlet-Neumann strips along a class of curves of sign-changing curvature, improving in this way an initial result of J. Dittrich and J. Kříž [12].

2.2.4 Nature of the essential spectrum

In analogy with the spectra of atoms, the energy spectrum of a locally curved quantum waveguide typically consists of the interval $[E_1, \infty)$ representing the continuous (or, more precisely, essential) spectrum and of a number of discrete eigenvalues below E_1. In principle, the structure of the essential spectrum can be quite complex: apart from the absolutely continuous part (representing propagating states), there might be embedded eigenvalues (bound states) and also a (physically obscure) singularly continuous spectrum. For scattering theory, it is important to know that the singularly continuous spectrum is not present and to have a control over the embedded eigenvalues.

In the joint work with R. Tiedra de Aldecoa [49] (Chapter 13), we make a thorough analysis of the essential spectrum of locally bent but untwisted waveguides (see Figure 2.5) of arbitrary cross-section in any dimension. Under suitable assumptions about the decay of curvatures at infinity, we prove that the singularly continuous spectrum is empty and that the set of eigenvalues is closed and countable, with possible accumulation points only at the thresholds given by the discrete set of Dirichlet eigenvalues of the cross-section. A limiting absorption principle follows as a consequence of the results. The conditions about the decay of curvatures are quite complex to state in an arbitrary dimension, therefore we present here our result for the special three-dimensional situation only.

Theorem 2.3. *Let Ω be untwisted, $d = 3$. Let $\mathcal{T} := \{E_n\}_{n=1}^{\infty}$ be the set of eigenvalues of $-\Delta_D^{\omega}$. Suppose*

1. $\kappa(s),\ \ddot{\kappa}(s) \longrightarrow 0 \quad as \quad |s| \to \infty,$
2. $\exists \vartheta \in (0,1]\,, \qquad \dot{\kappa}(s),\ \dddot{\kappa}(s),\ \tau(s),\ \ddot{\tau}(s) = \mathcal{O}\left(|s|^{-(1+\vartheta)}\right),$
3. $\dot{\tau} \in L^{\infty}(\mathbb{R}).$

Then

(i) $\sigma_{\mathrm{sc}}\left(-\Delta_D^{\Omega}\right) = \varnothing,$

(ii) $\sigma_{\mathrm{p}}\left(-\Delta_D^{\Omega}\right) \cup \mathcal{T}$ *is closed and countable,*

(iii) $\sigma_{\mathrm{p}}\left(-\Delta_D^{\Omega}\right) \setminus \mathcal{T}$ *is composed of finitely degenerated eigenvalues which can accumulate at points of $\mathcal{T}$ only.*

Our conditions needed in $d = 2$ can be easily deduced from these by formally putting $\tau = 0$.

In addition to the generalizations to any dimensions and to waveguides of arbitrary cross-section, the results of the paper [49] provide, in comparison with previous results established in [15] by different methods, alternative sufficient conditions to ensure the important spectral properties. An analogous analysis of the essential spectrum for twisted but unbent three-dimensional tubes was made only recently in [4].

Our analysis is based on Mourre conjugate operator method developed for acoustic multistratified domains in [2, 11]. As a technical preliminary, we carry out a spectral analysis for general Schrödinger-type operators of the form (1.26) in straight tubes Ω_0. To apply the general method, we establish the validity of the Mourre estimate

$$E_{\hat{H}}(J)\ i\big[\hat{H}, A\big]\ E_{\hat{H}}(J)\ \geq\ a\, E_{\hat{H}}(J) + K\,, \tag{2.11}$$

with a positive number a and a compact operator K, outside the set of thresholds $\mathcal{T}$. Here $E_{\hat{H}}(\cdot)$ is the projection-valued function determined by the spectral family of $\hat{H}$ and A is the dilation operator in the longitudinal (infinite) direction. Our strategy to show (2.11) consists in three steps. First, we show that (2.11) holds for the Dirichlet Laplacian $\hat{H}_0 := -\Delta_D^{\Omega_0}$ in a straight tube Ω_0 (*i.e.* $\kappa = 0 = \tau$) and $\hat{H}_0 \in C^{\infty}(A)$. Second (most difficult), we show that $\hat{H} \in C^{1+\vartheta}(A)$, under the stated hypotheses. Finally, we employ the fact that the difference of resolvents of $\hat{H}$ and $\hat{H}_0$ is compact.

We also apply the general result to strips embedded in abstract surfaces.

2.2.5 Diffusive processes

It is well known that some quantum properties (*e.g.* regularity of bound states, criticality of the Hamiltonian, *etc*) are better studied by considering the heat semigroup associated with the Hamiltonian instead of the Schrödinger unitary group. In the quantum-waveguide context, this consists in replacing (1.3) by the heat equation (1.2) (formally obtained by considering imaginary times in (1.3)). Moreover, (1.2) models diffusive processes in other areas of physics (*e.g.* heat flow, Brownian motion, *etc*).

In the joint work with E. Zuazua [51] (Chapter 14), we examine the influence of the existence of the Hardy inequality (2.9) in twisted three-dimensional waveguides on the *large-time behaviour* of the solutions to (1.2). First, we give a new proof of the Hardy inequality, which is more elegant than that presented in [17] and holds under less restrictive conditions on the geometry of Ω. Second, we establish a new Nash-type inequality, which

holds irrespectively of whether the tube is twisted or not. The latter can be used in energy estimates to derive a robust decay estimate for the solutions of (1.2), optimal for straight (*i.e.* untwisted) waveguides.

The main objective of the paper [51] is to show that a better decay estimate holds in twisted unbent waveguides (see Figure 2.6), as a consequence of (2.9). Unfortunately, energy estimates does not seem to be useful and we had to instead apply a refined version of the method of self-similar variables together with the theory of weighted Sobolev spaces in order to show that (2.9) indeed ends up enhancing the decay rate of the solutions. One version of our main results can be stated in terms of the the *decay rate* defined by

$$\Gamma(\Omega) := \sup\left\{\Gamma \;\middle|\; \exists C_\Gamma > 0,\, \forall t \geq 0,\; \left\|e^{\Delta_D^\Omega t}\right\|_{L^2_w(\Omega)\to L^2(\Omega)} \leq C_\Gamma\,(1+t)^{-\Gamma}\,e^{-E_1 t}\right\}, \tag{2.12}$$

where $L^2_w(\Omega) := L^2(\Omega, w(x)\,dx)$ with the Gaussian weight $w(x) := e^{|x|^2/4}$. Our main result reads as follows.

Theorem 2.4. *Let Ω be unbent. Let $\theta \in C^1(\mathbb{R})$ be such that $\dot\theta$ has compact support. Then*

$$\Gamma(\Omega) = \begin{cases} 1/4 & \text{if } \Omega \text{ is untwisted,} \\ 3/4 & \text{if } \Omega \text{ is twisted.} \end{cases}$$

(Strictly speaking, only the lower bound $\Gamma(\Omega) \geq 3/4$ is established in [51] for twisted tubes. However, the same upper bound follows easily by a comparison argument, *cf* [43].)

For solutions u of (1.2) in Ω, we clearly have that for every $\Gamma < \Gamma(\Omega)$, there exists a positive constant C_Γ (independent of u, u_0 and t) such that

$$\|u(t)\| \leq C_\Gamma\,(1+t)^{-\Gamma}\,e^{-E_1 t}\,\|u_0\|_w\,, \tag{2.13}$$

for each time $t \geq 0$ and any initial datum $u_0 \in L^2_w(\Omega)$, where $\|\cdot\|_w$ stands for the norm in the weighted space $L^2_w(\Omega)$. Then Theorem 2.4 can be reformulated as that the solutions exhibit an extra polynomial decay in twisted tubes. The result can be interpreted as that the twisting implies a faster cool-down/death of the medium/Brownian particle in the tube.

The part of Theorem 2.4 for untwisted tubes follows easily by separation of variables. The principal idea behind the main result, *i.e.* the better decay rate in twisted tubes, is the positivity of the right hand side in (2.9). We employ it by the following strategy.

I. First, by mapping the twisted tube Ω to the straight one Ω_0 by the change of variables of Section 1.4, we may consider the transformed (and shifted by E_1) equation

$$u_t - (\partial_1 - \dot\theta\,\partial_u)^2 u - \Delta' u - E_1 u = 0 \tag{2.14}$$

in Ω_0 instead of (1.2). Here $-\Delta' := -\partial_2^2 - \partial_3^2$, with $x = (x_1, x_2, x_3) \in \Omega_0$, denotes the "transverse" Laplacian (recall that $\partial_u := x_3\partial_2 - x_2\partial_3$ is the transverse angular derivative; the same notation u for the solutions is just a coincidence here).

II. The main ingredient in our analysis is the method of self-similar variables developed in the whole Euclidean space by Escobedo and Kavian [18]. Writing

$$\tilde u(y_1, y_2, y_3, s) = e^{s/4} u(e^{s/2} y_1, y_2, y_3, e^s - 1)\,, \tag{2.15}$$

the equation (2.14) is transformed to

$$\tilde u_s - \tfrac{1}{2}\,y_1\,\partial_1\tilde u - (\partial_1 - \sigma_s\,\partial_\tau)^2\tilde u - e^s\,\Delta'\tilde u - E_1\,e^s\,\tilde u - \tfrac{1}{4}\,\tilde u = 0 \tag{2.16}$$

in self-similarity variables $(y, s) \in \Omega_0 \times (0, \infty)$, where

$$\sigma_s(y_1) := e^{s/2}\dot\theta(e^{s/2} y_1)\,. \tag{2.17}$$

III. We reconsider (2.16) in the weighted space $L^2_w(\Omega)$ and show that the associated generator has purely discrete spectrum then (contrary to the generator of (2.14)!).

IV. Finally, we look at the asymptotic behaviour of (2.16) as the self-similar time s tends to infinity. Assume that the tube is twisted. The scaling coming from the self-similarity transformation is such that the function (2.17) converges in a distributional sense to a multiple of the delta function supported at zero as $s \to \infty$. The square of σ_s becomes therefore extremely singular at the section $\{0\} \times \omega$ of the tube for large times. At the same time, the prefactors e^s in (2.16) diverge exactly as if the cross-section of

the tube shrunk to zero as $s \to \infty$. Taking these two simultaneous limits into account, it is expectable that (2.16) will be approximated for large times by the essentially one-dimensional problem

$$\varphi_s - \tfrac{1}{2}\, y_1\, \varphi_{y_1} - \varphi_{y_1 y_1} - \tfrac{1}{4}\, \varphi = 0\,, \qquad s \in (0,\infty),\ y_1 \in \mathbb{R}\,, \tag{2.18}$$

with an extra Dirichlet boundary condition at $y_1 = 0$. This evolution equation is explicitly solvable in $L^2_w(\mathbb{R})$ and it is easy to see that

$$\|\varphi\|_{L^2_w(\mathbb{R})} \le e^{-\frac{3}{4}s}\, \|\varphi_0\|_{L^2_w(\mathbb{R})}\,, \tag{2.19}$$

for any initial datum φ_0. Here the exponential decay rate transfers to a polynomial one after returning to the original time t, and the number $3/4$ gives rise to that of the bound of Theorem 2.4 in the twisted case.

On the other hand, we get just $e^{-\frac{1}{4}s}$ in (2.19) provided that the tube is untwisted (which corresponds to imposing no extra condition at $y_1 = 0$).

It is this last step where we essentially use the existence of the Hardy inequality (2.9) in twisted tubes. In fact, the positivity of the right hand side of (2.9) is directly responsible for the extra Dirichlet boundary condition of (2.18).

In the follow-up work with E. Zuazua [52] (Chapter 15), we show similar results, by the same techniques, for the two-dimensional waveguide twisted via boundary conditions (see Figure 2.7), for which the existence of Hardy inequality was established in [44].

2.3 *Ad* Part III: Quantum traveller on manifolds

The problem of quantization on submanifolds of Riemannian manifolds has attracted a considerable attention from the beginning of quantum mechanics, partly because of the frustrating fact that different approaches lead to different results (see [36] for a concise comparison and [62] for a recent review with many references). Apart from a conceptual importance, the problem is motivated by several specific applications, such as molecular dynamics and physics of nanostructures.

A physically reasonable quantization procedure for the latter can be achieved by imposing a large confining potential in the vicinity of a submanifold of the Euclidean space and by the renormalization consisting in throwing away the exploding normal oscillations in the limit when the neighbourhood shrinks. This leads to an effective Hamiltonian that contains information on, *inter alia*, how the submanifold is embedded in the ambient space [37, 61] (so that the result is not really intrinsic).

Performing the same procedure for submanifolds in a *curved ambient space*, one reveals that the effective Hamiltonian additionally depends on the intrinsic curvature of the ambient Riemannian manifold [55]. An interesting conceptual question is how this curvature affects the quantum transport. The problem is equally interesting for other physical models, such as the heat flow or Brownian motion on curved manifolds.

In a series of papers [45, 46, 43] (Chapters 16–18), we attacked the above question in the simplest non-trivial model when the ambient manifold is an (abstract) surface $\mathcal{A}$ of Gauss curvature K (not necessarily embedded in the three-dimensional Euclidean space) and the submanifold is an infinite curve of (geodesic) curvature κ on it. The confining potential is represented by Dirichlet boundary conditions imposed at a fixed (not necessarily small) distance a from the curve.

Hence, we can again think about the spectral problem (1.4), however, it is important to keep in mind that the strip-like tubular neighbourhood $\Omega \subset \mathcal{A}$ is a manifold now and $-\Delta$ has the meaning of the Laplace-Beltrami operator. Our main strategy to study spectral properties of the operator is to express it in the Fermi (or geodesic parallel) coordinates (x_1, x_2) based on the reference curve (see Figure 2.9). It rigorously consists in defining the strip Ω as the image (1.18) via the exponential map

$$\mathcal{L}(x_1, x_2) := \exp_{\Gamma(x_1)}(N(x_1)\, x_2)\,. \tag{2.20}$$

Here $\Gamma : \mathbb{R} \to \mathcal{A}$ is a unit-speed curve and N is a unit normal vector field along Γ. Then we can use the general formulae of Section 1.4 and identify the Laplace-Beltrami operator with the operator H (respectively $\hat{H}$) on $\mathcal{H}$ (respectively $L^2(\Omega_0)$), provided that we use the metric induced by (2.20). By the generalized Gauss lemma (*cf* [32, Sec. 2]), we have

$$G := \begin{pmatrix} h^2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.21}$$

where h is obtained as the solution of the Jacobi equation

$$\partial_2^2 h + K\, h = 0 \qquad \text{with} \qquad \begin{cases} h(\cdot, 0) = 1\,, \\ \partial_2 h(\cdot, 0) = -\kappa\,. \end{cases} \tag{2.22}$$

In particular, (1.24) and (1.26) hold and we can employ similar techniques as in the case of Euclidean tubes (actually, we have already mentioned some applications in the context of [26] and [49]).

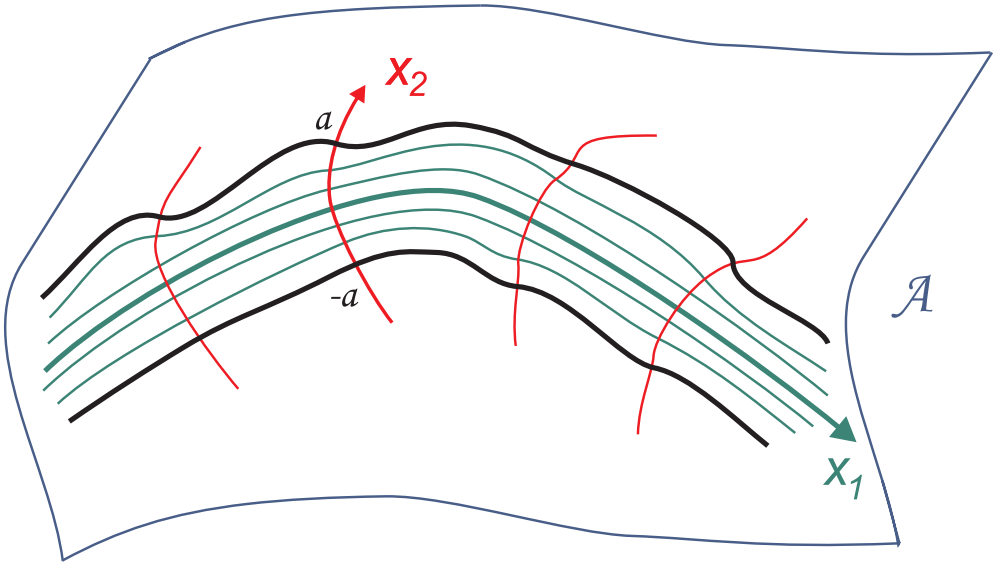


Figure 2.9: The parameterization of the strip Ω via Fermi coordinates $x = (x_1, x_2)$.

As in the case of three-dimensional Euclidean tubes, it turns out that the ambient curvature (namely its sign) has a significant influence on spectral properties: positive (respectively, negative) curvature acts as an attractive (respectively, repulsive) interaction. The contribution of our papers is to discover and mathematically describe the spectral-geometric phenomena.

2.3.1 Positive curvature

In [45] (Chapter 16), we start with analysing the effect of *non-negative* curvature K of the ambient manifold $\mathcal{A}$ (see Figure 2.10). It turns out that the positivity leads to stationary solutions (*i.e.* bound states) of the Schrödinger equation, hurting in this way the transport in the strip Ω. Let us mention that the presence of bound states was predicted by formal arguments in [9, 8]. More precisely, the following theorem holds.

Theorem 2.5 (Strips on positively curved manifolds)**.**

$$\begin{aligned} &\text{(i)} \quad \kappa(x_1),\, K(x) \xrightarrow[|x|\to\infty]{} 0 && \implies && \sigma_{\mathrm{ess}}(-\Delta_D^\Omega) = [E_1, \infty)\,, \\ &\text{(ii)} \quad K \geq 0 \;\;\&\;\; (\kappa \neq 0 \text{ or } K \neq 0) && \implies && \inf \sigma(-\Delta_D^\Omega) < E_1\,. \end{aligned}$$

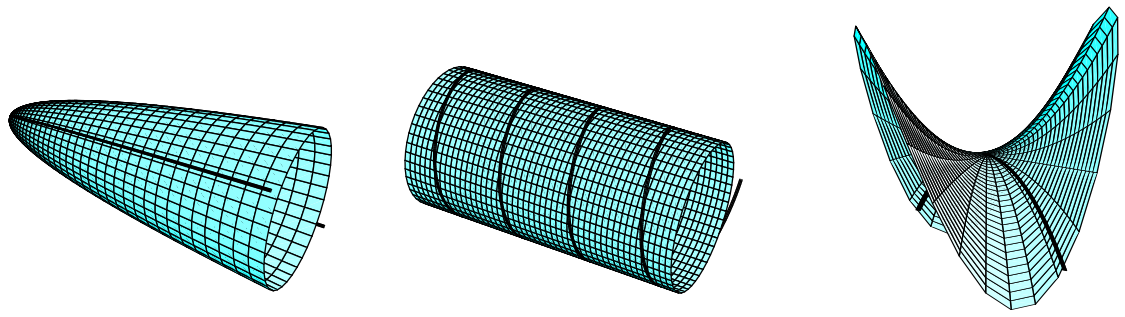
In particular, as a consequence of (i) and (ii), $-\Delta_D^\Omega$ possesses at least one isolated eigenvalue of finite multiplicity in all strips on locally positively curved surfaces.

These results are to be compared with Theorem 2.1, which follows as a special (*i.e.* $K = 0$) case of Theorem 2.5 in $d = 2$. Actually, only a lower bound to the essential spectrum is proved in [45]. Property (i) in the case $\kappa = 0$ is established in the subsequent work [43] and it is straightforward to extend the proof to the general case (in fact, the opposite case $K = 0$ is treated in [50]). The main result of [45] is property (ii) which tells us that there is a spectrum below energy E_1 for positively curved strips. In fact, a more general sufficient condition, requiring the positivity of K in an integral sense only, is established in [45]. Theorem 2.5 can be viewed as a generalization of the classical result of P. Exner and P. Šeba [19] to curved manifolds.

Also the proofs employ similar ideas as those for curved Euclidean tubes. Property (i) follows by a Neumann bracketing argument and the Weyl criterion (Theorem 1.2) adapted to quadratic forms (so that no derivatives of curvatures are needed). Property (ii) is established by a variational argument based on the minimax principle (Theorem 1.3), using a mollification of $1 \otimes \mathcal{J}_1$ in $\mathbb{R} \times \omega$ as a test function.

2.3.2 Negative curvature

The above paper [45] does not answer the question what happens with the spectrum (in particular with the discrete eigenvalues) if the curvature K is *non-positive* (see Figure 2.10), although it is conjectured there



$K > 0$ $K = 0$ $K < 0$

Figure 2.10: The paraboloid of revolution, cylinder and hyperboloid of one sheet as examples of surfaces of positive, zero and negative curvature, respectively. The thick lines denote examples of the reference curve Γ.

that the bound states can be eliminated by the presence of negative curvature. Indeed, the effect of negative curvature is more subtle, for similar reasons as the twisting in quantum waveguides (*cf* Section 2.2.2).

To get at least a partial insight into the problem, in the follow-up paper [46] (Chapter 17), we study a special class of non-positive ambient spaces: *ruled surfaces*. The corresponding strips Ω can be thought as obtained by translating and rotating a segment $\omega := (-a, a)$ along a space curve $\Gamma : \mathbb{R} \to \mathbb{R}^3$ (see Figure 2.11). In this case, we have explicit expressions for the exponential map (2.20)

$$\mathcal{L}(x) = \Gamma(x_1) + x_2 \left[e_2(x_1) \cos\theta(x_1) - e_3(x_1) \sin\theta(x_1) \right],$$

where $\theta \in C^1(\mathbb{R})$ (recall that e_2, e_3 denote the principal normal and binormal of Γ, respectively). The corresponding metric (2.21) and the Gauss curvature appearing in (2.22) respectively read

$$h(x) = \sqrt{\left[1 - x_2\, \kappa(x_1) \cos\theta(x_1)\right]^2 + x_2^2 \left[\tau(x_1) - \dot\theta(x_1)\right]^2}, \qquad K(x) = -\frac{\left[\tau(x_1) - \dot\theta(x_1)\right]^2}{h(x)^4}.$$

Note that K is always non-positive for ruled surfaces.

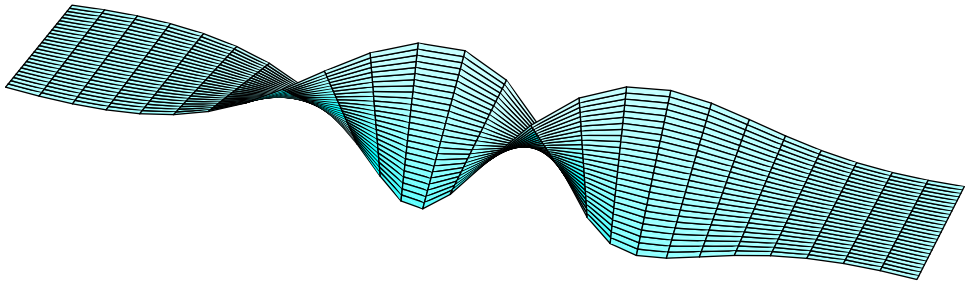


Figure 2.11: A ruled strip as a twisted ribbon along a straight line.

First, we establish the existence of Hardy-type inequalities (2.9) in ruled strips along geodesics (*i.e.* $\kappa = 0$), as a consequence of the presence of negative curvature.

Theorem 2.6 (Strips on ruled surfaces)**.** *Let Ω be a strip along a geodesic on a ruled surface. Assume $0 < a^2 \|\dot\theta\|_{L^\infty(\mathbb{R})} < 2$. Then* (2.9) *holds.*

The proof is based on the inequality (*cf* Section 1.4.7)

$$\hat H = -|G|^{-1/4} \partial_i |G|^{1/2} G^{ij} \partial_j |G|^{-1/4} \geq -|G|^{-1/4} \partial_2 |G|^{1/2} G^{22} \partial_2 |G|^{-1/4} = -\partial_2 G^{22} \partial_2 + V_2 \geq E_1 + V_2$$

in the form sense, where the last step employs the fact that $G^{22} = 1$ in the Fermi coordinates. The "transverse" potential

$$V_2(x) = \frac{\dot\theta(x_1)^2 \left[2 - x_2^2\, \dot\theta(x_1)^2\right]}{4\, h(x)^4}$$

clearly represents a non-negative non-trivial function whenever $a^2 \|\dot\theta\|_{L^\infty(\mathbb{R})} < 2$ and θ is not constant. To deduce from it the (everywhere) positive right hand side of (2.9), we employ the classical one-dimensional Hardy inequality for the Dirichlet Laplacian on the semi-axis $(0, \infty)$.

Second, we use the existence of the Hardy inequality to show that there are no discrete eigenvalues even if the reference curve is a mildly perturbed geodesic, improving in this sense the transport in Ω. The results provide a positive answer to some conjectures raised in [45].

2.3.3 Large-time behaviour

The objective of the most recent joint work with M. Kolb [43] (Chapter 18) is twofold. First, we extend the validity of Theorem 2.6 to a larger class of "negatively curved manifolds"; for instance, it is just enough to assume that the strip is negatively curved in a vicinity of the reference curve to have a Hardy-type inequality (2.9) for small a. Second, we specify the meaning of the "bad" and "good" transport in the positively and negatively curved strips, respectively. Our approach is probabilistic, viewing the traveller in Ω as a Brownian particle governed by (1.2) and the properties of the transport are interpreted in terms of the large-time behaviour of the solutions u. Roughly speaking, by the "good geometry" for transport we understand that which enables the Brownian traveller to reach his/her goal as soon as possible or "to escape from his/her starting point as far as possible".

If the curvature K is non-trivial non-negative, vanishing at the infinity of the strip Ω, then the existence of the ground-state energy $\lambda_1 < E_1$ (*cf* Theorem 2.5) implies that the decay rate is slower in comparison with the straight case ($K = 0 = \kappa$). This is as a direct consequence of the (sharp) spectral-type estimate

$$\|u(t)\|_{L^2(\Omega)} \leq e^{-\lambda_1 t}\, \|u_0\|_{L^2(\Omega)}\,. \tag{2.23}$$

On the other hand, the effect of negative curvature is more subtle and, in analogy with the heat equation in twisted waveguides (*cf* Section 2.2.5), we had to apply the machinery of self-similar variables and weighted Sobolev spaces in order to say something about the fine decay rate (2.12). Our main result can be stated as follows.

Theorem 2.7. *Let Ω be a strip along a geodesic (*i.e. *$\kappa = 0$) on a surface $\mathcal{A}$. Assume that K has compact support in Ω. Then*

$$\Gamma(\Omega) = \begin{cases} 1/4 & \text{if } K = 0, \\ 3/4 & \text{if } \Omega \text{ satisfies the Hardy-type inequality } (2.9). \end{cases}$$

Recall that (2.9) typically holds for negatively curved manifolds (*cf* Theorem 2.6). In particular, it follows from the results in [43] that the decay rate is indeed faster in comparison with the straight case, provided that the curvature K is non-trivial non-positive, compactly supported, and the strip Ω is sufficiently thin.

In addition to the norm-wise estimates (2.13), we also establish a number of point-wise results for probability densities. Overall, the moral of the story is that the negative curvature is "better for travelling", in the sense that the heat semigroup describing the Brownian motion gains an extra polynomial, geometrically induced decay rate. The latter is in fact a consequence of the existence of Hardy-type inequalities in negatively curved manifolds, which play a central role in our proof.

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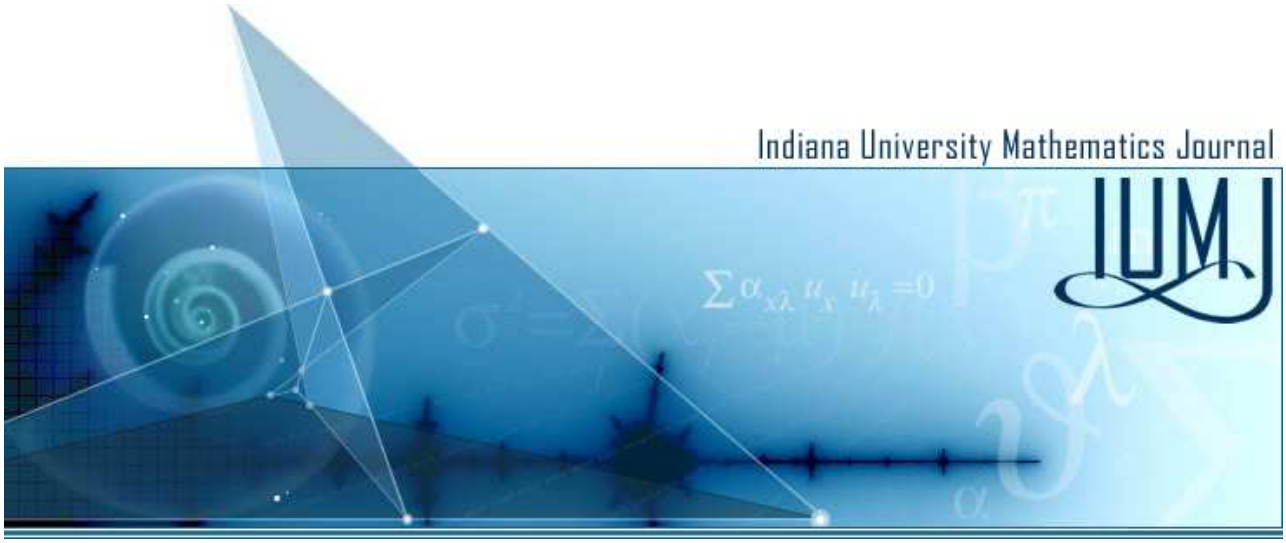
Part I

Vibrating systems

Chapter 3

Location of the nodal set for thin curved tubes



Published in: Indiana University Mathematics Journal **57** (2008), 343–376

Joint work with: Pedro Freitas

Location of the nodal set for thin curved tubes

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Abstract. The Dirichlet Laplacian in curved tubes of arbitrary constant cross-section rotating together with the Tang frame along a bounded curve in Euclidean spaces of arbitrary dimension is investigated in the limit when the volume of the cross-section diminishes. We show that spectral properties of the Laplacian are, in this limit, approximated well by those of the sum of the Dirichlet Laplacian in the cross-section and a one-dimensional Schrödinger operator whose potential is expressed solely in terms of the first curvature of the reference curve. In particular, we establish the convergence of eigenvalues, the uniform convergence of eigenfunctions and locate the nodal set of the Dirichlet Laplacian in the tube near nodal points of the one-dimensional Schrödinger operator. As a consequence, we prove the "nodal-line conjecture" for a class of non-convex and possibly multiply connected domains. The results are based on a perturbation theory developed for Schrödinger-type operators in a straight tube of diminishing cross-section.

3.1 Introduction

Consider the Dirichlet eigenvalue problem for the Laplacian in a bounded domain $U \subset \mathbb{R}^d$, $d \geq 2$:

$$\begin{cases} -\Delta u = \lambda u & \text{in} \quad U\,, \\ u = 0 & \text{on} \quad \partial U\,, \end{cases} \tag{3.1}$$

and let us arrange the eigenvalues in a non-decreasing sequence $\{\lambda_n\}_{n=1}^\infty$ with repetitions according to multiplicities. The set of corresponding eigenfunctions $\{u_n\}_{n=1}^\infty$ may be chosen in such a way that it forms an orthonormal basis for $L^2(U)$. Since the first eigenfunction u_1 does not vanish in U, all other eigenfunctions must change sign, and it makes thus sense to introduce the *nodal set* of u_n ($n \geq 2$):

$$\mathcal{N}(u_n) := u_n^{-1}(0)\,. \tag{3.2}$$

The connected components of $U \setminus \mathcal{N}(u_n)$ are called *nodal domains* of u_n.

Since the solutions of (3.1) are analytic in U, each $\mathcal{N}(u_n)$ decomposes into the disjoint union of an analytic $(d-1)$-dimensional manifold and a singular set contained in a countable number of analytic $(d-2)$-dimensional manifolds (*cf* [7]). The Courant nodal domain theorem then states that, if the boundary ∂U is sufficiently regular, the n^{th} eigenfunction u_n has at most n nodal domains (*cf* [11, 8, 3]). In particular, u_2 has exactly two nodal domains.

Apart from these basic results, not much is known regarding the structure of the nodal sets. One direction along which much work has been developed over the last three decades centres around a conjecture of Payne's from 1967 [49], which states that the nodal set of a second eigenfunction of problem (3.1) for any planar domain cannot consist of a closed curve. This conjecture can be extended in an obvious way to higher dimensions as follows:

Conjecture 3.1. *$\overline{\mathcal{N}(u_2)} \cap \partial U \neq \varnothing$ for any bounded domain $U \subset \mathbb{R}^d$.*

The most general result obtained so far was given by Melas in 1992 [47], who showed that Conjecture 3.1 holds in the case of planar convex domains (*cf* also [2]). This followed a string of results by several authors under some additional symmetry restrictions on the convex domain [50, 45, 52, 13].

Independently of Melas, Jerison had already announced a proof of Conjecture 3.1 in 1991 [33], in the case of planar convex domains which are sufficiently long and thin. In spite of the supplementary eccentricity condition, Jerison's method has some advantages over Melas's in that, on the one hand, it also applies to higher dimensions – *cf* [35] –, and, on the other hand, it is more quantitative, giving some indication as to the location of the nodal set. More precisely, in [34], Jerison located the nodal set in the two-dimensional case near the zero of an ordinary differential equation associated to the convex domain in a natural way (*cf* also [29]).

On the *negative* side, several counterexamples to Conjecture 3.1 have also been presented. The most significant is that in [30] showing that the result does not hold for multiply connected domains in general

(*cf* also [22]). Other counterexamples have been given illustrating different ways in which the conjecture may not hold. These include adding a potential [46] and the case of simply-connected surfaces [23]. The restriction to bounded domains is also crucial, since there are examples of simply-connected, unbounded, planar domains which furthermore satisfy the symmetry restrictions under which the conjecture holds for the bounded case, but for which the nodal set does not touch the boundary [24].

It thus remains an open question for which classes of domains Conjecture 3.1 holds. One possibility which seems reasonable is the following

Conjecture 3.2. *Conjecture 3.1 holds for simply-connected domains in Euclidean space.*

Let us also mention that the study of nodal sets and domains of eigenfunctions may of course be extended in a natural way to manifolds [10, 5, 15, 16, 17, 54, 8, 22]. In fact, in the list of open problems given in [54], Yau asks to what extent this type of results can be extended to manifolds – see Problem 45 in the Chapter *Open problems in differential geometry* in [54], for instance.

The main goal of this paper is to support the validity of Conjecture 3.2 by showing that it holds when U is a sufficiently thin curved (and therefore non-convex) *tube* in $\mathbb{R}^d$, for any $d \geq 2$. Note that we allow for the tube to have an arbitrary cross-section (rotated appropriately with respect to the Frenet frame of a reference curve), and thus we do not exclude the case of multiply connected domains either.

This result may be extended to higher eigenfunctions and we show that, given a natural number N greater than or equal to two, for any $2 \leq n \leq N$ there are precisely n nodal domains of u_n, and the closure of each of these domains has a non-empty intersection with ∂U, provided the tube U is sufficiently thin. Moreover, we locate the nodal set $\mathcal{N}(u_n)$ near the zeros of the solution of an ordinary differential equation which is associated to the tube in a natural way, via the geometry of the reference curve.

Although the nature of our results and the main idea behind them are somewhat similar to those of Jerison's paper [34], the technical approach is actually slightly different. While in that case the location of the nodal set in a convex domain is based on the usage of a trial function and refined applications of the maximum principle, we use the fact that the eigenvalue problem for the Laplacian in a curved tube can be transformed into an eigenvalue problem for a Schrödinger-type operator in a straight tube. This idea goes at least as far back as the paper of Exner and Šeba's from 1989 [21], where it was used to prove the existence of discrete eigenvalues in infinite tubes (*cf* also [18, 43]). This will enable us to develop a perturbation theory (in which the eigenfunctions of a comparison operator play the role of the trial function of Jerison's) and prove L^2-convergence results. However, the maximum principle (in addition to other techniques) will be also useful for us eventually, in order to obtain the necessary C^0-convergence results. More importantly, the main difference with respect to Jerison's paper lies in the different setting: while in Jerison's paper the cross-section varies wildly along a straight line (and part of the point is to show that under the convexity assumption this variation is not too "wild" for the problem at hand), it is constant in this paper (except for Remark 3.3) and the present complication is that the underlying manifold is curved.

The paper is organized as follows. In the next section we collect and comment our main results (Theorems 3.1 and 3.2) together with the main ideas behind them. Section 3.3 consists of a number of subsections devoted to the proof of Theorem 3.2 concerning the spectral properties of Schrödinger-type operators in a straight tube of shrinking cross-section and represents our main technical result. Section 3.4 is devoted to a detailed definition of a tube and the corresponding Laplacian (concisely introduced in Section 3.2.1), and to an application of Theorem 3.2 to this situation, with Theorem 3.1 as an outcome. We conclude the paper with Section 3.5 where we discuss an extension of the main result to strips on surfaces (Theorem 3.9).

3.2 Main results and ideas

Throughout the present paper we use the following notation: $d \geq 2$ is an integer denoting the dimension; $I := (0, L)$ is an open interval of length $L > 0$; ω is a bounded open connected subset of $\mathbb{R}^{d-1}$ with the centre of mass at the origin and with the boundary $\partial\omega$ of class C^∞;

$$a \equiv a(\omega) := \sup_{t \in \omega} |t| \,, \tag{3.3}$$

which estimates from above the half of diameter of ω; $\Omega := I \times \omega$ is a d-dimensional straight tube of length L and cross-section ω; and ε is a (small) positive parameter.

3.2.1 The Laplacian in thin curved tubes

We start by some geometric preliminaries and refer to Section 3.4 for more details.

Let $\Gamma : I \to \mathbb{R}^d$ be a regular curve (*i.e.* an immersion), which is parametrized by arc length. We assume that Γ is uniformly C^∞-smooth and that it possesses an appropriate uniformly C^∞-smooth Frenet frame $\{e_1, \ldots, e_d\}$ (*cf* Assumption B and Remark 3.6 below). Then the i^{th} curvature κ_i of Γ, with $i \in \{1, \ldots, d-1\}$, is also uniformly C^∞-smooth, and given an (arbitrary) positive constant C_Γ, we restrict ourselves to the class of curves satisfying

$$\|\kappa_1\|_{C^2(\overline{I})} \leq C_\Gamma \qquad \text{and} \qquad \|\kappa_\mu\|_{C^1(\overline{I})} \leq C_\Gamma, \quad \forall \mu \in \{2, \ldots, d-1\}. \tag{3.4}$$

For any $\varepsilon > 0$, we introduce the mapping $\mathcal{L} : \Omega \to \mathbb{R}^d$ by setting

$$\mathcal{L}(s,t) := \Gamma(s) + \varepsilon \sum_{\mu,\nu=2}^{d} t_\mu \, \mathcal{R}_{\mu\nu}(s) \, e_\nu(s) \,, \qquad s \in I, \quad t \equiv (t_2, \ldots, t_d) \in \omega, \tag{3.5}$$

where $\mathcal{R}_{\mu\nu}$ are coefficients of a uniformly C^∞-smooth family of rotation matrices in $\mathbb{R}^{d-1}$ yet to be specified. $\mathcal{L}$ is an immersion provided ε is small enough (namely, it satisfies (3.35) below) and induces therefore a Riemannian metric G on the *straight tube* Ω. Solving a system of ordinary differential equations governed by higher curvatures (*cf* (3.29) below), we choose the rotations $(\mathcal{R}_{\mu\nu})$ in such a special way that the metric is diagonal (*cf* Section 3.4.2 for more details). The explicit expression for the metric then reads as follows:

$$G = \operatorname{diag}\left(h^2, \varepsilon^2, \ldots, \varepsilon^2\right) \,, \tag{3.6}$$

where the function h is given by

$$h(s,t) := 1 - \varepsilon \, \kappa_1(s) \sum_{\mu=2}^{d} \mathcal{R}_{\mu 2}(s) \, t_\mu \,, \tag{3.7}$$

and $\mathcal{R}_{\mu 2}$ are determined by the system (3.29) below.

Definition 3.1. We define the *tube* $\mathcal{T}$ *of cross-section* $\varepsilon\omega := \{\varepsilon t \,|\, t \in \omega\}$ *about* Γ to be the Riemannian manifold (Ω, G), where the metric G is given by (3.6) with (3.7).

We refer to Remark 3.2 below for a discussion about the significance of the special class of rotations, and therefore tubes, we restrict to.

Remark 3.1. Notice that neither the tube $\mathcal{T}$ nor the curve Γ are required to be embedded in $\mathbb{R}^d$. However, if Γ is embedded and $\mathcal{L}$ is injective, then $\mathcal{L}$ induces a global diffeomorphism, $\mathcal{T}$ is also embedded and the image $U := \mathcal{L}(\Omega)$ is an open subset of $\mathbb{R}^d$, which has indeed a geometrical meaning of a non-self-intersecting tube. Moreover, $\mathcal{T}$ can be considered as U expressed in curvilinear "coordinates" (s,t) via (3.5).

We denote by $-\Delta_D^{\mathcal{T}}$ the *Dirichlet Laplacian* in the tube $\mathcal{T} \equiv (\Omega, G)$, *i.e.*, the self-adjoint operator in the Hilbert space $L^2(\mathcal{T})$ defined as the Friedrichs extension of the Laplacian on $C_0^\infty(\mathcal{T})$ (*cf* Section 3.4.4 for more details). Of course, in the spirit of Remark 3.1, it is clear that if $\mathcal{L}$ is injective, then $-\Delta_D^{\mathcal{T}}$ is nothing else than the usual Dirichlet Laplacian defined in the open set U, $-\Delta_D^U$, and expressed in the "coordinates" (s,t) via (3.5). The spectrum of $-\Delta_D^{\mathcal{T}}$ is purely discrete; we denote by $\{\lambda_n\}_{n=1}^\infty$ the set of its eigenvalues sorted in non-decreasing order and repeated according to multiplicity, and by $\{\Psi_n\}_{n=1}^\infty$ the set of corresponding eigenfunctions.

As $\varepsilon \to 0$, the image of the tube $U \equiv \mathcal{L}(\Omega)$ collapses in some sense into the reference curve Γ. However, it turns out that the Dirichlet Laplacian in Γ (which is in fact the Dirichlet Laplacian in I, $-\Delta_D^I$, because Γ is parametrized by arc length) is not the right operator governing the spectral properties of $-\Delta_D^{\mathcal{T}}$ in this limit. Instead, it will be clear in a moment that the right operator for this purpose is the one-dimensional Schrödinger operator

$$S := -\Delta_D^I + v_0 \,, \tag{3.8}$$

where the potential function is determined uniquely by the first curvature of Γ:

$$v_0 := -\frac{\kappa_1^2}{4} \tag{3.9}$$

(later on, we shall use the notation S also for other choices of v_0). The spectrum of S is purely discrete and simple for any bounded v_0; we denote by $\{\mu_n\}_{n=1}^\infty$ the set of its eigenvalues sorted in non-decreasing order and repeated according to multiplicity, and by $\{\phi_n\}_{n=1}^\infty$ the set of corresponding eigenfunctions. The Sturm oscillation theorems imply that each ϕ_n has exactly $n-1$ distinct zeros in I, which forms therefore the nodal set $\mathcal{N}(\phi_n)$.

Of course, as ε decreases to 0, the eigenvalues of $-\Delta_D^{\mathcal{T}}$ tend to infinity because of the Dirichlet boundary conditions, but it turns out that there is a simple way of regularizing this singularity. Namely, the shifted eigenvalues $\lambda_n - \varepsilon^{-2}E_1$ remain bounded as ε goes to 0, where E_1 is the first eigenvalue of the Dirichlet Laplacian in ω, $-\Delta_D^{\omega}$; we denote by $\mathcal{J}_1$ a corresponding eigenfunction.

The connection between the tube $\mathcal{T}$ and the operator S is then given by the following theorem.

Theorem 3.1. *Given a positive constant C_Γ, let Γ be any curve as above satisfying (3.4), and let $\mathcal{T}$ be the tube of (shrinking) cross-section $\varepsilon\omega$ about it. For any integer $N \geq 1$, there exist positive constants ε_0 and C depending on N, L, C_Γ, ω and d such that for all $\varepsilon \leq \varepsilon_0$,*

(i) the set of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of $-\Delta_D^{\mathcal{T}}$ sorted in non-decreasing order and repeated according to multiplicity satisfies for any $n \in \{1, \ldots, N\}$,

$$\left|\lambda_n - (\mu_n + \varepsilon^{-2}E_1)\right| \leq C\,\varepsilon\,;$$

(ii) the set of corresponding eigenfunctions $\{\Psi_n\}_{n=1}^{N}$ of $-\Delta_D^{\mathcal{T}}$ can be chosen in such a way that, for any $n \in \{1, \ldots, N\}$,

$$\forall (s,t) \in \Omega, \quad |\Psi_n(s,t) - \Psi_n^0(s,t)| \leq C\,\varepsilon\ \operatorname{dist}(t, \partial\omega)\,,$$

where $\Psi_n^0 := \varepsilon^{-(d-1)/2}\, h^{-1/2}\, \phi_n \otimes \mathcal{J}_1$;

(iii) with the above choice of eigenfunctions, we have for any $n \in \{1, \ldots, N\}$,

$$\forall (s,t) \in \Omega, \quad \operatorname{dist}\big(s, \mathcal{N}(\phi_n)\big) > C\,\varepsilon \implies \operatorname{sgn}\Psi_n(s,t) = \operatorname{sgn}\phi_n(s)\,.$$

As a consequence of (i), we get that, as the cross-section of the tube diminishes, the first N eigenvalues of $-\Delta_D^{\mathcal{T}}$ converge to the first N eigenvalues of S shifted by $\varepsilon^{-2}E_1$. In particular, the former eigenvalues must also be simple for all sufficiently small ε.

Property (ii) implies that Ψ_n is well approximated by Ψ_n^0 in the topology of uniformly continuous functions for all sufficiently small ε and that this is also true for "transverse" derivatives on the boundary.

Property (iii) follows from (ii) and the Courant nodal domain theorem (*cf* Section 3.3.8 for more details). In particular, we get that for any $k \in \{1, \ldots, n\}$, the k^{th} nodal domain of Ψ_n projects surjectively to ω and converges in some sense to the Cartesian product of the k^{th} nodal domain of ϕ_n with ω. At the same time, we get that the nodal set $\mathcal{N}(\Psi_n)$ projects surjectively to ω and converges to the Cartesian-product $\mathcal{N}(\phi_n) \times \omega$. The Courant nodal domain theorem therefore yields

Corollary 3.1. *Under the hypotheses of Theorem 3.1, each $\overline{\mathcal{N}(\Psi_n)}$ has a non-empty intersection with the boundary $\partial\Omega$ (and this at exactly two points in the case $d = 2$ and $n = 2$) and each Ψ_n has exactly n nodal domains.*

Consequently, Conjecture 3.2 holds for all sufficiently thin tubes provided the mapping $\mathcal{L}$ is injective, since then the above results extend to the Dirichlet Laplacian in the open set $U \equiv \mathcal{L}(\Omega)$ in view of Remark 3.1. In particular, if u_n is the n^{th} eigenfunction of $-\Delta_D^U$ (*e.g.* one can choose $u_n := \Psi_n \circ \mathcal{L}^{-1}$), the nodal set $\mathcal{N}(u_n)$ converges to the image $\mathcal{L}\big(\mathcal{N}(\phi_n) \times \omega\big)$ in the sense that

$$\forall x \in U, \quad x \in \mathcal{N}(u_n) \implies \operatorname{dist}\big(s, \mathcal{N}(\phi_n)\big) \leq C\,\varepsilon\,,$$

where s is determined as the first component of $\mathcal{L}^{-1}(x)$.

The observation that the operator S has something to do with the spectral properties of thin tubes is not new. Indeed, it is involved in attempts to justify quantization on submanifolds (*cf* [40, 32, 56, 12, 48] and [25] for formal and rigorous treatments, respectively), in the proofs of the existence of discrete spectrum in curved quantum wires (*cf* [21, 18, 41, 9] and [20] for different mathematical models, respectively) and in asymptotics of eigenvalues of tubular neighbourhoods (*cf* [28, 36]).

Moreover, paper [18] (where the part (i) of Theorem 3.1 is in fact proved for the case $I = \mathbb{R}$ and $d = 2, 3$) inspired us to use L^2-perturbation theory to handle the problem. However, L^2-convergence results are not sufficient to get an estimate on the location of nodal sets and we had to develop new ideas to establish the C^0-(or even some sort of Lipschitz-) convergence result included in the part (ii) of Theorem 3.1. To our knowledge, the results in the present paper are the first convergence results of this type for eigenfunctions and their nodal sets in tubes.

Remark 3.2 (Other tubes)**.** Assume that $\mathcal{L}$ is injective and recall Remark 3.1. It should be stressed here that while the shape of the image U is not influenced by a special choice of the rotation $(\mathcal{R}_{\mu\nu})$ provided ω is rotationally symmetric, this may not be longer true for a general cross-section. The geometrical meaning of

our special choice for rotations ($\mathcal{R}_{\mu\nu}$) is that we restrict to *non-twisted* tubes in the language of [19], which simplifies the analysis considerably. It has been noticed recently in [19, 6] that other choices for the rotation may change the spectral picture, too. Namely, in view of the eigenvalue asymptotics obtained in [6], it seems to be reasonable to conjecture that a version of our Theorem 3.1 will still hold for twisted tubes, provided that v_0 is replaced by a more complicated potential, depending also on the higher curvatures of Γ and the geometry of ω.

Remark 3.3 (Tubes of variable cross-section)**.** While Theorem 3.1 applies exclusively to tubes of uniform cross-section $\varepsilon\omega$, it is possible to obtain similar convergence results for non-uniform cases, too. Let us fix a tube $\mathcal{T}$ of uniform cross-section $\varepsilon\omega$ for which the results of Theorem 3.1 hold true. Consider its deformation $\mathcal{T}_\delta$ obtained by replacing ε in (3.5) by $\varepsilon + \delta r(s)$, where r is a uniformly C^∞-smooth function, ε is fixed and δ plays the role of small parameter now. Then it is possible to verify by methods of the present paper that the spectral properties of $-\Delta_D^{\mathcal{T}_\delta}$ can be approximated by those of $-\Delta_D^{\mathcal{T}}$ in the limit $\delta \to 0$, and obtain an analogy of Theorem 3.1 in this sense. However, let us stress that this problem is much simpler because the comparison operator is independent of the perturbation parameter, while the one treated in the present paper is singular in ε.

3.2.2 Schrödinger-type operators in a thin straight tube

The main idea behind the proof of Theorem 3.1 is that the Laplacian $-\Delta_D^{\mathcal{T}}$ in a curved tube is unitarily equivalent to a Schrödinger-type operator in the straight tube Ω (*cf* Section 3.4.4). This observation leads us to consider in Section 3.3 the self-adjoint operator T in the Hilbert space $L^2(\Omega)$ defined by

$$T := -\partial_1 a_\varepsilon \partial_1 + \varepsilon^{-2}(-\Delta' - E_1) + V_\varepsilon\,, \qquad \mathfrak{D}(T) := \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^2(\Omega)\,. \tag{3.10}$$

Here $-\Delta'$ is the Laplacian in the variables $(t_2, \ldots, t_d)$, with $(s, t_2, \ldots, t_d) \in \Omega$, $\mathcal{H}_0^1(\Omega)$ and $\mathcal{H}^2(\Omega)$ are the usual Sobolev spaces (*cf* [1]), and a_ε and V_ε are real-valued functions satisfying:

Assumption A. There exists a positive constant C_{A} (independent of ε) such that for all $\varepsilon \in (0, C_{\mathrm{A}}^{-1}]$,

(i) $a_\varepsilon, V_\varepsilon \in C^\infty(\overline{\Omega})$,

(ii) $\inf_\Omega a_\varepsilon \geq C_{\mathrm{A}}^{-1}$,

(iii) $\|a_\varepsilon - 1\|_{C^1(\overline{\Omega})} + \|V_\varepsilon - V_0\|_{C^0(\overline{\Omega})} \leq C_{\mathrm{A}}\,\varepsilon$,

where $V_0 := v_0 \otimes 1$ with v_0 being a function from $C^\infty(\overline{I})$ independent of ε and satisfying

(iv) $\|v_0\|_{C^0(\overline{I})} \leq C_{\mathrm{A}}$.

Remark 3.4. In the case of tubes, a_ε and V_ε will be expressed in terms of the function h given by (3.7) (*cf* (3.38) and (3.40) below) and v_0 will be given by (3.9). However, we do not restrict ourselves to this particular setting neither here nor in Section 3.3. Let us also point out that the subtraction of $\varepsilon^{-2}E_1$ in (3.10) is a technically useful trick in order to deal with bounded eigenvalues as $\varepsilon \to 0$ (*cf* Theorem 3.2.(i) below).

The spectrum of T is purely discrete; we denote by $\{\sigma_n\}_{n=1}^\infty$ the set of its eigenvalues sorted in non-decreasing order and repeated according to multiplicity, and by $\{\psi_n\}_{n=1}^\infty$ the set of corresponding eigenfunctions. In view of Assumption A.(iii), it is reasonable to expect that, as $\varepsilon \to 0$, the eigenvalues and eigenfunctions of T will be approximated by those of the decoupled operator

$$S \otimes 1 + 1 \otimes \varepsilon^{-2}\big(-\Delta_D^\omega - E_1\big) \qquad \text{in} \qquad L^2(I) \otimes L^2(\omega)\,. \tag{3.11}$$

Here S is the one-dimensional operator (3.8) with v_0 being determined only by Assumption A; we adopt the notation $\{\mu_n\}_{n=1}^\infty$ and $\{\phi_n\}_{n=1}^\infty$ for its set of eigenvalues (sorted in non-decreasing order and repeated according to multiplicity) and corresponding eigenfunctions, respectively; recall also the notation $\mathcal{J}_1$ for the first eigenfunction of $-\Delta_D^\omega$. It is indeed the case:

Theorem 3.2. *Suppose Assumption A holds. For any integer $N \geq 1$, there exist positive constants ε_0 and C depending on N, L, C_A, ω and d such that for all $\varepsilon \leq \varepsilon_0$,*

(i) the set of eigenvalues $\{\sigma_n\}_{n=1}^\infty$ of T sorted in non-decreasing order and repeated according to multiplicity satisfies for any $n \in \{1, \ldots, N\}$,

$$|\sigma_n - \mu_n| \;\leq\; C\,\varepsilon\,;$$

(ii) the set of corresponding eigenfunctions $\{\psi_n\}_{n=1}^N$ *of* T *can be chosen in such a way that for any* $n \in \{1,\dots,N\}$,

$$\forall (s,t) \in \Omega, \quad |\psi_n(s,t) - \psi_n^0(s,t)| \;\leq\; C\,\varepsilon\;\operatorname{dist}(t,\partial\omega)\,,$$

where $\psi_n^0 := \phi_n \otimes \mathcal{J}_1$ *;*

(iii) with the above choice of eigenfunctions, we have for any $n \in \{1,\dots,N\}$,

$$\forall (s,t) \in \Omega, \quad \operatorname{dist}\big(s,\mathcal{N}(\phi_n)\big) > C\,\varepsilon \implies \operatorname{sgn}\psi_n(s,t) = \operatorname{sgn}\phi_n(s)\,.$$

Theorem 3.1 therefore follows as a consequence of Theorem 3.2 and the unitary equivalence between $-\Delta_D^{\mathcal{T}} - \varepsilon^{-2}E_1$ and T with a particular choice of a_ε and V_ε (*cf* Sections 3.4.4 and 3.4.5).

The proof of Theorem 3.2 consists of six steps:

1. Convergence of eigenvalues;
2. Convergence of eigenfunctions in $L^2(\Omega)$;
3. Convergence of eigenfunctions in $\mathcal{H}^2(\Omega)$;
4. Convergence of eigenfunctions in $C^0(\Omega)$;
5. Convergence of transverse derivatives of eigenfunctions in $C^0(I \times \partial\omega)$;
6. Convergence of nodal domains.

Step 1, *i.e.* part (i) of Theorem 3.2, is established in Section 3.3.3 directly through the *minimax principle*. Step 2 is a consequence of the fact that T converges to the operator (3.11) in a generalized sense, which is established in Section 3.3.4 by means of *perturbation theory*. Step 3 is deduced from Step 2 in Section 3.3.5 by using ideas of *elliptic regularity theory* in a refined way. In view of the Sobolev embedding theorem, Step 3 already implies Step 4 in the case of $d = 2$ or 3. For higher dimensions, however, we have to use a different argument to deduce Step 4 in Section 3.3.6, namely, the *maximum principle*. In Section 3.3.6, the latter is also used to establish Step 5, *i.e.*, more precisely, the part (ii) of Theorem 3.2. Finally, in Section 3.3.8, Step 6, *i.e.* the part (iii) of Theorem 3.2, is deduced from Step 5 by means of the *Courant nodal domain theorem*.

The last reasoning also yields

Corollary 3.2. *Under the hypotheses of Theorem 3.2, each* $\overline{\mathcal{N}(\psi_n)}$ *has a non-empty intersection with the boundary* $\partial\Omega$ *(and this at exactly two points in the case* $n = 2$ *and* $d = 2$*) and each* ψ_n *has exactly* n *nodal domains.*

3.2.3 Comments on notation

Here we point out some special convention frequently used throughout the paper.

Since the straight tube Ω is of the form $I \times \omega$, with $I \subset \mathbb{R}$ and $\omega \subset \mathbb{R}^{d-1}$, we consistently split variables into $(s,t) \in \mathbb{R} \times \mathbb{R}^{d-1}$, where $t \equiv (t_2,\dots,t_d)$.

The symbol $\partial_i^n := \partial^n/\partial x_i^n$ stands for the partial derivative of the n^{th}-order with respect to the i^{th}-variable, $i \in \{1,\dots,d\}$, and we use the identification $(x_1,x_2,\dots,x_d) \equiv (s,t_2,\dots,t_d)$. ∇' denotes the gradient in the "transverse" variables $(t_2,\dots,t_d)$.

We set $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ where $\mathbb{N} = \{0,1,\dots\}$.

If U is an open set, we denote by $-\Delta_D^U$ the Dirichlet Laplacian in U, *i.e.* the self-adjoint operator associated on $L^2(U)$ with the quadratic form Q_D^U defined by $Q_D^U[\psi] := \int_U |\nabla\psi|^2$, $\mathfrak{D}(Q_D^U) := \mathcal{H}_0^1(U)$.

We shall usually not distinguish between multiplication operators and the corresponding generating functions, between differential operators and the corresponding differential expressions, *etc.* In fact, all the operators we consider act in the classical sense.

3.3 Schrödinger-type operators in shrinking straight tubes

This section is devoted to the study of spectral properties of the operator T defined by (3.10) in the limit as ε goes to 0. While the cross-section of Ω is ω (and not $\varepsilon\omega$) and is thus actually fixed, we refer to T as a (Schrödinger-type) operator in a shrinking tube, since that is indeed the case after an obvious change of variables. Our main goal is Theorem 3.2, which is established in the following subsections as described in Section 3.2.2.

3.3.1 The Schrödinger-type operator

The operator T is properly introduced as follows. Let t be the sesquilinear form in the Hilbert space $\mathcal{H} := L^2(\Omega)$ defined by

$$t(\phi,\psi) := (\partial_1\phi, a_\varepsilon\partial_1\psi)_{\mathcal{H}} + \varepsilon^{-2}(\nabla'\phi,\nabla'\psi)_{\mathcal{H}} - \varepsilon^{-2}E_1(\phi,\psi)_{\mathcal{H}} + (\phi, V_\varepsilon\psi)_{\mathcal{H}}\,,$$
$$\phi,\psi\in\mathfrak{D}(t) := \mathcal{H}_0^1(\Omega)\,.$$

In view of the properties (i) and (ii) from Assumption A, the form t is clearly densely defined, closed, symmetric and bounded from below for any positive ε. Consequently, it follows by the first representation theorem (*cf* [40, Sec. VI.2.1]) that there exists a unique self-adjoint operator T which is also bounded from below and satisfies

$$T\psi = \tau\psi\,, \qquad \psi\in\mathfrak{D}(T) = \{\psi\in\mathcal{H}_0^1(\Omega)\,|\,\tau\psi\in\mathcal{H}\}\,,$$

where τ is the differential expression in Ω defined by

$$\tau := -\partial_1 a_\varepsilon\partial_1 + \varepsilon^{-2}(-\Delta' - E_1) + V_\varepsilon$$

and the derivatives must be interpreted in the distributional sense. However, since the boundary $\partial\Omega$ is sufficiently regular, it follows by standard methods for regularity of weak solutions of elliptic equations (*cf* [44, Chap. III]) that indeed

$$\mathfrak{D}(T) = \mathcal{H}_2 := \mathcal{H}_0^1(\Omega)\cap\mathcal{H}^2(\Omega) \tag{3.12}$$

and T acts as τ in the classical sense on functions from $\mathfrak{D}(T)$.

Hence, $\{T\,|\,\varepsilon\in(0,C_{\mathrm{A}}^{-1}]\}$ forms a family of elliptic self-adjoint ε-dependent operators which is uniformly bounded from below by $-(1+C_{\mathrm{A}})$ (*cf* (3.17) below for the last statement). The spectrum of T is purely discrete (*cf* [29, Sec. 8.12]), and recall that we have denoted by $\{\sigma_n\}_{n=1}^\infty$ the set of its eigenvalues (sorted in non-decreasing order and repeated according to multiplicity). The set of corresponding eigenfunctions $\{\psi_n\}_{n=1}^\infty$ can be chosen in such a way that all ψ_n are real. Moreover, applying the elliptic regularity theorem repeatedly, we know that each $\psi_n\in C^\infty(\overline{\Omega})$.

3.3.2 The comparison operator

In view of Assumption A.(iii), it is reasonable to expect that, as $\varepsilon\to 0$, T "converges" in a suitable sense to the self-adjoint operator T_0 defined by

$$T_0 := -\partial_1^2 + \varepsilon^{-2}(-\Delta' - E_1) + V_0\,, \qquad \mathfrak{D}(T_0) := \mathcal{H}_2\,. \tag{3.13}$$

We use the quotation marks because T_0 itself forms a family of ε-dependent operators. T_0 is bounded from below by $-C_{\mathrm{A}}$. The spectrum of T_0 is purely discrete too and we denote by $\{\sigma_n^0\}_{n=1}^\infty$ the set of its eigenvalues sorted in non-decreasing order and repeated according to multiplicity.

T_0 is a good comparison operator since it is naturally decoupled as (3.11). Consequently, we know that (*cf* [53, Corol. of Thm. VIII.33])

$$\sigma(T_0) = \{\mu_n\}_{n=1}^\infty + \{\varepsilon^{-2}(E_n - E_1)\}_{n=1}^\infty\,, \tag{3.14}$$

where $\{\mu_n\}_{n=1}^\infty$ (respectively $\{E_n\}_{n=1}^\infty$) denotes the set of eigenvalues of the operator S defined by (3.8) and with Assumption A being the only restriction on v_0 (respectively of $-\Delta_D^\omega$), sorted in non-decreasing order and repeated according to multiplicity.

The longitudinal operator

Let ϕ_n be a real eigenfunction of S corresponding to the eigenvalue μ_n; we choose ϕ_1 to be positive and normalize all ϕ_n to 1 in $L^2(I)$. Assumption A ensures that $\phi_n\in C^\infty(\overline{I})$. By the Sturm oscillation theorems [31, Chap. X], the spectrum of S is simple and ϕ_n has exactly $n+1$ zeros in $\overline{I}$ which we denote by

$$0\equiv s_0^0(n) < s_0^1(n) < \cdots < s_0^{n-1}(n) < s_0^n(n) \equiv L\,. \tag{3.15}$$

That is, $\mathcal{N}(\phi_n) = \{s_0^k(n)\}_{k=1}^{n-1}$. We define $I_n := I\setminus\mathcal{N}(\phi_n)$, *i.e.* I without the *nodal points* of ϕ_n. We also introduce the open subintervals

$$I_n^k := \big(s_0^k(n),\, s_0^{k+1}(n)\big)\,, \qquad k\in\{0,\ldots,n-1\}\,.$$

The importance of the following proposition relies on the fact that the constant c is independent of the particular choice of v_0.

Proposition 3.1. *Suppose Assumption A holds. For any $n \in \mathbb{N}^*$, there exists a positive constant $c = c(n, L, C_A)$ such that*

(i) $(n\pi/L)^2 - C_A \le \mu_n \le (n\pi/L)^2 + C_A$;

(ii) $\|\phi_n\|_{C^2(\overline{I})} \le c^{-1}$;

(iii) $\mu_{n+1} - \mu_n \ge c$;

(iv) $\forall k \in \{0, \dots, n-1\}, \quad s_0^{k+1}(n) - s_0^k(n) \ge c$;

(v) $\forall s \in I, \quad |\phi_n(s)| \ge c \operatorname{dist}(s, \partial I_n)$.

Proof. The estimates (i) follow directly by means of the minimax principle.

The bound (ii) is deduced in three steps. Firstly, using the eigenvalue equation for S and (i), we obtain an *a priori* estimate $\|\phi_n\|_{\mathcal{H}^2(I)} \le 1 + (n\pi/L)^2 + 2C_{\mathrm{A}}$. Secondly, $\|\phi_n\|_{C^1(\overline{I})} \le C_1 \|\phi_n\|_{\mathcal{H}^2(I)}$ with a positive constant $C_1 = C_1(L)$ coming from the Sobolev embedding theorem (*cf* [1, Thm. 5.4]). Finally, using again the eigenvalue equation for S and (i), we can estimate $\|\phi_n''\|_{C^0(\overline{I})} \le ((n\pi/L)^2 + 2C_{\mathrm{A}}) \|\phi_n\|_{C^0(\overline{I})}$. This proves (ii) with some $c_1 = c_1(n, L, C_{\mathrm{A}}) > 0$.

As for (iii), the existence of a positive constant $c_2 = c_2(n, L, C_{\mathrm{A}})$ estimating from below the eigenvalue gaps uniformly in the class of potentials bounded by a constant C_{A} follows from [38]. (Alternatively, one can use the Rellich-Kondrachov embedding theorem in the spirit of [4, proof of Thm. II.1], where (iii) is proved for $n = 1$; the generalization to higher eigenvalues is straightforward.)

To prove (iv), notice that for any given $n \in \mathbb{N}^*$, ϕ_n satisfies the eigenvalue problem

$$\begin{cases} -\phi_n'' + v_0 \phi_n = \mu_n \phi_n & \text{in} \quad I_n^k \,, \\ \phi_n = 0 & \text{on} \quad \partial I_n^k \,, \end{cases} \tag{3.16}$$

for every $k \in \{0, \dots, n-1\}$. Furthermore, ϕ_n does not change sign in I_n^k. Combining (3.16) with (i), we arrive at

$$n^2\pi^2/L^2 + C_{\mathrm{A}} \ge \mu_n \ge \pi^2 / \big(s_0^{k+1}(n) - s_0^k(n)\big)^2 - C_{\mathrm{A}} \,,$$

which gives (iv) with a positive constant $c_3 = c_3(n, L, C_{\mathrm{A}})$.

Finally, given $n \in \mathbb{N}^*$, fix $k \in \{0, \dots, n-1\}$ and let $s_{\max} \in I_n^k$ be such that $\max_{I_n^k} |\phi_n| = |\phi_n(s_{\max})|$. Combining (3.16) with (i) and using obvious estimates, we arrive at

$$\begin{aligned} \pi^2 / \big(s_0^{k+1}(n) - s_0^k(n)\big)^2 &\le \int_{s_0^k(n)}^{s_0^{k+1}(n)} \phi_n'(s)^2 ds = \int_{s_0^k(n)}^{s_0^{k+1}(n)} [\mu_n - v_0(s)] \phi_n(s)^2 ds \\ &\le (n^2\pi^2/L^2 + 2C_{\mathrm{A}}) \big(s_0^{k+1}(n) - s_0^k(n)\big) \, \phi_n(s_{\max})^2 \,. \end{aligned}$$

Since $s_0^{k+1}(n) - s_0^k(n) \le L$, it follows that $|\phi_n(s_{\max})| \ge c_4$ with a positive constant $c_4 = c_4(n, L, C_{\mathrm{A}})$. Let w be the solution to

$$\begin{cases} Aw := -w'' + C_{\mathrm{A}} w = 0 & \text{in} \quad (s_0^k(n),\, s_{\max}) \,, \\ w = 0 & \text{at} \quad s_0^k(n) \,, \\ w = c_4 & \text{at} \quad s_{\max} \,. \end{cases}$$

Since $Aw \le A|\phi_n|$ in $(s_0^k(n),\, s_{\max})$ and $w \le |\phi_n|$ at the boundary points, the maximum principle (*cf* [26, Thm. 3.3]) yields $w \le |\phi_n|$ in $(s_0^k(n),\, s_{\max})$. Using the explicit form of w, we estimate

$$w(s) \equiv \frac{c_4 \sinh[\sqrt{C_{\mathrm{A}}}(s - s_0^k(n))]}{\sinh[\sqrt{C_{\mathrm{A}}}(s_{\max} - s_0^k(n))]} \ge \frac{c_4 \sqrt{C_{\mathrm{A}}}\,(s - s_0^k(n))}{\sinh[\sqrt{C_{\mathrm{A}}} L]} \ge \frac{c_4 \sqrt{C_{\mathrm{A}}}\, \operatorname{dist}(s, \partial I_n)}{\sinh[\sqrt{C_{\mathrm{A}}} L]} \,,$$

and therefore $|\phi_n| \ge c_5 \operatorname{dist}(\cdot, \partial I_n)$ in $(s_0^k(n),\, s_{\max})$ with a positive constant $c_5 = c_5(n, L, C_{\mathrm{A}})$. A similar comparison argument on $(s_{\max}, s_0^{k+1}(n))$ yields the same lower bound, with the same constant c_5, and (v) is proved.

The constant c is eventually chosen as the smallest one among $c_1, \dots, c_5$. □

The transversal operator

Let $\mathcal{J}_n$ denote a real eigenfunction of $-\Delta_D^\omega$ corresponding to E_n; we choose $\mathcal{J}_1$ to be positive and normalize all $\mathcal{J}_n$ to 1 in $L^2(\omega)$. Since we assume that the boundary $\partial\omega$ is of class C^∞, elliptic regularity theory yields that the eigenfunctions belong to $C^\infty(\overline{\omega})$ (they are even analytic in the interior of ω). Moreover, since $\mathcal{J}_1$ is a positive eigenfunction of a Dirichlet problem, we have

Proposition 3.2. *There exists a positive constant $c = c(\omega)$ such that*

$$\forall t \in \omega, \qquad \mathcal{J}_1(t) \,\geq\, c\ \mathrm{dist}(t, \partial\omega)\,.$$

Next, as a consequence of the variational definition of E_1, we have the following Poincaré inequality for ω:

$$\forall \psi \in \mathcal{H}_0^1(\omega), \qquad \|\nabla\psi\|^2_{L^2(\omega)} \,\geq\, E_1\,\|\psi\|^2_{L^2(\omega)}\,. \tag{3.17}$$

This inequality will play a major role in what follows.

Spectrum of the comparison operator

Proposition 3.3. *Suppose Assumption A holds. One has $\sigma_1^0 = \mu_1$. Moreover, for any integer $N \geq 2$, there exists $\varepsilon_0 = \varepsilon_0(N, L, C_A, \omega) > 0$ such that for all $\varepsilon \leq \varepsilon_0$,*

$$\forall n \in \{1, \ldots, N\}, \qquad \sigma_n^0 = \mu_n\,.$$

Proof. Recall (3.14). The assertion for $n = 1$ is obvious. Let $n \geq 2$ and assume by induction that $\sigma_{n-1}^0 = \mu_{n-1}$. Then $\sigma_n^0 = \min\big\{\mu_n, \mu_{n-1} + \varepsilon^{-2}(E_2 - E_1)\big\}$ and the assertion of Proposition follows at once. It is only important to notice that $E_2 - E_1$ depends on ω, and $\mu_n - \mu_{n-1}$ can be estimated by means of Proposition 3.1.(i). □

Let ψ_n^0 denote a real eigenfunction of T_0 corresponding to σ_n^0. In view of Proposition 3.3 and the fact that T_0 is decoupled as (3.11), there is a natural choice for ψ_n^0 if ε is small enough. Namely, we always choose

$$\psi_n^0 := \phi_n \otimes \mathcal{J}_1 \tag{3.18}$$

provided the conclusion of Proposition 3.3 holds for the first n eigenvalues. ψ_n^0 is then normalized to 1 in $\mathcal{H}$. As a consequence of (3.18), Proposition 3.1.(v) and Proposition 3.2, we get

Proposition 3.4. *Suppose Assumption A holds. For any $N \in \mathbb{N}^*$, there exist positive constants $\varepsilon_0 = \varepsilon_0(N, L, C_A, \omega)$ and $c = c(N, L, C_A, \omega)$ such that for all $\varepsilon \leq \varepsilon_0$ and any $n \in \{1, \ldots, N\}$,*

(i) $\mathcal{N}(\psi_n^0) = \mathcal{N}(\phi_n) \times \omega\,,$

(ii) $\forall (s,t) \in \Omega, \quad |\psi_n^0(s,t)| \,\geq\, c\ \mathrm{dist}(s, \partial I_n)\ \mathrm{dist}(t, \partial\omega)\,.$

3.3.3 Convergence of eigenvalues

The purpose of this subsection is to show that T converges to T_0 in the sense of their spectra.

Theorem 3.3. *Suppose Assumption A holds. For any $N \in \mathbb{N}^*$, there exist positive constants $\varepsilon_0 = \varepsilon_0(N, L, C_A, \omega)$ and $C = C(N, L, C_A)$ such that for all $\varepsilon \leq \varepsilon_0$,*

$$\forall n \in \{1, \ldots, N\}, \qquad |\sigma_n - \sigma_n^0| \;\leq\; C\,\varepsilon\,.$$

Proof. Using Assumption A, we estimate $T^- \leq T \leq T^+$ in the form sense, where $T^\pm$ are defined by $\mathfrak{D}(T^\pm) := \mathcal{H}_2$ and

$$T^\pm := (1 \pm C_{\mathrm{A}}\varepsilon)(-\partial_1^2 + V_0) + \varepsilon^{-2}(-\Delta' - E_1) \pm C_{\mathrm{A}}(1 + C_{\mathrm{A}})\varepsilon\,.$$

Assuming that $\varepsilon < C_{\mathrm{A}}^{-1}$, so that T^- is bounded from below, the minimax principle gives $\sigma_n^- \leq \sigma_n \leq \sigma_n^+$ for all $n \in \mathbb{N}^*$, where $\{\sigma_n^\pm\}_{n=1}^\infty$ denotes the set of eigenvalues of $T^\pm$ sorted in non-decreasing order and repeated according to multiplicity. However, using a similar argument to that leading to Proposition 3.3, we know that if ε is so small that the inequality $(1 + C_{\mathrm{A}}\varepsilon)\,\varepsilon^2 \leq (E_2 - E_1)/(\mu_n - \mu_1)$ holds true, where $E_2 - E_1$ depends on ω and $\mu_n - \mu_{n-1}$ can be estimated by means of Proposition 3.1.(i), then

$$\sigma_n^\pm = (1 \pm C_{\mathrm{A}}\varepsilon)\mu_n \pm C_{\mathrm{A}}(1 + C_{\mathrm{A}})\varepsilon = \sigma_n^0 \pm C_{\mathrm{A}}(\mu_n + 1 + C_{\mathrm{A}})\varepsilon\,.$$

This proves the claim, since μ_n can be estimated by Proposition 3.1.(i). □

Remark 3.5. The C^1-norm of $a_\varepsilon - 1$ in Assumption A.(iii) could be weakened to the mere C^0-norm in order to prove Theorem 3.3 by the above method.

The first part of Theorem 3.2 follows immediately as a consequence of Theorem 3.3 and Proposition 3.3.

3.3.4 L^2-convergence of eigenfunctions

While we have proved Theorem 3.3 using just the minimax principle, we shall have to use a stronger technique to establish the convergence of eigenfunctions in $\mathcal{H}$, namely, perturbation theory.

Let us start by introducing some notation. Given two Hilbert spaces X and X', we denote by $\mathfrak{B}(X,X')$ the set of bounded operators from X to X'; we also denote $\mathfrak{B}(X) := \mathfrak{B}(X,X)$. For every complex number z in the resolvent set of T [respectively T_0], we introduce $R(z) := (T-z)^{-1} \in \mathfrak{B}(\mathcal{H})$ [respectively $R_0(z) := (T_0-z)^{-1} \in \mathfrak{B}(\mathcal{H})$]. The space $\mathcal{H}_2$ is introduced in (3.12) and we equip it with the usual $\mathcal{H}^2(\Omega)$-norm.

We shall need two technical lemmata.

Lemma 3.1. *Suppose Assumption A holds. One has*

$$\|T - T_0\|_{\mathfrak{B}(\mathcal{H}_2,\mathcal{H})} \leq 2\, C_A\, \varepsilon\,.$$

Proof. Using Assumption A, one has the obvious estimates

$$\begin{aligned}\|(T-T_0)\psi\|_{\mathcal{H}}^2 \equiv \|(1-a_\varepsilon)\partial_1^2\psi - (\partial_1 a_\varepsilon)\partial_1\psi + (V_\varepsilon - V_0)\psi\|_{\mathcal{H}}^2 \\ \leq 4\, C_{\mathrm{A}}^2\, \varepsilon^2 \left(\|\partial_1^2\psi\|_{\mathcal{H}}^2 + \|\partial_1\psi\|_{\mathcal{H}}^2 + \|\psi\|_{\mathcal{H}}^2\right) \leq 4\, C_{\mathrm{A}}^2\, \varepsilon^2\, \|\psi\|_{\mathcal{H}_2}^2\end{aligned}$$

for any $\psi \in \mathcal{H}_2$. □

Lemma 3.2. *Suppose Assumption A holds. There exists a negative number $z_0 = z_0(C_A,\omega)$ such that for all $\varepsilon \leq 1$,*

$$\forall z \leq z_0, \qquad \|R_0(z)\|_{\mathfrak{B}(\mathcal{H},\mathcal{H}_2)} \leq 2\,.$$

Proof. For any $z \in \mathbb{R} \setminus \sigma(T_0)$ and $f \in C_0^\infty(\Omega)$, let ψ be the (unique) solution to

$$\begin{cases} (T_0 - z)\psi = f & \text{in} \quad \Omega\,, \\ \psi = 0 & \text{on} \quad \partial\Omega\,. \end{cases}$$

Assumption A ensures that $\psi \in C^\infty(\overline{\Omega})$. Restricting ourselves to $z \leq -C_{\mathrm{A}}$ so that $V_0 - z \geq 0$, and integrating by parts, we have

$$\begin{aligned}\|f\|_{\mathcal{H}}^2 &= \|(T_0-z)\psi\|_{\mathcal{H}}^2 \\ &= \|\partial_1^2\psi\|_{\mathcal{H}}^2 + \|(V_0-z)\psi\|_{\mathcal{H}}^2 - 2z\|\partial_1\psi\|_{\mathcal{H}}^2 - 2\Re(\partial_1^2\psi, V_0\psi)_{\mathcal{H}} \\ &\quad + 2\varepsilon^{-2}\big(\|\nabla'\partial_1\psi\|_{\mathcal{H}}^2 - E_1\|\partial_1\psi\|_{\mathcal{H}}^2\big) \\ &\quad + 2\varepsilon^{-2}\big(\|\nabla'(V_0-z)^{1/2}\psi\|_{\mathcal{H}}^2 - E_1\|(V_0-z)^{1/2}\psi\|_{\mathcal{H}}^2\big) \\ &\quad + \varepsilon^{-4}\big(\|\Delta'\psi\|_{\mathcal{H}}^2 - 2E_1\|\nabla'\psi\|_{\mathcal{H}}^2 + E_1^2\|\psi\|_{\mathcal{H}}^2\big)\,.\end{aligned}$$

It is important to notice that the last three lines are non-negative. This is clear for the last one since the expression in brackets is just $\|(-\Delta' - E_1)\psi\|_{\mathcal{H}}^2$ after a factorization and an integration by parts, while the precedent two are non-negative due to (3.17). We also note that $2|\Re(\partial_1^2\psi, V_0\psi)_{\mathcal{H}}| \leq \frac{1}{2}\|\partial_1^2\psi\|_{\mathcal{H}}^2 + 2\|V_0\psi\|_{\mathcal{H}}^2$ by a consecutive usage of the Schwarz and Cauchy inequalities. Consequently, restricting ourselves to $\varepsilon \leq 1$, rearranging the norms, recalling that $\nabla' V_0 = 0$ and using obvious estimates, we can write

$$\begin{aligned}\|f\|_{\mathcal{H}}^2 &\geq \tfrac{1}{2}\|\partial_1^2\psi\|_{\mathcal{H}}^2 + 2\|\nabla'\partial_1\psi\|_{\mathcal{H}}^2 + \|\Delta'\psi\|_{\mathcal{H}}^2 \\ &\quad + 2(-z - E_1)\|\partial_1\psi\|_{\mathcal{H}}^2 + 2(-z - \|V_0\|_{L^\infty(\Omega)} - E_1)\|\nabla'\psi\|_{\mathcal{H}}^2 \\ &\quad + \big[(-z - \|V_0\|_{L^\infty(\Omega)} + E_1)^2 - 2\|V_0\|_{L^\infty(\Omega)}^2\big]\|\psi\|_{\mathcal{H}}^2\,.\end{aligned}$$

Choosing z sufficiently large negative, namely $z \leq z_0 := -(1 + 3C_{\mathrm{A}} + E_1)$, we therefore conclude by

$$\|f\|_{\mathcal{H}}^2 \;\geq\; \tfrac{1}{2}\|\Delta\psi\|_{\mathcal{H}}^2 + \|\nabla\psi\|_{\mathcal{H}}^2 + \|\psi\|_{\mathcal{H}}^2 \;\geq\; \tfrac{1}{2}\|\psi\|_{\mathcal{H}_2}^2\,.$$

Then the assertion of Lemma follows by the density of $C_0^\infty(\Omega)$ in $\mathcal{H}$. □

Now we are ready to prove the main result of this subsection.

Theorem 3.4. *Suppose Assumption A holds. Given $n \in \mathbb{N}^*$, let $\mathcal{C}$ be the circle in $\mathbb{C}$ centred in μ_n and of radius r defined by*

$$4r := \begin{cases} \min\big\{c(n-1,L,C_A), c(n,L,C_A)\big\} & \text{if} \quad n \geq 2\,, \\ c(1,L,C_A) & \text{if} \quad n = 1\,, \end{cases}$$

where c is the positive constant determined by Proposition 3.1. Then there exist positive constants $\varepsilon_0 = \varepsilon_0(n,L,C_A,\omega)$ and $C = C(n,L,C_A)$ such that for all $\varepsilon \leq \varepsilon_0$, one has $\mathcal{C} \cap [\sigma(T_0) \cup \sigma(T)] = \varnothing$ and

$$\forall z \in \mathcal{C}, \qquad \|R(z) - R_0(z)\|_{\mathfrak{B}(\mathcal{H})} \;\leq\; C\,\varepsilon\,.$$

Proof. Fix $n \in \mathbb{N}^*$. Let ε be sufficiently small so that the conclusion of Proposition 3.3 holds true for $n+1$ eigenvalues of T_0. In particular, $\operatorname{dist}(\sigma(T_0), \mathcal{C}) = r$. We also assume that $\varepsilon \leq 1$, so that the conclusion of Lemma 3.2 holds true with some $z_0 < 0$. Then the first resolvent identity [58, Thm. 5.13.(a)], the embedding $\mathcal{H}_2 \hookrightarrow \mathcal{H}$ and obvious estimates yield for any $z \in \mathcal{C}$,

$$\begin{aligned}\|R_0(z)\|_{\mathfrak{B}(\mathcal{H},\mathcal{H}_2)} &\leq \|R_0(z_0)\|_{\mathfrak{B}(\mathcal{H},\mathcal{H}_2)} \big(1 + (|z| + |z_0|)\, \|R_0(z)\|_{\mathfrak{B}(\mathcal{H})}\big) \\ &\leq 2\big(1 + (\mu_n + r + |z_0|)\, r^{-1}\big) \leq C_1\end{aligned}$$

where $C_1 = C_1(n, L, C_{\mathrm{A}})$ comes from an upper bound to μ_n due to Proposition 3.1.(i). Let us now assume in addition that ε is sufficiently small so that σ_n lies inside the circle concentric with $\mathcal{C}$ but with half the radius and other eigenvalues of T lie outside the circle concentric with $\mathcal{C}$ but with twice the radius (this will be true provided $\varepsilon \leq \varepsilon_0(n+1, L, C_{\mathrm{A}}, \omega)$ and $C(n+1, L, C_{\mathrm{A}})\, \varepsilon < r/2$, where ε_0 and C are determined by Theorem 3.3). In particular, $\operatorname{dist}(\sigma(T), \mathcal{C}) \geq r/2$. Then the second resolvent identity [58, Thm. 5.13.(c)] and Lemma 3.1 yield

$$\begin{aligned}\|R(z) - R_0(z)\|_{\mathfrak{B}(\mathcal{H})} &\leq \|R(z)\|_{\mathfrak{B}(\mathcal{H})}\, \|T - T_0\|_{\mathfrak{B}(\mathcal{H}_2,\mathcal{H})}\, \|R_0(z)\|_{\mathfrak{B}(\mathcal{H},\mathcal{H}_2)} \\ &\leq 2\, r^{-1}\; 2\, C_{\mathrm{A}}\, \varepsilon\; C_1\,,\end{aligned}$$

which concludes the proof. □

As a consequence of Theorem 3.4, we get that, as $\varepsilon \to 0$, T converges to T_0 in the *generalized sense* [37, Sec. IV.2.6] (also referred to as the *norm resolvent sense* [14, Sec. 2.6]). This implies (*cf* [37, Sec. IV.3.5]) the continuity of eigenvalues of T at $\varepsilon = 0$ (*cf* also our Theorem 3.3) and of the corresponding spectral projections

$$P_n := -\frac{1}{2\pi i} \int_{\mathcal{C}} R(z)\, dz\,,$$

where $\mathcal{C}$ is determined in Theorem 3.4 for a given $n \in \mathbb{N}^*$ and $\varepsilon \leq \varepsilon_0$. Furthermore, one gets the continuity of eigenfunctions if they are normalized suitably; we choose

$$\psi_n := \Re\left(P_n \psi_n^0\right) \tag{3.19}$$

and stress that ψ_n is not normalized to 1 in $\mathcal{H}$ with this choice.

Corollary 3.3. *Suppose Assumption A holds. For any $N \in \mathbb{N}^*$, there exist positive constants $\varepsilon_0 = \varepsilon_0(N, L, C_A, \omega)$ and $C = C(N, L, C_A)$ such that for all $\varepsilon \leq \varepsilon_0$,*

$$\forall n \in \{1, \ldots, N\}, \qquad \|\psi_n - \psi_n^0\|_{\mathcal{H}} \;\leq\; C\, \varepsilon\,.$$

3.3.5 $\mathcal{H}^2$-convergence of eigenfunctions

Since ψ_n and ψ_n^0 are eigenfunctions of elliptic operators, it is possible to deduce from Corollary 3.3 a stronger convergence result. To do so, let us point out several facts. Fix $n \in \mathbb{N}^*$. Combining the eigenvalue equations for T and T_0, we verify that the difference

$$\psi := \psi_n - \psi_n^0 \tag{3.20}$$

satisfies the Dirichlet problem

$$\begin{cases} -\partial_1(a_\varepsilon\, \partial_1 \psi) + \varepsilon^{-2}\, (-\Delta' - E_1)\, \psi + b_\varepsilon \psi = f_\varepsilon & \text{in} \quad \Omega\,, \\ \psi = 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{3.21}$$

where

$$b_\varepsilon := V_\varepsilon - \sigma_n\,, \qquad f_\varepsilon := (a_\varepsilon - 1)\partial_1^2 \psi_n^0 + (\partial_1 a_\varepsilon)\partial_1 \psi_n^0 + (\sigma_n - \sigma_n^0 - V_\varepsilon + V_0)\psi_n^0\,.$$

The functions b_ε and f_ε belong to $C^\infty(\overline{\Omega})$ due to Assumption A. Moreover, using in addition (3.18), Proposition 3.1 and Theorem 3.3, it is easy to check that

$$\|b_\varepsilon\|_{C^0(\overline{\Omega})} \leq C'_{\mathrm{A}} \qquad \text{and} \qquad \|f_\varepsilon\|_{C^0(\overline{\Omega})} \leq C'_{\mathrm{A}}\, \varepsilon \tag{3.22}$$

with some positive $C'_{\mathrm{A}} = C'_{\mathrm{A}}(n, L, C_{\mathrm{A}}, \omega)$. We also recall that $\psi \in C^\infty(\overline{\Omega})$ due to Assumption A. Now we are ready to prove

Theorem 3.5. *Suppose Assumption A holds. For any $N \in \mathbb{N}^*$, there exist positive constants $\varepsilon_0 = \varepsilon_0(N, L, C_A, \omega)$ and $C = C(N, L, C_A, \omega)$ such that for all $\varepsilon \leq \varepsilon_0$,*

$$\forall n \in \{1, \ldots, N\}, \qquad \|\psi_n - \psi_n^0\|_{\mathcal{H}^2(\Omega)} \;\leq\; C\, \varepsilon\,.$$

Proof. Fix $n \in \mathbb{N}^*$ and assume that ε is sufficiently small so that the conclusion of Corollary 3.3 holds true with a positive constant C_1.

Multiplying the first equation of (3.21) by ψ, integrating by parts in Ω and using obvious estimates, we arrive at

$$\begin{aligned}\big\|a_\varepsilon^{1/2}\partial_1\psi\big\|_{\mathcal{H}}^2 + \varepsilon^{-2}\big(\|\nabla'\psi\|_{\mathcal{H}}^2 - E_1\|\psi\|_{\mathcal{H}}^2\big)& \\ = -(\psi, b_\varepsilon\psi)_{\mathcal{H}} + (\psi, f_\varepsilon)_{\mathcal{H}} \le \|b_\varepsilon\|_{C^0(\overline{\Omega})}\|\psi\|_{\mathcal{H}}^2 + \|\psi\|_{\mathcal{H}}^2 + \|f_\varepsilon\|_{\mathcal{H}}^2 \le C_2^2\,\varepsilon^2\,,&\end{aligned}$$

where $C_2 = C_2(n, L, C_{\mathrm{A}}, \omega)$ is a positive constant determined by C'_{A}, C_1 and the volume of Ω. Since $\big\|a_\varepsilon^{1/2}\partial_1\psi\big\|_{\mathcal{H}}^2 \ge C_{\mathrm{A}}^{-1}\|\partial_1\psi\|_{\mathcal{H}}^2$ by Assumption A.(ii) and $\|\nabla'\psi\|_{\mathcal{H}}^2 - E_1\|\psi\|_{\mathcal{H}}^2$ is non-negative by (3.17), we get that

$$\|\partial_1\psi\|_{\mathcal{H}} \le C_3\,\varepsilon\,, \qquad \|\nabla'\psi\|_{\mathcal{H}} \le C_3\,\varepsilon\,, \tag{3.23}$$

where $C_3 = C_3(n, L, C_{\mathrm{A}}, \omega)$ is a positive constant determined by C_2, C_{A}, C_1 and E_1. This proves an $\mathcal{H}^1$-convergence of ψ.

Now we rewrite the first equation of (3.21) as

$$-\partial_1^2\psi + \varepsilon^{-2}\,(-\Delta' - E_1)\,\psi = (a_\varepsilon - 1)\partial_1^2\psi + f'_\varepsilon\,, \tag{3.24}$$

where $f'_\varepsilon := f_\varepsilon - b_\varepsilon\psi + (\partial_1 a_\varepsilon)\partial_1\psi$. Taking the norm of both sides of (3.24) and using obvious estimates, we arrive at

$$\begin{aligned}\|\partial_1^2\psi\|_{\mathcal{H}}^2 - 2\varepsilon^{-2}\big(\partial_1^2\psi, (-\Delta' - E_1)\psi\big)_{\mathcal{H}} + \varepsilon^{-4}\big\|(-\Delta' - E_1)\psi\big\|_{\mathcal{H}}^2& \\ = \big\|(a_\varepsilon - 1)\partial_1^2\psi + f'_\varepsilon\big\|_{\mathcal{H}}^2 \quad\le\quad 2\|a_\varepsilon - 1\|_{C^0(\overline{\Omega})}^2\|\partial_1^2\psi\|_{\mathcal{H}}^2 + 2\|f'_\varepsilon\|_{\mathcal{H}}^2& \\ \le\quad C_4^2\,\varepsilon^2\big(\|\partial_1^2\psi\|_{\mathcal{H}}^2 + 1\big)\,,&\end{aligned}$$

where $C_4 = C_4(n, L, C_{\mathrm{A}}, \omega)$ is a positive constant determined by C_{A}, C'_{A}, C_1, C_3 and the volume of Ω. Since an integration by parts yields

$$\begin{aligned}-\big(\partial_1^2\psi, (-\Delta' - E_1)\psi\big)_{\mathcal{H}} &= \|\nabla'\partial_1\psi\|_{\mathcal{H}}^2 - E_1\|\partial_1\psi\|_{\mathcal{H}}^2\,,\\ \big\|(-\Delta' - E_1)\psi\big\|_{\mathcal{H}}^2 &= \|\Delta'\psi\|_{\mathcal{H}}^2 - 2E_1\|\nabla'\psi\|_{\mathcal{H}}^2 + E_1^2\|\psi\|_{\mathcal{H}}^2\,,\end{aligned}$$

where the expression on the right hand side of the first line is non-negative by (3.17), we conclude that if $C_4\,\varepsilon < 1$ then

$$\|\partial_1^2\psi\|_{\mathcal{H}} \le C_5\,\varepsilon\,, \qquad \|\nabla'\partial_1\psi\|_{\mathcal{H}} \le C_5\,\varepsilon\,, \qquad \|\Delta'\psi\|_{\mathcal{H}} \le C_5\,\varepsilon\,, \tag{3.25}$$

where $C_5 = C_5(n, L, C_{\mathrm{A}}, \omega)$ is a positive constant determined by C_4, C_3, C_1 and E_1. Summing up, Corollary 3.3, (3.23) and (3.25) establish the assertion of Theorem. □

3.3.6 C^0-convergence of eigenfunctions

To show that the convergence result of Theorem 3.5 holds actually in the topology of $C^0(\overline{\Omega})$ in all dimensions, we shall use the fact that ψ is a classical solution of (3.21) satisfying the maximum principle. In particular, our method is based on:

Proposition 3.5 (Generalized Maximum Principle)**.** *Let M be a linear elliptic second-order differential operator with bounded coefficients in a bounded open set $U \subset \mathbb{R}^d$; the principal part of M is assumed to be formed by a* negative *definite matrix. Suppose that there are two functions $u, w \in C^2(U) \cap C^0(\overline{U})$ satisfying:*

$$\begin{aligned} Mu &\le 0 \quad \text{in} \quad U\,, & Mw &\ge 0 \quad \text{in} \quad U\,,\\ u &\ge 0 \quad \text{in} \quad U\,, & w &> 0 \quad \text{in} \quad \overline{U}\,.\end{aligned}$$

Then

$$\sup_U(u/w) \le \sup_{\partial U}(u/w)\,.$$

To prove this, observe that the quotient u/w is a subsolution of an elliptic operator for which the usual maximum principle holds true (*cf* [51, Sec. 2.5]).

We shall also need two simple lemmata. Writing $B(t, r)$ for an open $(d-1)$-dimensional ball of radius $r > 0$ centred at $t \in \mathbb{R}^{d-1}$ and abbreviating $B_r := B(0, r)$, we denote by $\nu_1(r)$ the first eigenvalue of the Dirichlet Laplacian in B_r.

Lemma 3.3. *Given a positive constant C, let $r > 0$ be sufficiently small so that $\nu_1(r) > C$. Then for any $f \in C^0(\partial B_r)$, the boundary problem*

$$\begin{cases} -\Delta v - Cv = 0 & \text{in} \quad B_r\,, \\ v = f & \text{on} \quad \partial B_r\,, \end{cases}$$

has a unique solution $v \in C^0(\overline{B_r}) \cap C^2(B_r)$ and

$$v(0) = \alpha \, \frac{1}{|\partial B_r|} \int_{\partial B_r} f$$

with some positive constant $\alpha = \alpha(r, C, d)$ such that $r \mapsto \alpha(r, C, d)$ is increasing. Furthermore, $f > 0$ implies $\inf v > 0$.

Lemma 3.4. *Let $u \in L^2(\mathbb{R}^{d-1})$ be such that* $\operatorname{supp} u \subseteq \overline{\omega}$, *and let $\delta > 0$. Then for any $t \in \overline{\omega}$, there exists $r = r(t, u, \delta) \in (0, \delta]$ such that*

$$\frac{1}{|\partial B_r|} \int_{\partial B(t,r)} |u| \;\leq\; \frac{1}{|B_\delta|^{1/2}} \, \|u\|_{L^2(\omega)}\,.$$

Lemma 3.3 can be established by standard arguments, using the positive and rotationally symmetric eigenfunction of the Dirichlet Laplacian in a larger ball (*cf* also [35, Rem. 1]), while Lemma 3.4 follows easily by Fubini's theorem.

Now we are ready to prove

Theorem 3.6. *Suppose Assumption A holds. For any $N \in \mathbb{N}^*$, there exist positive constants $\varepsilon_0 = \varepsilon_0(N, L, C_A, \omega)$ and $C = C(N, L, C_A, \omega, d)$ such that for all $\varepsilon \leq \varepsilon_0$,*

$$\forall n \in \{1, \dots, N\}, \qquad \|\psi_n - \psi_n^0\|_{C^0(\overline{\Omega})} \;\leq\; C\,\varepsilon\,.$$

Proof. Fix $n \in \mathbb{N}^*$, assume that ε is sufficiently small so that the conclusion of Theorem 3.5 holds true with a positive constant C_1 and recall the definition (3.20). Defining the operator

$$M := -\partial_1 a_\varepsilon \partial_1 + \varepsilon^{-2}\,(-\Delta' - E_1) + b_\varepsilon\,,$$

the first equation of (3.21) together with (3.22) yields

$$M(\psi^\pm + \varepsilon) = \pm f_\varepsilon - (\varepsilon^{-1} E_1 - b_\varepsilon \varepsilon) \leq 0 \qquad \text{in} \quad \Omega^\pm := \{(s,t) \in \Omega \,|\, \psi^\pm > 0\}$$

provided $\varepsilon^2 \leq E_1/(2C'_A)$, where $\psi^\pm := \max\{\pm\psi, 0\}$. That is, $\psi^\pm + \varepsilon$ is a positive subsolution of M in $\Omega^\pm$. Our strategy will be to find a supersolution appropriate for the comparison argument included in Proposition 3.5. Defining

$$u(t) := \begin{cases} \sup_{s \in I} |\psi(s,t)| & \text{if} \quad t \in \overline{\omega}\,, \\ 0 & \text{if} \quad t \in \mathbb{R}^{d-1} \setminus \overline{\omega}\,, \end{cases}$$

we obviously have

$$\|u\|_{L^2(\omega)} \leq L^{1/2}\, \|\partial_1 \psi\|_{\mathcal{H}} \leq L^{1/2}\, C_1\, \varepsilon\,. \tag{3.26}$$

Let $\delta > 0$ be (uniquely) determined by the condition $\nu_1(\delta) = 3E_1$. Fix $t_0 \in \overline{\omega}$, and let $r = r(t_0, u, \delta)$ be the corresponding radius determined by Lemma 3.4. Finally, define a function w on $I \times B(t_0, r)$ by putting $w := 1 \otimes v$, where v is the solution to

$$\begin{cases} -\Delta v - 2E_1 v = 0 & \text{in} \quad B(t_0, r)\,, \\ v = u + \varepsilon & \text{on} \quad \partial B(t_0, r)\,. \end{cases}$$

By Lemma 3.3, v indeed exists and it is continuous and positive up to the boundary. Moreover, the minimum of v is achieved on the boundary because v is superharmonic. Since

$$Mw = (\varepsilon^{-2} E_1 + b_\varepsilon) w \geq 0 \qquad \text{in} \quad I \times B(t_0, r)$$

provided $\varepsilon^2 \leq E_1/C'_{\mathrm{A}}$, and since $\psi^\pm$ is either equal to zero or not greater than $w - \varepsilon$ on the boundary of $U^\pm := \Omega^\pm \cap [I \times B(t_0, r)]$, Proposition 3.5 yields

$$\forall (s,t) \in \overline{U^\pm}, \qquad \psi^\pm(s,t) + \varepsilon \leq v(t)\,.$$

In particular, Lemma 3.3 gives

$$\forall s \in \overline{I}, \qquad |\psi(s,t_0)| + \varepsilon \le v(t_0) = \alpha(r, 2E_1, d)\, \frac{1}{|\partial B_r|} \int_{\partial B(t_0,r)} (u+\varepsilon)\,,$$

where α is the coefficient of Lemma 3.3. Using Lemmata 3.3, 3.4 and (3.26), we conclude that

$$\forall (s,t_0) \in \overline{\Omega}, \qquad |\psi(s,t_0)| \le \alpha(\delta, 2E_1, d)\, \big(1 + |B_\delta|^{-1/2}\, L^{1/2}\, C_1\big)\, \varepsilon$$

because t_0 was chosen arbitrarily. □

3.3.7 C^0-convergence of transverse derivatives on the boundary

Now we use the maximum principle to derive a Lipschitz-type condition for ψ, which will play a crucial role in our proof of convergence of nodal sets.

Theorem 3.7. *Suppose Assumption A holds. For any $N \in \mathbb{N}^*$, there exist positive constants $\varepsilon_0 = \varepsilon_0(N, L, C_A, \omega)$ and $C = C(N, L, C_A, \omega, d)$ such that for all $\varepsilon \le \varepsilon_0$,*

$$\forall n \in \{1, \dots, N\}, \qquad \forall t_0 \in \partial\omega, \quad \sup_{(s,t)\in I\times\omega} \left| \frac{\psi_n(s,t)}{|t-t_0|} - \frac{\psi_n^0(s,t)}{|t-t_0|} \right| \;\le\; C\,\varepsilon\,.$$

Proof. Rewrite the first line of (3.21) as

$$M\psi := -\varepsilon^2 \partial_1(a_\varepsilon\, \partial_1\psi) - \Delta'\psi = \varepsilon^2 f_\varepsilon - \varepsilon^2 b_\varepsilon \psi + E_1\psi =: F_\varepsilon\,.$$

Theorem 3.6 and (3.22) imply that for all $\varepsilon \le \varepsilon_1$,

$$|M\psi| \le \|F_\varepsilon\|_{C^0(\overline{\Omega})} \le C_1\varepsilon \qquad \text{in} \quad \Omega$$

with some positive constants $\varepsilon_1 = \varepsilon_1(n, L, C_{\mathrm{A}}, \omega)$ and $C_1 = C_1(n, L, C_{\mathrm{A}}, \omega, d)$. We are now inspired by [26, Prob. 3.6] to construct a supersolution suitable for a comparison argument. Since the boundary $\partial\omega$ is of class C^∞, there exists a positive number $r = r(\omega)$ such that for every boundary point $t_0 \in \partial\omega$, there is an exterior point $\tau \in \mathbb{R}^{d-1} \setminus \overline{\omega}$ satisfying $\overline{B(\tau, r)} \cap \overline{\omega} = \{t_0\}$. Recall also the definition (3.3) of a. In the cylindrical layer

$$U := I \times \big[B(\tau, r+2a) \setminus \overline{B(\tau, r)}\big] \supset \Omega\,,$$

consider a (positive) function w defined by

$$w := 1 \otimes v\,, \qquad v(t) := \beta\, \big(e^{-\alpha r} - e^{-\alpha|t-\tau|}\big)\,,$$

where α and β are positive parameters yet to be determined. Direct calculation gives for $(s,t) \in U$,

$$(Mw)(s,t) = \beta\,\alpha\, e^{-\alpha|t-\tau|}\big[\alpha - (d-2)\,|t-\tau|^{-1}\big] \ge \beta\,\alpha\, e^{-\alpha(r+2a)}\big[\alpha - (d-2)\,r^{-1}\big]\,,$$

where the inequality hold provided $\alpha \ge (d-2)\,r^{-1}$. Choosing, *e.g.*,

$$\alpha := (d-1)\,r^{-1} \qquad \text{and} \qquad \beta := \alpha^{-1}\, r\, e^{\alpha\,(r+2a)}\, \|F_\varepsilon\|_{C^0(\overline{\Omega})}\,,$$

we therefore have

$$Mw \ge \|F_\varepsilon\|_{C^0(\overline{\Omega})} \qquad \text{in} \quad U\,.$$

Since $\psi = 0$ on $\partial\Omega$ and w is non-negative on ∂U, the maximum principle (*cf* [26, Thm. 3.3]) yields $|\psi| \le w$ in the closure of Ω. However, for $(s,t) \in U$,

$$w(s,t) \le \alpha\,\beta\, e^{-\alpha r}\, \big(|t-\tau| - |\tau - t_0|\big) \le \alpha\,\beta\, e^{-\alpha r}\, |t-t_0|\,,$$

and the claim of Theorem therefore follows with $C := C_1\, r e^{(d-1)r^{-1}2a}$. □

The part (ii) of Theorem 3.2 follows as a consequence of Theorem 3.7.

3.3.8 Convergence of nodal domains

Now we are in a position to establish the main result of this section, referring to Section 3.3.2 for the definition of the one-dimensional eigenfunction ϕ_n and the discussion of its nodal set.

Theorem 3.8. *Suppose Assumption A holds. For any $N \in \mathbb{N}^*$, there exist positive constants $\varepsilon_0 = \varepsilon_0(N, L, C_A, \omega)$ and $C = C(N, L, C_A, \omega, d)$ such that for all $\varepsilon \leq \varepsilon_0$ and any $n \in \{1, \dots, N\}$,*

(i) $\forall (s,t) \in \Omega, \quad (s,t) \in \mathcal{N}(\psi_n) \implies \operatorname{dist}\big(s, \mathcal{N}(\phi_n)\big) \leq C\,\varepsilon\,,$

(ii) $\forall (s,t) \in \Omega, \quad \operatorname{dist}\big(s, \mathcal{N}(\phi_n)\big) > C\,\varepsilon \implies \operatorname{sgn}\psi_n(s,t) = \operatorname{sgn}\phi_n(s)\,.$

Proof. Fix $n \in \mathbb{N}$, $n \geq 2$, and recall the definitions of I_n, I_n^k and $s_0^k(n)$ introduced in Section 3.3.2. Let ε be sufficiently small so that conclusions of Proposition 3.4.(ii) and Theorem 3.7 hold true with positive constants c_1 and C_1, respectively. Combining Theorem 3.7, Proposition 3.4.(ii) and (3.18), we have for all $(s,t) \in I_n \times \omega$,

$$\frac{\psi_n(s,t)}{\operatorname{dist}(t, \partial\omega)} \operatorname{sgn}\phi_n(s) \;\geq\; c_1 \operatorname{dist}(s, \partial I_n) - C_1\,\varepsilon \;>\; 0 \tag{3.27}$$

provided $\operatorname{dist}(s, \partial I_n) > C\,\varepsilon$, with $C := c_1^{-1} C_1$. This establishes (i) and (ii) with $\mathcal{N}(\phi_n)$ being replaced by $\partial I_n = \mathcal{N}(\phi_n) \cup \{0, L\}$.

It remains to show that $\mathcal{N}(\psi_n) \cap (I_n' \times \omega) = \varnothing$, where

$$I_n' := \big(0, s_0^1(n) - C\,\varepsilon\big) \cup \big(s_0^{n-1}(n) + C\,\varepsilon, L\big)\,.$$

Let c_2 be the positive constant determined by Proposition 3.1. If $\varepsilon \leq c_2/(4C)$ so that by Proposition 3.1.(iv) there does exist an open non-empty interval $J_n^k \subset I_n^k$ satisfying $\operatorname{dist}(J_n^k, \partial I_n) > C\,\varepsilon$ for all $k \in \{0, \dots, n-1\}$, (3.27) also implies that each $J_n^k \times \omega$ belongs to one distinct nodal domain of ψ_n (*i.e.* connected component of $\Omega \setminus \mathcal{N}(\psi_n)$). In particular, ψ_n has already n nodal domains, and Courant's nodal domain theorem [3] implies that ψ_n cannot change sign in each of the connected components of $I_n' \times \omega$. But then ψ_n cannot vanish in $I_n' \times \omega$, *e.g.*, because of the Harnack inequality (*cf* [26, Thm. 8.20]). □

This concludes the proof of Theorem 3.2. Corollary 3.2 is a consequence of the property (iii) of Theorem 3.2 and Courant's nodal domain theorem [3].

3.4 Curved tubes

In this section, we consider the Dirichlet Laplacian in curved tubes of shrinking cross-section. Using special curvilinear coordinates, we transform the Laplacian in a tube to a unitarily equivalent Schrödinger-type operator of the form (3.10) in a straight tube, and apply Theorem 3.2. The necessary geometric preliminaries follow the lines of Section 3.2.1, but we proceed in more details.

3.4.1 The reference curve

Let us precise what we meant by the *appropriate* Frenet frame of the reference curve Γ in the beginning of Section 3.2.1:

Assumption B. Γ possesses a positively oriented Frenet frame $\{e_1, \dots, e_d\}$ with the properties that

(i) $e_1 = \dot\Gamma\,,$

(ii) $\forall i \in \{1, \dots, d\}, \quad e_i \in C^\infty(\overline{I}; \mathbb{R}^d)\,,$

(iii) $\forall i \in \{1, \dots, d-1\},\ \forall s \in \overline{I}, \quad \dot e_i(s)$ lies in the span of $e_1(s), \dots, e_{i+1}(s)\,.$

Here and in the sequel, the *dot* denotes the derivative.

Remark 3.6. Recall [42, Sec. 1.2] that a Frenet frame is by definition a moving (orthonormal) frame such that for all $i \in \{1, \dots, d\}$ and $s \in \overline{I}$, $\Gamma^{(i)}(s)$ lies in the span of $e_1(s), \dots, e_i(s)$. A sufficient condition to ensure the existence of the Frenet frame of Assumption B is to require that for all $s \in \overline{I}$, the vectors $\dot\Gamma(s), \Gamma^{(2)}(s), \dots, \Gamma^{(d-1)}(s)$ are linearly independent (*cf* [39, Prop. 1.2.2]). This is always satisfied if $d = 2$. However, we do not assume *a priori* this non-degeneracy condition for $d \geq 3$ because it excludes, *e.g.*, the curves such that $\Gamma(J)$ is a straight segment for some subinterval $J \subseteq I$.

Let $\mathcal{K} \equiv (\mathcal{K}_{ij})_{i,j=1}^{d}$ be the matrix-valued function defined by the (Serret-Frenet) formulae

$$\dot{e}_i = \sum_{j=1}^{d} \mathcal{K}_{ij}\, e_j\,, \qquad i \in \{1,\dots,d\}\,. \tag{3.28}$$

By virtue of Assumption B, $\mathcal{K}$ has a skew symmetry

$$\mathcal{K}_{ij} = -\mathcal{K}_{ji}\,, \qquad i,j \in \{1,\dots,d\}$$

and $\mathcal{K}_{ij} = 0$ for $j > i+1$. We define by $\kappa_i := \mathcal{K}_{i,i+1}$ the i^{th} curvature of Γ. It will be also convenient to introduce the submatrix $\mathcal{K}' := (\mathcal{K}_{\mu\nu})_{\mu,\nu=2}^{d}$.

3.4.2 The Tang frame

We introduce now another moving frame along Γ, which better reflects the geometry of the curve. We shall refer to it as the *Tang frame* because it is a natural generalization of the Tang frame known from the theory of three-dimensional waveguides [57]. Our construction follows the generalization introduced in [9].

Let $\mathcal{R}' \equiv (\mathcal{R}_{\mu\nu})_{\mu,\nu=2}^{d}$ be the $(d-1)\times(d-1)$ matrix-valued function defined by the initial-value problem

$$\begin{cases} \dot{\mathcal{R}}' + \mathcal{R}'\,\mathcal{K}' = 0 & \text{in} \quad [0,L]\,, \\ \mathcal{R}' = \mathcal{R}'_0 & \text{at} \quad 0\,, \end{cases} \tag{3.29}$$

where $\mathcal{R}'_0$ is a rotation matrix in $\mathbb{R}^{d-1}$, *i.e.*,

$$\det(\mathcal{R}'_0) = 1\,, \qquad \mathcal{R}'_0\,\mathcal{R}'^{\mathrm{T}}_0 = 1\,. \tag{3.30}$$

Here "T" denotes the transpose operation and 1 stands both for a scalar number and an identity matrix. The solution of (3.29) exists, it is unique (for a given $\mathcal{R}'_0$), belongs to $C^\infty(\overline{I};\mathbb{R}^{(d-1)^2})$ and satisfies the conditions (3.30) in all $\overline{I}$ (*cf* [9, Sec. 2.2] for more details).

We extend $\mathcal{R}'$ to a $d\times d$ matrix-valued function $\mathcal{R} \equiv (\mathcal{R}_{ij})_{i,j=1}^{d}$ by setting

$$\mathcal{R} := \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{R}' \end{pmatrix}, \tag{3.31}$$

and introduce the Tang frame $\{\tilde{e}_1,\dots,\tilde{e}_d\}$ by

$$\tilde{e}_i := \sum_{j=1}^{d} \mathcal{R}_{ij}\, e_j\,, \qquad i \in \{1,\dots,d\}\,. \tag{3.32}$$

Combining (3.29) and (3.28) together with the properties of $\mathcal{K}$, one easily finds

$$\dot{\tilde{e}}_1 = \kappa_1\, e_2 \qquad \text{and} \qquad \dot{\tilde{e}}_\mu = -\kappa_1\,\mathcal{R}_{\mu 2}\, e_1\,, \qquad \mu \in \{2,\dots,d\}\,. \tag{3.33}$$

3.4.3 Tubes

As in Section 3.2.1, we introduce a mapping $\mathcal{L}$ from a straight tube Ω to $\mathbb{R}^d$ by (3.5). Recalling the relation (3.32) between the Frenet and Tang frames, it is clear that the image $\mathcal{L}(\Omega)$ is obtained by "translating" the cross-section $\varepsilon\omega$ along the curve Γ in a special way (it "rotates" with respect to the Tang frame).

Obviously, $\mathcal{L} \in C^\infty(\overline{\Omega};\mathbb{R}^d)$ for all $\varepsilon > 0$. Furthermore, $\mathcal{L}$ is an immersion provided ε is small enough. This can be seen as follows. Let $G \equiv (G_{ij})_{i,j=1}^{d}$ be the metric induced by the mapping $\mathcal{L}$, *i.e.*, $G_{ij} := (\partial_i\mathcal{L})\cdot(\partial_j\mathcal{L})$, where "$\cdot$" denotes the inner product in $\mathbb{R}^d$. Using the orthonormality of the Tang frame, relations (3.33) and Assumption B.(i), one easily establishes the formulae (3.6) and (3.7). Consequently,

$$|G| := \det(G) = \varepsilon^{2(d-1)}\, h^2\,.$$

Since $\mathcal{R}$ satisfies the orthogonality condition (3.30) and $|t| \le a$ (recall that ω is assumed to have its centre of mass at the origin of $\mathbb{R}^{d-1}$ and a is given by (3.3)), the restriction (3.4) yields

$$1 - C_\Gamma\, a\,\varepsilon \;\le\; h \;\le\; 1 + C_\Gamma\, a\,\varepsilon\,. \tag{3.34}$$

In particular, h does not vanish in Ω provided

$$\varepsilon < (C_\Gamma\, a)^{-1}\,, \tag{3.35}$$

and it follows by the inverse function theorem that $\mathcal{L}$ induces a local C^∞-diffeomorphism. This shows that $\mathcal{L}$ is an immersion for all positive ε satisfying (3.35).

In Definition 3.1, we have introduced the tube $\mathcal{T}$ to be the manifold Ω equipped with the Riemannian metric G. The symbol $d\mathrm{vol}$ will denote the volume measure on $\mathcal{T}$, *i.e.*,

$$d\mathrm{vol} := |G(s,t)|^{1/2}\, ds\, dt = \varepsilon^{d-1}\, h(s,t)\, ds\, dt\,,$$

where $dt \equiv dt_2 \dots dt_d$ stays for the $(d-1)$-dimensional Lebesgue measure on ω.

3.4.4 The Laplacian

It is a general fact that the Laplacian $-\Delta_G$ (as a differential expression) in a manifold Ω equipped with the metric G can be written as

$$-\Delta_G \;=\; -|G|^{-1/2} \sum_{i,j=1}^{d} \partial_i |G|^{1/2} G^{ij} \partial_j\,,$$

where G^{ij} denote the coefficients of the inverse matrix G^{-1}. We introduce the *Dirichlet Laplacian* $-\Delta_D^{\mathcal{T}}$ (as a differential operator) in the tube $\mathcal{T} \equiv (\Omega, G)$ to be the operator in the Hilbert space $L^2(\mathcal{T}) \equiv L^2(\Omega, d\mathrm{vol})$ defined by

$$-\Delta_D^{\mathcal{T}}\psi := -\Delta_G\psi\,, \qquad \psi \in \mathfrak{D}(-\Delta_D^{\mathcal{T}}) := \mathcal{H}_0^1(\mathcal{T}) \cap \mathcal{H}^2(\mathcal{T})\,. \tag{3.36}$$

It can be verified directly that $-\Delta_D^{\mathcal{T}}$ is self-adjoint. Alternatively, this can be deduced from the following unitary equivalence we shall need anyway.

Let $\mathcal{U}$ be the unitary operator defined by

$$\mathcal{U} : L^2(\mathcal{T}) \to L^2(\Omega) : \big\{\psi \mapsto |G|^{1/4}\psi\big\}\,. \tag{3.37}$$

Setting $H := \mathcal{U}\,(-\Delta_D^{\mathcal{T}})\,\mathcal{U}^{-1}$, one can check that

$$H = -\partial_1 h^{-2}\partial_1 - \varepsilon^{-2}\Delta' + V\,, \qquad \mathfrak{D}(H) = \mathcal{H}_2\,, \tag{3.38}$$

where the space $\mathcal{H}_2$ is defined in (3.12) and

$$V \;:=\; -\frac{5}{4}\,\frac{(\partial_1 h)^2}{h^4} + \frac{1}{2}\,\frac{\partial_1^2 h}{h^3} - \frac{1}{4}\,\frac{|\nabla' h|^2}{\varepsilon^2 h^2} + \frac{1}{2}\,\frac{\Delta' h}{\varepsilon^2 h}\,. \tag{3.39}$$

Actually, (3.38) with (3.39) is a general formula valid for any C^∞-smooth metric of the form (3.6). In our special case when h is given by (3.7), we find easily that $\partial_\mu h = -\varepsilon\,\kappa_1\,\mathcal{R}_{\mu 2}$ and $\partial_\mu\partial_\nu h = 0$ for any $\mu,\nu \in \{2,\dots,d\}$, and therefore

$$V \;=\; -\frac{1}{4}\,\frac{\kappa_1^2}{h^2} + \frac{1}{2}\,\frac{\partial_1^2 h}{h^3} - \frac{5}{4}\,\frac{(\partial_1 h)^2}{h^4}\,. \tag{3.40}$$

Moreover, (3.29) gives

$$\begin{aligned}
\partial_1 h(\cdot,t) \;&=\; \varepsilon \sum_{\mu,\nu,\rho=2}^{d} t_\mu\, \mathcal{R}_{\mu\nu}\big(\dot{\mathcal{K}}_{\nu 1} - \mathcal{K}_{\nu\rho}\mathcal{K}_{\rho 1}\big)\,,\\
\partial_1^2 h(\cdot,t) \;&=\; \varepsilon \sum_{\mu,\nu,\rho,\sigma=2}^{d} t_\mu\, \mathcal{R}_{\mu\nu}\big(\ddot{\mathcal{K}}_{\nu 1} - \dot{\mathcal{K}}_{\nu\rho}\mathcal{K}_{\rho 1} - 2\,\mathcal{K}_{\nu\rho}\dot{\mathcal{K}}_{\rho 1} + \mathcal{K}_{\nu\rho}\mathcal{K}_{\rho\sigma}\mathcal{K}_{\sigma 1}\big)\,.
\end{aligned}$$

Using (3.34), it is easy to see that H is uniformly elliptic with uniformly C^∞-smooth coefficients for all ε satisfying (3.35). Consequently, H (and therefore $-\Delta_D^{\mathcal{T}}$) is self-adjoint by the same reasoning as in Section 3.3.1.

3.4.5 Thin tubes

It remains to apply the results of Section 3.3, namely Theorem 3.2, to H. Let us define the operator

$$T := H - \varepsilon^{-2}E_1\,, \tag{3.41}$$

which is indeed of the form (3.10), with $a_\varepsilon := h^{-2}$ and $V_\varepsilon := V$. It is important to notice that, while the eigenvalues of T are just eigenvalues of H shifted by $-\varepsilon^{-2}E_1$, the operators have in fact the same eigenfunctions.

Let us now verify Assumption A for (3.41). Assuming (3.35), the functions h^{-2} and V obviously belong to $C^\infty(\overline{\Omega})$ (they are even analytic in t), since it is true for $\mathcal{L}$ (as a consequence of the geometric Assumption B); this

checks Assumption A.(i). In view of (3.34) and (3.35), Assumption A.(ii) holds true with $C_{\mathrm{A}} = 4$. Strengthen now (3.35) to $\varepsilon \le (2C_\Gamma a)^{-1}$, so that $h \ge 1/2$ uniformly in ε by virtue of (3.34). Since

$$\begin{aligned}
\|1-h^2\|_{C^0(\overline{\Omega})} &\le \varepsilon\, a\, \|\kappa_1\|_{C^0(\overline{I})} \big(2 + \varepsilon\, a\, \|\kappa_1\|_{C^0(\overline{I})}\big)\,,\\
\|\partial_1 h\|_{C^0(\overline{\Omega})} &\le \varepsilon\, a\, \big(\|\dot\kappa_1\|_{C^0(\overline{I})} + \|\mathcal{K}\|^2_{C^0(\overline{I};\mathbb{R}^{d^2})}\big)\,,\\
\|\partial_1^2 h\|_{C^0(\overline{\Omega})} &\le \varepsilon\, a\, \big(\|\ddot\kappa_1\|_{C^0(\overline{I})} + 3\|\dot{\mathcal{K}}\|_{C^0(\overline{I};\mathbb{R}^{d^2})}\|\mathcal{K}\|_{C^0(\overline{I};\mathbb{R}^{d^2})} + \|\mathcal{K}\|^3_{C^0(\overline{I};\mathbb{R}^{d^2})}\big)\,,\\
\|\partial_\mu h\|_{C^0(\overline{\Omega})} &\le \varepsilon\, a\, \|\kappa_1\|_{C^0(\overline{I})}\,, \qquad \mu \in \{2,\dots,d\}\,,
\end{aligned}$$

it is easy to see from (3.39) by applying (3.4) that Assumptions A.(iii) and (iv) hold true with some positive C_{A} depending on C_Γ, a and d, and with v_0 as defined in (3.9).

The application of Theorem 3.2 to the operator (3.41) is now straightforward. In order to deduce from it Theorem 3.1, it remains to recall that H and $-\Delta_D^{\mathcal{T}}$ have the same eigenvalues and their eigenfunctions are related by (3.37).

3.5 Strips on surfaces

As another application of Theorem 3.2, let us consider the situation where the ambient space of the tube is a general Riemannian manifold instead of the Euclidean space $\mathbb{R}^d$. We restrict ourselves to the case $d = 2$, *i.e.*, the tube $\mathcal{T}$ will be a strip about a curve in an (abstract) two-dimensional surface. We refer to [41, 43, 42] for geometric details and basic spectral properties of $-\Delta_D^{\mathcal{T}}$ in the infinite case $I = \mathbb{R}$.

Consider a C^∞-smooth connected complete non-compact two-dimensional Riemannian manifold $\mathcal{A}$ of bounded Gauss curvature K, and a C^∞-smooth curve $\Gamma : \overline{I} \to \mathcal{A}$ which is assumed to be parametrized by arc length and embedded. Let N be the unit normal vector field along Γ, which is uniquely determined as the C^∞-smooth mapping from $\overline{I}$ to the tangent bundle of $\mathcal{A}$ by requiring that $N(s)$ is orthogonal to the derivative $\dot\Gamma(s)$ and that $\{\dot\Gamma(s), N(s)\}$ is positively oriented for all $s \in \overline{I}$ (*cf* [55, Sec. 7.B]). We denote by κ the corresponding curvature of Γ defined by the Frenet formula and note that its sign is uniquely determined up to the re-parametrization of Γ (κ is the geodesic curvature of Γ if $\mathcal{A}$ is embedded in $\mathbb{R}^3$).

Without loss of generality, ω can be chosen as $(-1, 1)$. For sufficiently small positive ε, we define a mapping $\mathcal{L}$ from Ω to $\mathcal{A}$ by setting

$$\mathcal{L}(s,t) := \exp_{\Gamma(s)} \big(\varepsilon\, t\, N(s)\big)\,, \tag{3.42}$$

where $\exp_p$ is the exponential map of $\mathcal{A}$ at $p \in \mathcal{A}$. Note that $s \mapsto \mathcal{L}(s,t)$ traces the curves parallel to Γ at a fixed distance $\varepsilon|t|$, while the curve $t \mapsto \mathcal{L}(s,t)$ is a geodesic orthogonal to Γ for any fixed s. Since Γ is compact, $\mathcal{L}$ induces a diffeomorphism of Ω onto the image $\mathcal{L}(\Omega)$ provided ε is small enough (*cf* [27, Sec. 3.1]). Consequently, $\mathcal{L}$ induces a Riemannian metric G on Ω, and we define the *strip* $\mathcal{T}$ about Γ to be the manifold (Ω, G). It follows by the generalized Gauss lemma [27, Sec. 2.4] that the metric acquires the diagonal form:

$$G = \mathrm{diag}\,\big(h^2, \varepsilon^2\big)\,,$$

where h is a uniformly C^∞-smooth function on Ω satisfying the Jacobi equation

$$\partial_2^2 h + \varepsilon^2\, K\, h = 0 \qquad \text{with} \qquad \begin{cases} h(\cdot,0) = 1\,,\\ \partial_2 h(\cdot,0) = -\varepsilon\,\kappa\,, \end{cases} \tag{3.43}$$

where K is considered as a function of the (Fermi) "coordinates" (s,t) determined by (3.42).

Since the metric G is of the form (3.6), the Dirichlet Laplacian in the manifold $\mathcal{T}$ (defined in the same manner as (3.36)), is unitarily equivalent to the operator H defined by (3.38) with (3.39). It is easy see that Theorem 3.2 applies to the shifted operator $H - \varepsilon^{-2}E_1$. In particular, using (3.43), we verify that Assumption A holds true with some positive constant depending on the norms $\|\kappa\|_{C^2(\overline{I})}$ and $\|K\|_{C^2(\overline{I})}$, and with the function v_0 given this time by (*cf* [41, Sec. 3]):

$$v_0(s) := -\frac{\kappa(s)^2}{4} - \frac{K(s,0)}{2}\,.$$

The latter determines spectral properties of the one-dimensional operator S introduced in (3.8), namely, the set of eigenvalues $\{\mu_n\}_{n=1}^\infty$ and corresponding eigenfunctions $\{\phi_n\}_{n=1}^\infty$.

Theorem 3.9. *Let $\mathcal{T}$ be the strip of width 2ε defined above as a tubular neighbourhood about a curve Γ embedded in a two-dimensional Riemannian manifold $\mathcal{A}$. For any integer $N \ge 1$, there exist positive constants ε_0 and C depending on N, L and the geometries of Γ and $\mathcal{A}$ such that for all $\varepsilon \le \varepsilon_0$, the claims (i)–(iii) of Theorem 3.1 hold true.*

Acknowledgements. The authors would like to thank Diogo Gomes and Denis I. Borisov for useful remarks. Most of the results presented here were obtained during the stay of one of the authors (D.K.) at the Departamento de Matemática, Instituto Superior Técnico, Lisboa, Portugal; the author expresses his gratitude to the host. The work was partially supported by FCT/POCTI/FEDER, Portugal. The second author (D.K.) was also supported by the Czech Academy of Sciences and its Grant Agency within the projects IRP AV0Z10480505 and A100480501, and by the project LC06002 of the Ministry of Education, Youth and Sports of the Czech Republic.

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Chapter 4

Unbounded planar domains whose second nodal line does not touch the boundary

Published in: Mathematical Research Letters **14** (2007), 107–111

Joint work with: Pedro Freitas

Unbounded planar domains whose second nodal line does not touch the boundary

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Abstract. We show the existence of simply-connected unbounded planar domains for which the second nodal line of the Dirichlet Laplacian does not touch the boundary.

4.1 Introduction

Consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in} \quad \Omega\,, \\ u = 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{4.1}$$

where Ω is a domain (*i.e.* open connected set) in $\mathbb{R}^2$. We interprete (4.1) in a weak sense as the eigenvalue problem for the Dirichlet Laplacian $-\Delta_D^\Omega$ acting in the Hilbert space $L^2(\Omega)$, and recall that $-\Delta_D^\Omega$ is the non-negative self-adjoint operator associated with the quadratic form

$$Q_D^\Omega[v] := \|\nabla v\|^2_{L^2(\Omega)}\,, \qquad v \in \mathfrak{D}(Q_D^\Omega) := \mathcal{H}^1_0(\Omega)\,.$$

We denote by $\{\lambda_k(\Omega)\}_{k=1}^\infty$ the non-decreasing sequence of numbers corresponding to the spectral problem of $-\Delta_D^\Omega$ according to the Rayleigh-Ritz variational formula [10, Sec. 4.5]. Each $\lambda_k(\Omega)$ represents either a discrete eigenvalue or the threshold of the essential spectrum (if Ω is not bounded). All the eigenvalues below the essential spectrum of the boundary-value problem (4.1) may be characterized by this variational principle.

The nodal line of a real eigenfunction u of the problem (4.1) is defined by

$$\mathcal{N}(u) = \overline{\{x \in \Omega : u(x) = 0\}}\,,$$

and the connected components into which Ω is divided by $\mathcal{N}(u)$ are called the nodal domains of u. By the Courant nodal domain theorem an eigenfunction corresponding to the k^{th} eigenvalue below the essential spectrum has at most k nodal domains (see [1] for the proof in the bounded case, the generalization to the unbounded case being straightforward). In particular, since the first eigenvalue below the essential spectrum is always simple and the corresponding eigenfunction can be chosen to be positive, any eigenfunction corresponding to the second eigenvalue below the essential spectrum will have exactly two nodal domains.

Apart from this result, very little is known regarding the structure of the nodal lines, but much work has been developed over the last three decades around a conjecture of Payne's which states that a second eigenfunction of the above problem cannot have a closed nodal line. This conjecture is also quite often stated as follows:

Nodal Line Conjecture. The nodal line of any second eigenfunction of the Laplacian intersects the boundary $\partial\Omega$ at exactly two points.

The most general result obtained so far was given by Melas in 1992, who showed that the above conjecture holds in the case of bounded planar convex domains [13]. This followed a string of results obtained under some symmetry restrictions by several authors (Payne himself included) – see, for instance, [6] and the references therein.

On the other hand, several counterexamples have also been presented, of which the most significant is that in [9] showing that the result does not hold for multiply connected planar domains in general. Other counterexamples have been given illustrating other ways in which the conjecture may not hold. These include adding a potential ([12]) and the case of simply-connected surfaces ([6]).

The purpose of this note is to give examples showing that if one does not require the domain to be bounded, then the nodal line need not touch the boundary even under the same assumptions that have been previously used in the bounded case to prove the conjecture. More precisely we will prove the following

Theorem. There exists a simply-connected unbounded planar domain Ω which is convex and symmetric with respect to two orthogonal directions, and for which the nodal line of a second eigenfunction does not touch the boundary $\partial\Omega$.

This domain can be chosen as one of the following two types:

(i) the distance between the nodal line of a second eigenfunction and the boundary $\partial\Omega$ is bounded away from zero, but the spectrum is not purely discrete;

(ii) the spectrum consists only of discrete eigenvalues, but the infimum of the distance between a point on the nodal line of a second eigenfunction and the boundary $\partial\Omega$ is zero.

The idea behind both examples is to start from a bounded convex domain Ω_0 which is invariant under reflections through two orthogonal lines r and $r^\perp$, and which we will assume to be sufficiently long in the direction $r^\perp$, such that its second eigenvalue is simple and any corresponding eigenfunction is antisymmetric with respect to r. In fact, its second nodal line will be given by the closure of $\Omega_0 \cap r$. We then append two sufficiently thin semi-infinite strips to Ω_0 in neighbourhoods of the points where its second nodal line touches the boundary, in such a way that the nodal line coincides with the axis r and thus stays within these strips without touching the boundary – see Figure 2.2.

In order to establish case (i), we will consider domains which are asymptotically cylindrical – see the classification of Euclidean domains in [30, § 49], where these sets are called quasi-cylindrical. This means that there will also exist essential spectrum, and so it will be necessary to prove that the domain does indeed possess a second discrete eigenvalue in this case. In order for condition (ii) to be satisfied, we will need to consider what are referred to in [7] as quasi-bounded domains. This means that the domains are asymptotically narrow and thus, although the nodal line does not touch the boundary, it does get asymptotically close to it.

It should be stressed here that while the nodal line in both our examples does not touch the boundary, it is not closed.

4.2 The Proof

Let Ω_0 be a bounded open convex subset of $\mathbb{R}^2$ which is simultaneously invariant under the reflection through the coordinate axes $r := \{0\} \times \mathbb{R}$ and $r^\perp := \mathbb{R} \times \{0\}$, *i.e.*,

$$\forall (x_1, x_2) \in \mathbb{R}^2, \qquad (x_1, x_2) \in \Omega_0 \quad \Longrightarrow \quad \begin{cases} (x_1, -x_2) \in \Omega_0 \,, \\ (-x_1, x_2) \in \Omega_0 \,. \end{cases}$$

Of course, the first eigenvalue $\lambda_1(\Omega_0) > 0$ is simple and the corresponding eigenfunction can be chosen to be positive. We assume that also the second eigenvalue $\lambda_2(\Omega_0)$ is simple and that the nodal line of the corresponding eigenfunction is the closure of $\Omega_0 \cap r$ (by [10, 33] and the symmetry, these always happen if Ω_0 is sufficiently long in the direction $r^\perp$).

Let $h : [0, +\infty) \to (0, 1]$ be a convex function. For any $\varepsilon > 0$, we define an open tubular neighbourhood of the axis r by

$$\mathcal{T}_\varepsilon := \big\{ (x_1, x_2) \in \mathbb{R}^2 : \, |x_1| < \varepsilon \, h(|x_2|) \big\} \,,$$

and introduce the unbounded open connected set

$$\Omega_\varepsilon := \Omega_0 \cup \mathcal{T}_\varepsilon \,. \tag{4.2}$$

Note that Ω_ε is invariant both under the reflections through r and $r^\perp$, and convex both along r and $r^\perp$. It is also worth to notice that the boundary $\partial\Omega_\varepsilon$ is necessarily at least of class $C^{0,1}$, *cf* [16, Sec. V.4.1].

Using the minimax principle and a Dirichlet-Neumann bracketing argument, it is easy to see that

$$\inf \sigma_{\rm ess}(-\Delta_D^{\Omega_\varepsilon}) \geq \pi^2/(2\varepsilon)^2 \tag{4.3}$$

and that one can produce an arbitrary number of eigenvalues below the essential spectrum by making ε small enough. Furthermore, if h tends to zero at infinity then $\sigma_{\rm ess}(-\Delta_D^{\Omega_\varepsilon}) = \varnothing$ and the spectrum of $-\Delta_D^{\Omega_\varepsilon}$ consists of discrete eigenvalues only. In any case, one has the following convergence result.

Lemma 4.1. $\forall k \in \mathbb{N} \setminus \{0\}, \quad \lim\limits_{\varepsilon \to 0} \lambda_k(\Omega_\varepsilon) = \lambda_k(\Omega_0)$.

Proof. It follows from [3] that $-\Delta_D^{\Omega_\varepsilon}$ converges to $-\Delta_D^{\Omega_0}$ in the generalized sense of Kato [11], which implies, in particular, continuity of eigenvalues below the essential spectrum. □

In view of (4.3) and Lemma 4.1, $\lambda_1(\Omega_\varepsilon)$ and $\lambda_2(\Omega_\varepsilon)$ are discrete simple eigenvalues for all sufficiently small ε. Since the eigenfunction corresponding to $\lambda_1(\Omega_\varepsilon)$ can be chosen to be positive, the eigenfunction $u_{2,\varepsilon}$ corresponding to $\lambda_2(\Omega_\varepsilon)$ has to change sign in Ω_ε. We now prove a result which will give us immediately the conclusions of the Theorem.

Proposition 4.1. $\exists \varepsilon_0 > 0$, $\forall \varepsilon \in (0, \varepsilon_0)$, $\quad \mathcal{N}(u_{2,\varepsilon}) = r$.

Proof. Let ε be so small that $\lambda_2(\Omega_\varepsilon)$ is a simple discrete eigenvalue. Due to the symmetry of Ω_ε, the corresponding eigenfunction $u_{2,\varepsilon}$ must be symmetric or antisymmetric with respect to r, and symmetric or antisymmetric with respect to $r^\perp$. This observation and the Courant nodal domain theorem yield that the nodal set $\mathcal{N}(u_{2,\varepsilon})$ is either the closure of $r^\perp \cap \Omega_0$, the axis r or a closed loop. The last possibility is excluded by mimicking the argument given in [2] (see also [14]) for bounded domains with the required symmetry and convexity. We will exclude the first possibility by using the fact that the second eigenvalue of a domain is the first eigenvalue of any of the nodal subdomains. Let us assume that there is a positive sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$, converging to zero as $j \to \infty$, such that $\mathcal{N}(u_{2,\varepsilon_j}) = \overline{r^\perp \cap \Omega_0}$ for all $j \in \mathbb{N}$. Then

$$\lambda_2(\Omega_{\varepsilon_j}) = \lambda_1\big(\Omega_{\varepsilon_j} \cap [\mathbb{R} \times (0, +\infty)]\big) \longrightarrow \lambda_1\big(\Omega_0 \cap [\mathbb{R} \times (0, +\infty)]\big) \qquad \text{as} \quad j \to \infty$$

by a convergence argument analogous to Lemma 4.1. On the other hand, we know that

$$\lambda_2(\Omega_{\varepsilon_j}) \longrightarrow \lambda_2(\Omega_0) = \lambda_1\big(\Omega_0 \cap [(0, +\infty) \times \mathbb{R}]\big) \qquad \text{as} \quad j \to \infty$$

by Lemma 4.1 and the assumption we have made about Ω_0. This implies that $\lambda_2(\Omega_0)$ is degenerate (there is one eigenfunction antisymmetric with respect to r and one eigenfunction antisymmetric with respect to $r^\perp$), a contradiction. □

It follows that the nodal line $\mathcal{N}(u_{2,\varepsilon})$ does not touch the boundary of Ω_ε for all sufficiently small $\varepsilon > 0$. Furthermore, if we choose $h \equiv 1$ then the distance between the nodal line and the boundary is equal to ε, which establishes part (i) of the Theorem. Part (ii) follows by taking a function h which tends to zero at infinity.

Acknowledgements. Partially supported by FCT/POCTI/FEDER, Portugal, by the Czech Academy of Sciences and its Grant Agency within the projects IRP AV0Z10480505 and A100480501, and by the project LC06002 of the Ministry of Education, Youth and Sports of the Czech Republic.

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Chapter 5

The growth of the isoperimetric constant of convex domains

PROCEEDINGS OF THE AMERICAN MATHEMATICAL SOCIETY

Published in: Proceedings of the American Mathematical Society **136** (2008), 2997–3006

Joint work with: Pedro Freitas

A sharp upper bound for the first Dirichlet eigenvalue and the growth of the isoperimetric constant of convex domains

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Abstract. We show that as the ratio between the first Dirichlet eigenvalues of a convex domain and of the ball with the same volume becomes large, the same must happen to the corresponding ratio of isoperimetric constants. The proof is based on the generalization to arbitrary dimensions of Pólya and Szegö's 1951 upper bound for the first eigenvalue of the Dirichlet Laplacian on planar star-shaped domains which depends on the support function of the domain.

As a by-product, we also obtain a sharp upper bound for the spectral gap of convex domains.

5.1 Introduction

Let Ω be a bounded domain (*i.e.* open connected set) in the d-dimensional Euclidean space $\mathbb{R}^d$, with $d \geq 1$, and denote by $\lambda_1(\Omega)$ the first eigenvalue of the Dirichlet Laplacian in $L^2(\Omega)$. The Faber-Krahn inequality provides a lower bound for $\lambda_1(\Omega)$, namely,

$$\lambda_1(\Omega) \;\geq\; \lambda_1(B_1)\left(\frac{|B_1|}{|\Omega|}\right)^{2/d}, \tag{5.1}$$

where B_1 is the d-dimensional ball of unit radius and the absolute-value signs denote d-dimensional Lebesgue measure (later on we use the same notation for $(d-1)$-dimensional Hausdorff measure of the boundary of Ω as well). Modulus a set of zero capacity, equality in (5.1) is attained if and only if Ω is a ball and it is thus of interest to understand how strong this connection is. More precisely, and assuming for the time being that we have a way of measuring how far a set is from the ball of the same volume in terms of elementary geometric quantities, we would like to answer questions such as the following: if a set is *far away* from the ball, must its first Dirichlet eigenvalue be much larger than that of the ball of the same volume?

This particular question was given a positive answer in [G], where the measure of deviation of a convex domain from the ball which was used was based on the support function of the domain. More recently, in [M, BC, FMP] the authors addressed the question of whether a domain for which $\lambda_1(\Omega)|\Omega|^{2/d}$ is close to the corresponding quantity for the ball must be close to the ball in the sense of Fraenkel asymmetry (*i.e.* Hausdorff distance if Ω is convex), again providing a positive answer.

In this paper, we consider the issue of whether having a large first Dirichlet eigenvalue implies being away from the corresponding ball. Using the trivial upper bound $\lambda_1(\Omega) \leq \lambda_1(B_{\rho_\Omega})$, where ρ_Ω is the inradius of Ω, it is possible to give an immediate answer to this question in terms of ρ_Ω. However, this bound is not very good in general, say for long parallelepipeds, and so our purpose was to obtain a different characterization which would behave better precisely when away from the ball.

The main result of this paper in this direction is the following estimate using the isoperimetric constant as a measure of deviation of Ω from B:

Theorem 5.1. *Let Ω be a bounded convex domain of $\mathbb{R}^d$. Then*

$$\frac{|\partial\Omega|}{|\Omega|^{1-1/d}} \;\geq\; \frac{|\partial B|}{|B|^{1-1/d}}\sqrt{\frac{\lambda_1(\Omega)}{\lambda_1(B)}}\,\frac{\pi}{2\sqrt{\lambda_1(B_1)}},$$

where B is the ball of volume $|\Omega|$.

The proof is based on the following result:

Theorem 5.2. *Let Ω be a bounded convex domain of $\mathbb{R}^d$. Then*

$$\lambda_1(\Omega) \;\leq\; \lambda_1(B_1)\,\frac{|\partial\Omega|}{d\,\rho_\Omega\,|\Omega|}.$$

This upper bound is a consequence of a stronger upper bound for $\lambda_1(\Omega)$ holding in the more general case of star-shaped domains and which we believe to be of interest in its own right (*cf* Theorem 5.3 below). This is an extension to arbitrary dimensions of an upper bound for $\lambda_1(\Omega)$ appearing in Pólya and Szegö's 1951 book [PS] in the planar case. As in the case of [G], this bound also depends on the support function of the domain in a non-elementary way. Due to this, we postpone the statement of this result to the next section where we provide the necessary background, its proof being then given in Section 5.3.

The proof of Theorem 5.1 together with a brief discussion of optimality, and other applications of our bounds are given in Section 5.4 where, in particular, we obtain a sharp upper bound for the spectral gap. Finally, in the last section we recall some two-dimensional upper bounds and conjectures which will be used for comparison and consider some examples.

5.2 An upper bound for the first eigenvalue of star-shaped domains

To state Pólya and Szegö's result and its generalization to arbitrary dimension we need to introduce a geometric quantity which measures how far away we are from the ball, and which may be expressed in terms of the support function of the given domain. To be more precise, let Ω be a *star-shaped* domain with respect to a point $\xi \in \Omega$, *i.e.*, for each point $x \in \partial\Omega$ the segment joining ξ with x lies in $\Omega \cup \{x\}$ and is transversal to $\partial\Omega$ at the point x. Assume now that the boundary $\partial\Omega$ is locally Lipschitz. Then the outward unit normal vector field $N : \partial\Omega \to \mathbb{R}^d$ can be uniquely defined almost everywhere on $\partial\Omega$. At those points $x \in \partial\Omega$ for which $N(x)$ is uniquely defined, we introduce the *support function*

$$h_\xi(x) := (x - \xi) \cdot N(x)\,,$$

where the dot denotes the standard scalar product in $\mathbb{R}^d$. We say that Ω is *strictly star-shaped* with respect to the point $\xi \in \Omega$ if Ω is star-shaped with respect to ξ and the support function is uniformly positive, *i.e.*,

$$\operatorname*{ess\,inf}_{x\in\partial\Omega} h_\xi(x) > 0\,. \tag{5.2}$$

In this case, we shall denote by ω the set of points with respect to which Ω is strictly star-shaped, and define the following intrinsic quantity of the domain

$$F(\Omega) := \inf_{\xi\in\omega} \int_{\partial\Omega} h_\xi^{-1}.$$

Our main result reads as follows:

Theorem 5.3. *Let Ω be a bounded strictly star-shaped domain in $\mathbb{R}^d$ with locally Lipschitz boundary $\partial\Omega$. Then*

$$\lambda_1(\Omega) \;\le\; \lambda_1(B_1)\,\frac{F(\Omega)}{d\,|\Omega|}\,.$$

Remark 5.1. If $d = 2$ Theorem 5.3 coincides with the Pólya-Szegö bound [PS, Sec. 5.6].

Remark 5.2. Combining the upper bound of Theorem 5.3 with the Faber-Krahn inequality given in (5.1), we see that $F(\Omega)$ is bounded from below by

$$F(\Omega) \ge \frac{d\,|B_1|^{2/d}}{|\Omega|^{2/d-1}}\,,$$

with equality only when Ω is a ball. In the two-dimensional case this was shown in [A] by a different method.

If we now restrict ourselves to convex domains it is possible to simplify the discussion somewhat. To begin with, we have that the boundary of a convex open subset of $\mathbb{R}^d$ is locally Lipschitz (*cf* [EE, Sec. V.4.1]). Furthermore, $\omega = \Omega$. Indeed, for any $\xi \in \Omega$ one has

$$\operatorname*{ess\,inf}_{x\in\partial\Omega} h_\xi(x) \ge \operatorname{dist}(\xi, \partial\Omega) \qquad \text{if } \Omega \text{ is convex,} \tag{5.3}$$

which follows from the geometrical meaning of $h_\xi(x)$ being the distance from ξ to the tangent space $T_x(\partial\Omega)$. Finally, (5.3) can be used to obtain a simple upper bound to $F(\Omega)$

$$F(\Omega) \le \frac{|\partial\Omega|}{\rho_\Omega} \qquad \text{if } \Omega \text{ is convex,} \tag{5.4}$$

where ρ_Ω is the inradius of Ω. This approximation readily establishes Theorem 5.2, a weaker version of Theorem 5.3 but, on the other hand, it allows us to write the upper bound explicitly in terms of more elementary geometric quantities.

Remark 5.3. Since [BZ, Thm. 35.1.2]

$$\rho_\Omega \, |\partial\Omega| \le d \, |\Omega| \,, \tag{5.5}$$

we see that Theorem 5.2 is, in general, an improvement to the trivial upper bound given by $\lambda_1(\Omega) \le \lambda_1(B_{\rho_\Omega})$. We note, however, that there are certain classes of domains such as triangles and regular polygons for which equality holds in (5.5) (for a recent discussion of these classes in the two- and three- dimensional cases we refer to [AM1, AM2], respectively). Moreover, for triangles it can be checked directly that we also have equality in (5.4), so that Theorem 5.3 is equivalent to Theorem 5.2 and our method gives just the trivial result in this case (see [A] for explicit formulae).

Remark 5.4. When the domain is star-shaped but not necessarily convex, the bound given by Theorem 5.3 may be worse than the trivial bound, as exemplified by the Swiss Cross example considered in [HS] – see Example 5.4 below.

Although our proof of Theorem 5.2 only holds for the convex case, we conjecture that this bound holds for general bounded domains in $\mathbb{R}^d$.

5.3 Proof of Theorem 5.3

Henceforth let us assume that $d \ge 2$ (for one-dimensional intervals we clearly have equalities in the upper bounds of Theorems 5.2 and 5.3).

We need to introduce some notation concerning the geometry of the hypersurface $\partial\Omega$. By assumption, each point x on the boundary $\partial\Omega$ has a neighbourhood (an open subset of $\mathbb{R}^d$) whose intersection with the boundary, denoted by $V \subset \partial\Omega$, is $C^{0,1}$-diffeomorphic to an open subset $U \subset \mathbb{R}^{d-1}$ by means of a chart $\Gamma : U \to V$. Having in mind that, by Rademacher's theorem, Γ is differentiable almost everywhere in U, let g_{ij} denote the coefficients of the metric tensor of $\partial\Omega$ induced by these local diffeomorphisms, *i.e.*,

$$g_{ij} := (\partial_i \Gamma) \cdot (\partial_j \Gamma) \,, \qquad i,j \in \{1,\dots,d-1\} \,.$$

Recall that the cross-product $[\partial_1\Gamma,\dots,\partial_{d-1}\Gamma]$ is perpendicular to $\partial\Omega$ and that its magnitude is equal to the square root of $|g| := \det(g_{ij})$.

Let Ω be strictly star-shaped with respect to $\xi \in \Omega$. We parameterize $\Omega \setminus \{\xi\}$ by means of the mapping

$$\mathcal{L} : \partial\Omega \times (0,1) \to \mathbb{R}^d : \big\{(x,t) \mapsto \xi + (x-\xi)t\big\} \,. \tag{5.6}$$

Notice that the "shrunk boundary" $\mathcal{L}(\partial\Omega \times \{t\})$ is indeed contained in Ω for any $t \in (0,1)$. Using the local parametrization of the boundary by Γ and properties of the cross-product, the Jacobian of the transformation $\mathcal{L}$ can be locally identified with

$$J(\cdot,t) = \begin{vmatrix} (\partial_1\Gamma^1)\,t & \dots & (\partial_{d-1}\Gamma^1)\,t & \Gamma^1 \\ \vdots & & \vdots & \vdots \\ (\partial_1\Gamma^d)\,t & \dots & (\partial_{d-1}\Gamma^d)\,t & \Gamma^d \end{vmatrix} = [\partial_1\Gamma,\dots,\partial_{d-1}\Gamma] \cdot \Gamma \; t^{d-1} \,.$$

Hence

$$|J(u,t)| = |g|^{1/2}(u) \; h_\xi\big(\Gamma(u)\big) \; t^{d-1} \tag{5.7}$$

for every $t \in (0,1)$ and almost every $u \in U$. By virtue of the inverse function theorem and assumption (5.2), we therefore conclude that $\mathcal{L} : \partial\Omega \times (0,1) \to \Omega \setminus \{\xi\}$ is indeed a diffeomorphism.

In other words, the Euclidean domain Ω with the point ξ removed can be identified with the Riemannian manifold

$$M := \big(\partial\Omega \times (0,1), G\big) \,,$$

where the metric G is induced by (5.6). Since the coefficients of G are locally given by

$$G_{ij} := (\partial_i \mathfrak{L}) \cdot (\partial_j \mathfrak{L}) \,, \qquad i,j \in \{1,\dots,d\} \,,$$

where $\mathfrak{L} := \mathcal{L} \circ (\Gamma \otimes 1)$ with 1 being the identity function on the interval $(0,1)$, we plainly have

$$\big(G_{ij}(\cdot,t)\big) = \begin{pmatrix} g_{11}\,t^2 & \dots & g_{1d-1}\,t^2 & \Gamma\cdot(\partial_1\Gamma)\,t \\ \vdots & & \vdots & \vdots \\ g_{d-11}\,t^2 & \dots & g_{d-1d-1}\,t^2 & \Gamma\cdot(\partial_{d-1}\Gamma)\,t \\ \Gamma\cdot(\partial_1\Gamma)\,t & \dots & \Gamma\cdot(\partial_{d-1}\Gamma)\,t & |\Gamma|^2 \end{pmatrix} .$$

Using the relation $|G| := \det(G_{ij}) = J^2$ and formula (5.7), we see that the volume element $d\mathrm{vol}$ of the manifold M is decoupled as follows:

$$d\mathrm{vol} = h_\xi(x)\, d\sigma(x)\; t^{d-1}\, dt\,, \tag{5.8}$$

where $d\sigma$ and dt denote the respective measures on $\partial\Omega$ and $(0,1)$. At the same time, denoting by G^{ij} the coefficients of the matrix inverse to (G_{ij}), we locally find that

$$G^{dd}(u,t) = h_\xi^{-2}\big(\Gamma(u)\big)\,.$$

If ψ is any differentiable function on $(0,1)$ and 1 denotes the identity function on $\partial\Omega$, then the last result implies

$$\big|\nabla_G(1\otimes\psi)\big|_G = h_\xi^{-1}\,|\psi'|\,, \tag{5.9}$$

where ∇_G and $|\cdot|_G$ stand for the gradient and norm on M, respectively.

Using the above geometric preliminaries, the Hilbert space $L^2(\Omega)$ can be identified with $L^2(M) \equiv L^2\big(\partial\Omega\times(0,1), d\mathrm{vol}\big)$ and the Dirichlet Laplacian in the former is unitarily equivalent to the self-adjoint operator associated in the latter with the quadratic form

$$Q[\Psi] := \big\|\,|\nabla_G\Psi|_G\big\|^2_{L^2(M)}\,, \qquad \Psi\in D(Q) := H^1_0(M)\,.$$

Recall that the Sobolev space $H^1_0(M)$ is the completion of $C_0^\infty\big(\partial\Omega\times(0,1)\big)$ with respect to the norm $(Q[\cdot] + \|\cdot\|^2_{L^2(M)})^{1/2}$. Let ψ be a non-zero function from $H^1_0\big((0,1), t^{d-1}\,dt\big)$ and let 1 denote the identity function on $\partial\Omega$. Then the function $1\otimes\psi$ belongs to the form domain $D(Q)$ and we can use it as a test function in the variational formulation for the first eigenvalue of the operator associated to Q. Employing (5.8) and (5.9), we get

$$\lambda_1(\Omega) \;\le\; \frac{Q[1\otimes\psi]}{\|1\otimes\psi\|^2_{L^2(\Omega)}} \;=\; \frac{\int_{\partial\Omega} h_\xi(x)^{-1} d\sigma(x)}{\int_{\partial\Omega} h_\xi(x)\, d\sigma(x)}\;\frac{\int_0^1 |\psi'(t)|\, t^{d-1} dt}{\int_0^1 |\psi(t)|^2\, t^{d-1} dt} =: \lambda_1(\Omega;\psi,\xi)\,.$$

Now, if Ω is the ball of radius 1 centered at ξ, the above result reduces to

$$\lambda_1(B_1) \le \frac{\int_0^1 |\psi'(t)|\, t^{d-1} dt}{\int_0^1 |\psi(t)|^2\, t^{d-1} dt}\,.$$

But we know that the equality sign holds if, and only if, ψ is chosen as the radial component of the first eigenfunction of the Dirichlet Laplacian in the ball. Indeed, this eigenfunction is radially symmetric and can be written as $1\otimes\psi$ in our coordinates. Consequently,

$$\min_\psi \lambda_1(\Omega;\psi,\xi) = \lambda_1(B_1)\;\frac{\int_{\partial\Omega} h_\xi(x)^{-1} d\sigma(x)}{\int_{\partial\Omega} h_\xi(x)\, d\sigma(x)}\,.$$

Minimizing the integral in the numerator with respect to ξ, we arrive at the quantity $F(\Omega)$ of Theorem 5.3. It remains to realize that the integral of support function is actually independent of ξ because, by (5.8) and Fubini's theorem,

$$|\Omega| = \int_{\partial\Omega\times(0,1)} d\mathrm{vol} = \frac{1}{d}\int_{\partial\Omega} h_\xi(x)\, d\sigma(x)\,.$$

This concludes the proof of Theorem 5.3.

5.4 Applications

5.4.1 Torsional rigidity

The method of the present paper applies to other Sobolev-inequality-type problems, too. For instance, let $P(\Omega)$ be the *torsional rigidity* of Ω defined by

$$\frac{1}{P(\Omega)} := \inf_{\psi\in H^1_0(\Omega)\setminus\{0\}} \frac{\|\nabla\psi\|^2_{L^2(\Omega)}}{\|\psi\|^2_{L^1(\Omega)}}\,.$$

Then, following the lines of Section 5.3, we have

Proposition 5.1. *Let Ω be a bounded strictly star-shaped domain in $\mathbb{R}^d$ with locally Lipschitz boundary $\partial\Omega$. Then*

$$\frac{1}{P(\Omega)} \;\le\; \frac{|B_1|}{P(B_1)}\,\frac{F(\Omega)}{d\,|\Omega|^2}\,.$$

Again, the equality is attained for Ω being a ball. For $d=2$ this result coincides with the Pólya-Szegö bound [PS, Sec. 5.5].

5.4.2 The growth of the isoperimetric constant

We shall now prove Theorem 5.1. The idea is to estimate the inradius appearing in the bound of Theorem 5.2 by means of the following lower bound due to Protter [Pr]

$$\lambda_1(\Omega) \geq \frac{\pi^2}{4\rho_\Omega^2}\,.$$

(Protter's bound actually includes an extra term depending on the diameter, but for our purposes it is sufficient to consider the expression above.) This leads to

$$\frac{|\partial\Omega|}{d\,|\Omega|} \geq \sqrt{\frac{\lambda_1(\Omega)}{\lambda_1(B_1)}}\,\frac{\pi}{2\sqrt{\lambda_1(B_1)}}\,,$$

which is equivalent to the inequality of Theorem 5.1 due to the scaling properties $\lambda_1(B) = \lambda_1(B_1)r^{-2}$ and $|B| = |B_1|r^d$ where r is the radius of B, and $|\partial B_1| = d|B_1|$.

Since Protter's bound used in the proof is not sharp for the ball, the inequality in the above theorem is not an improvement upon the classical isoperimetric inequality for $\lambda_1(\Omega)/\lambda_1(B)$ close to one. However, it is clear that this will be the case when this ratio becomes large. At the same time, our bound is optimal in the sense that there exist d-dimensional domains for which the growth of $\sqrt{\lambda_1(\Omega)/\lambda_1(B)}$ cannot be improved. More precisely, it is not difficult to obtain that if $\mathcal{R}$ is a d-dimensional parallelepiped we have

$$\frac{|\partial\mathcal{R}|}{|\mathcal{R}|^{1-1/d}} \leq \frac{|\partial B|}{|B|^{1-1/d}}\sqrt{\frac{\lambda_1(\mathcal{R})}{\lambda_1(B)}}\,\frac{2\sqrt{\lambda_1(B_1)}}{\pi\sqrt{d}}\,.$$

The question of the optimal constant multiplying $\sqrt{\lambda_1(\Omega)/\lambda_1(B)}$ in Theorem 5.1 remains open.

5.4.3 The second eigenvalue and the gap

Combining the bound in Theorem 5.2 with the Ashbaugh-Benguria bound [AB] for the spectral quotient $\lambda_2(\Omega)/\lambda_1(\Omega)$ gives a similar upper bound for the second eigenvalue of a convex domain. This, together with the Faber-Krahn inequality yields, in turn, an upper bound for the spectral gap.

Proposition 5.2. *Let Ω be a bounded convex domain of $\mathbb{R}^d$. Then the second Dirichlet eigenvalue $\lambda_2(\Omega)$ satisfies*

$$\lambda_2(\Omega) \;\leq\; \lambda_2(B_1)\,\frac{|\partial\Omega|}{d\,\rho_\Omega\,|\Omega|}\,.$$

As a consequence, the spectral gap satisfies

$$\lambda_2(\Omega) - \lambda_1(\Omega) \;\leq\; \lambda_2(B_1)\,\frac{|\partial\Omega|}{d\,\rho_\Omega\,|\Omega|} - \lambda_1(B_1)\left(\frac{|B_1|}{|\Omega|}\right)^{2/d}\,.$$

For simplicity we used the bound for convex sets, but of course that the bound in Theorem 5.3 provides a better result valid for star-shaped domains. Note also that both upper bounds give equality for the ball. For a numerical study regarding upper and lower bounds of the gap see [AF2], which also contains some new conjectures for this problem.

5.5 Discussion of the upper bounds and examples

In spite of the fact that due to the existence of Rayleigh's variational formulation upper bounds for the first eigenvalue are, in principle, easier to obtain than lower bounds, it is slightly more delicate to obtain sharp upper bounds depending on the geometric quantities used here and which are valid in arbitrary dimensions. We shall thus now compare Theorems 5.2 and 5.3 to other existing bounds and conjectures for some particular examples.

With this in mind we begin by recalling some two-dimensional bounds. Among these there is the family of upper bounds based on the method of parallel coordinates, which includes Pólya's 1960 bound for simply-connected domains [P] (sharp asymptotically on infinite rectangular strips)

$$\lambda_1(\Omega) \;\leq\; \frac{\pi^2}{4}\frac{|\partial\Omega|^2}{|\Omega|^2}\,,$$

and its improvement by Payne and Weinberger [PW], namely,

$$\lambda_1(\Omega) \;\le\; \frac{\pi j_{0,1}^2}{|\Omega|}\left[1+\left(\frac{1}{J_1^2(j_{0,1})}-1\right)\left(\frac{|\partial\Omega|^2}{4\pi|\Omega|}-1\right)\right]. \tag{5.10}$$

Here J_1 and $j_{0,1}$ denote, respectively, the Bessel function of the first kind of order one, and the first positive zero of J_0, the Bessel function of the first kind of order zero. The Payne-Weinberger bound (5.10) is an explicit expression in terms of the area and perimeter which is obtained from their stronger bound

$$\lambda_1(\Omega) \;\le\; \frac{4\pi^2}{|\partial\Omega|^2}\,k(p)^2\,, \qquad p:=1-\frac{4\pi|\Omega|}{|\partial\Omega|^2}\,, \tag{5.11}$$

where $k=k(p)$ is the first zero of the transcendental equation

$$J_0(k)Y_1(pk)=Y_0(k)J_1(pk)\,.$$

Here Y_0 and Y_1 denote the Bessel functions of the second kind of order zero and one, respectively. It is this stronger bound that we will consider throughout this section for comparison, and we shall refer to it as the PW-bound. Note that (5.11) is sharp for the disc and asymptotically for infinite rectangular strips. While a generalization of Pólya's bound to arbitrary dimensions can be found in [S], the proof of the stronger result (5.11) does not seem to have a straightforward extension to higher dimensions.

We also remark that although (5.10) does give equality on the disc, the numerical study carried out in [AF1] suggests that this bound might still be improved and it is conjectured there that the optimal bound depending explicitly on the area and the perimeter and valid for simply-connected two-dimensional domains should be

$$\lambda_1(\Omega) \;\le\; \frac{\pi j_{01}^2}{|\Omega|}+\frac{\pi^2}{4}\frac{|\partial\Omega|^2-4\pi|\Omega|}{|\Omega|^2}\,, \tag{5.12}$$

providing now equality not only for the disc but also asymptotically on infinite rectangular strips.

Along different lines, Maz'ya and Shubin have recently proved upper and lower bounds depending on the interior capacity radius [MS].

We shall now consider some examples for which we compare the upper bounds given by Theorems 5.2 and 5.3 with the PW-bound (5.11) and conjecture (5.12).

Example 5.1 (Rectangular parallelepipeds)**.** Given positive numbers $a_1,\dots,a_d$, let $\mathcal{R}:=(-a_1,a_1)\times\cdots\times(-a_d,a_d)$. Elementary calculations show that the infimum in the definition of $F(\mathcal{R})$ is attained for the intuitive choice $\xi=0$, with the result

$$F(\mathcal{R})=|\mathcal{R}|\left(a_1^{-2}+\cdots+a_d^{-2}\right).$$

For rectangles, conjecture (5.12) is better than the PW-bound (5.11) for all the values of the parameter $c:=a_1/a_2\in(0,1]$. Theorem 5.3 (respectively Theorem 5.2) provides a better upper bound than conjecture (5.12) in the range of $c\in(0.3,1]$ (respectively $c\in(0.7,1]$). The largest discrepancy between the upper bound of Theorem 5.3 (respectively conjecture (5.12)) and the actual value of $\lambda_1(\mathcal{R})$ is in the limit $c\to0$ (respectively for $c=1$) when it is about 15% (respectively 26%).

Example 5.2 (Ellipsoids)**.** Given positive numbers $a_1,\dots,a_d$, let $\mathcal{E}$ be the domain enclosed by an ellipsoid, *i.e.* the surface determined by the implicit equation $f(x):=(x_1/a_1)^2+\cdots+(x_d/a_d)^2-1=0$. First of all, by symmetry, it is possible to conclude that the infimum in the definition of $F(\mathcal{E})$ is attained for $\xi=0$. Since $\nabla f/|\nabla f|$ is either $+N$ or $-N$ uniformly on $\partial\Omega$, we have

$$h_0^{-1}(x)=N(x)\cdot\frac{\nabla f(x)}{x\cdot\nabla f(x)}=N(x)\cdot\left(\frac{x_1}{a_1^2},\dots,\frac{x_d}{a_d^2}\right).$$

Now using the divergence theorem, we arrive at

$$F(\mathcal{E})=|\mathcal{E}|\left(a_1^{-2}+\cdots+a_d^{-2}\right).$$

That is, we formally obtain the same upper bound as in the case of parallelepipeds (notice that the volume terms in the bound of Theorem 5.3 cancel). However, the present bound is better because $\mathcal{E}\subset\mathcal{R}$, so that $\lambda_1(\mathcal{E})\ge\lambda_1(\mathcal{R})$ by monotonicity of Dirichlet eigenvalues.

For ellipses, Theorem 5.3 provides a better upper bound than conjecture (5.12) (which is again better than the PW-bound (5.11)) for all the values of the parameter $c:=a_1/a_2\in(0,1]$. Theorem 5.2 provides a better upper bound than the conjecture (5.12) in the regime of $c\in(0,0.1]$. In fact, the upper bound of Theorem 5.3

for two-dimensional ellipses seems to be familiar in the applied sciences and it is known that for $c \geq 0.5$ the discrepancy between this and $\lambda_1(\mathcal{E})$ does not exceed 1% (*cf* [Po, Sec. 7.3.4–3]).

Finally, let us mention that we obtain the same formula for $F(\mathcal{E})$ (and therefore for the upper bound of Theorem 5.3) also in the case when $\mathcal{E}$ is a tube of elliptical cross-section, *i.e.* the domain determined by $f(x_1, \dots, x_{d-1}, 0) = 0$ and $x_d \in (-a_d, a_d)$.

Example 5.3 (Stadium)**.** Given positive numbers a and b, let $\mathcal{S} \subset \mathbb{R}^2$ be the union of the rectangle $(-b, b) \times (-a, a)$ and two discs of radius a centered at the points $(-b, 0)$ and $(b, 0)$. We put $c := b/a \in [0, +\infty)$. Again, by symmetry, it is possible to conclude that the infimum in the definition of $F(\mathcal{S})$ is attained for $\xi = 0$. Since the boundary of $\mathcal{S}$ is composed of straight and arc segments, the integral of the inverse of the support function can be computed explicitly:

$$F(\mathcal{S}) = \begin{cases} 4c + \dfrac{8}{\sqrt{1-c^2}} \arctan \sqrt{\dfrac{1-c}{1+c}} & \text{if} \quad c < 1\,, \\ 8 & \text{if} \quad c = 1\,, \\ 4c + \dfrac{4}{\sqrt{c^2-1}} \log\left(c + \sqrt{c^2-1}\right) & \text{if} \quad c > 1\,. \end{cases}$$

In this example, Theorem 5.3 provides a better upper bound than conjecture (5.12) (which is again better than the PW-bound (5.11)) for $c \in [0, 2.6]$, while Theorem 5.2 is worse than both the conjecture and PW-bound for all the values of the parameter.

It is also possible to consider the asymmetric domain $\{(x_1, x_2) \in \mathcal{S} \,|\, x_1 > 0\}$. Then the position of ξ minimizing the infimum in the definition of $F(\mathcal{S})$ significantly depends on the value of c.

Example 5.4 (Swiss cross)**.** As an example of a non-convex domain (but strictly star-shaped with respect to the origin), let $\mathcal{C} \subset \mathbb{R}^2$ be the union of the two rectangles $(-b-a, b+a) \times (-a, a)$ and $(-a, a) \times (-b-a, b+a)$. We put $c := b/a \in [0, +\infty)$. An explicit calculation yields

$$F(\mathcal{C}) = 8\,\frac{1 + c + c^2}{1 + c}\,.$$

In this example, Theorem 5.3 provides a better upper bound than conjecture (5.12) (which is again better than the PW-bound (5.11)) for $c \in [0, 3.8]$. The case $c = 2$ was numerically analysed in [HS] and it was shown that the discrepancy between the bound and $\lambda_1(\mathcal{C})$ is less than 39%.

Acknowledgements. Partially supported by FCT, Portugal, through programs POCTI/MAT/60863/2004, POCTI/POCI2010 and SFRH/BPD/11457/2002. The second author was also supported by the Czech Academy of Sciences and its Grant Agency within the projects IRP AV0Z10480505 and A100480501, and by the project LC06002 of the Ministry of Education, Youth and Sports of the Czech Republic.

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Chapter 6

Instability results for the damped wave equation

Published in: Journal of Differential Equations **211** (2005), 168–186

Joint work with: Pedro Freitas

Instability results for the damped wave equation in unbounded domains

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Abstract. We extend some previous results for the damped wave equation in bounded domains in $\mathbb{R}^d$ to the unbounded case. In particular, we show that if the damping term is of the form αa with bounded a taking on negative values on a set of positive measure, then there will always exist unbounded solutions for sufficiently large positive α.

In order to prove these results, we generalize some existing results on the asymptotic behaviour of eigencurves of one-parameter families of Schrödinger operators to the unbounded case, which we believe to be of interest in their own right.

6.1 Introduction

Let Ω be a domain in $\mathbb{R}^d$ (bounded or unbounded), with $d \geq 1$, and let S_0 denote the self-adjoint operator generated on $L^2(\Omega)$ by the differential expression $\mathfrak{S}_0 := -\Delta + b$, subject to Dirichlet boundary conditions, where b is a real-valued measurable function on Ω; s_0 will denote the associated quadratic form. In this paper, we are interested in the instability of the solutions $t \mapsto \psi(t,\cdot) \in C^0\big([0,\infty);\mathfrak{D}(S_0)\big) \cap C^1\big([0,\infty);\mathfrak{D}(s_0)\big) \cap C^2\big([0,\infty);L^2(\Omega)\big)$ of the following initial value problem

$$\begin{cases} \psi_{tt} + \alpha\, a\, \psi_t + S_0\psi = 0\,, \\ \psi(0,\cdot) = \phi_1 \in \mathfrak{D}(S_0)\,, \\ \psi_t(0,\cdot) = \phi_2 \in \mathfrak{D}(s_0)\,, \end{cases} \tag{6.1}$$

where $\alpha \geq 0$ is a parameter (not necessarily small) and a is a real-valued measurable function on Ω.

In the case where the damping αa remains non-negative and bounded, the asymptotic behaviour of solutions of (6.1) is well understood [23]. However, there are situations for which this will not be the case and the damping term will change sign, precluding the usage of energy methods and other tools which rely on the positivity of the functional

$$\int_\Omega a(x)\,|u(x)|^2\,dx\,.$$

Such a situation might arise, for instance, when linearizing semilinear damped wave equations around a stationary solution – see, for instance [9]. Because of the change in sign, it is not clear what will now happen to the stability of these solutions.

In 1991 Chen *et al.* [6] conjectured that for bounded intervals and under certain extra conditions on the damping the trivial solution of (6.1) would remain stable. This was disproved in 1996 by the first author who showed that in the case of bounded Ω in $\mathbb{R}^d$ this sign-changing condition is sufficient to cause the existence of unbounded solutions of (6.1), provided that the L^∞-norm of the damping is large enough [11]. Heuristically, this behaviour can be understood from the fact that, when a changes sign, as the parameter α increases equation (6.1) (formally) approaches a backward–forward heat equation and thus one does expect the appearance of points in the spectrum on the positive side of the real axis. On the other hand, and still at the heuristic level, note that while for bounded domains the result is not unexpected from the point of view of geometric optic rays either, for unbounded domains this is not as clear. Think, for instance, of the case of $\Omega = \mathbb{R}^d$ and assume that $a(x)$ is negative inside the unit ball, positive outside, and goes to zero as $|x|$ goes to infinity. Then Theorem 6.1 below still gives that for large enough α the trivial solution of (6.1) becomes unstable.

The purpose of the present paper is to extend the results for the bounded case to unbounded domains and, in fact, the idea behind the results given here is the same as that used in [11]. The key point is the observation (*cf* Lemma 6.2) that it is possible to obtain information on the real part of the spectrum of the damped wave equation operator by studying some spectral properties of the one-parameter family of self-adjoint Schrödinger operator S_μ generated on $L^2(\Omega)$ by the differential expression

$$\mathfrak{S}_\mu := -\Delta + b + \mu\, a\,, \tag{6.2}$$

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where μ is a real parameter. However, the situation now becomes more complex due to the presence of essential spectrum, and, in particular, requires the extension of some results for S_μ to this setting which we believe to be interesting in their own right – see Appendix 6.6.

To state the main instability result, we now need to introduce some notation and basic assumptions. Throughout the paper, we assume that the damping coefficient a is bounded and that the potential b is bounded from below, does not have local singularities but can grow at infinity, namely,

$\langle$H1$\rangle$ $\quad a \in L^\infty(\Omega),\ \ b \in L^\infty_{\mathrm{loc}}(\Omega) \ \ \textit{and} \ \ b_{\min} := \operatorname{ess\,inf} b > -\infty\,.$

We also use the notation

$$a_{\min} := \operatorname{ess\,inf} a \qquad \text{and} \qquad a_{\max} := \operatorname{ess\,sup} a\,.$$

Under the stated assumptions, the initial value problem (6.1) has a unique solution (*cf* Corollary 6.2 in Appendix 6.7). The main result of the paper implies that if the damping a is sign-changing then there are initial conditions of (6.1) for which the corresponding solution are unbounded for sufficiently large α.

Theorem 6.1. *Assume* $\langle$H1$\rangle$*. If* $a_{\min} < 0$*, then there exists* $\alpha_0 > 0$ *such that the system* (6.1) *is unstable for all* $\alpha > \alpha_0$*.*

The instability property is obtained by studying a spectral problem for a non-self-adjoint operator A_α generated by the matrix-differential expression

$$\begin{pmatrix} 0 & 1 \\ -\mathfrak{S}_0 & -\alpha\, a \end{pmatrix}, \tag{6.3}$$

which appears in a first-order evolution equation (*cf* (6.7)) equivalent to (6.1), and proving that under these conditions there are positive points in its spectrum (Theorem 6.2). If the point in the spectrum is an eigenvalue, then this result shows that there are initial conditions of (6.1) for which the corresponding solution grows exponentially. If the point belongs to the essential spectrum, then Theorem 6.1 follows as a consequence of the global instability property (GI-property) for a local semi-dynamical system associated with (6.1), *cf* [20]. In order to apply the last result, we need to ensure that A_α generates a C_0-semigroup (*cf* Appendix 6.7).

The organization of the paper is as follows. In the preliminary Section 6.2, we define properly the operators S_μ and A_α, and state some basic properties. The real spectrum of A_α is investigated in Section 6.3, where we also prove Theorem 6.2 which implies the instability result of Theorem 6.1. Section 6.4 is devoted to a more precise discussion of the real essential spectrum of A_α in unbounded domains if the damping a is asymptotically constant; we also discuss there the relation between the essential spectrum and the form of Ω or the behaviour of b. In Section 6.5, we discuss the application of the results obtained to the semilinear case, and consider some open questions. The paper is concluded by two appendices, where we establish the asymptotics of eigenvalues of S_μ below the essential spectrum for large $|\mu|$ and prove the C_0-semigroup property of A_α.

Given a closed operator A in a Hilbert space, we denote by $\sigma(A)$, $\sigma_{\mathrm{p}}(A)$, $\sigma_{\mathrm{d}}(A)$, $\sigma_{\mathrm{e}}(A)$, repectively, the spectrum, the point spectrum, the discrete spectrum and the essential spectrum of A. There are various definitions of the essential spectrum for non-self-adjoint operators in the literature, *cf* [14]. We define $\sigma_{\mathrm{e}}(A) := \sigma(A) \setminus \sigma_{\mathrm{d}}(A)$ and recall that $\lambda \in \sigma_{\mathrm{d}}(A)$ if and only if it is an isolated point of $\sigma(A)$ consisting of an eigenvalue with finite algebraic multiplicity and with the range of $A - \lambda 1$ being closed.

We also point out some special conventions that we use throughout the paper: $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, where $\mathbb{N} = \{0, 1, 2, \dots\}$, and $\mathbb{R}^*_\pm = (0, \pm\infty)$.

6.2 Preliminaries

Denote by $(\cdot,\cdot)$ the inner product in the Hilbert space $L^2(\Omega)$ and by $\|\cdot\|$ the corresponding norm, and let Ω be an (arbitrary) open connected set in $\mathbb{R}^d$, with $d \geq 1$.

6.2.1 The auxiliary Schrödinger operator

In this section we consider the family of one-parameter Schrödinger operators S_μ on $L^2(\Omega)$, subject to Dirichlet boundary conditions, generated by the differential expression (6.2), where μ is a real parameter and $a, b : \Omega \to \mathbb{R}$ are measurable functions satisfying $\langle$H1$\rangle$.

The operators S_μ are introduced as follows. Let s_0 be the sesquilinear form on $L^2(\Omega)$ defined by

$$s_0(\phi,\psi) := (\nabla\phi, \nabla\psi) + (\phi, b\,\psi)\,, \qquad \phi,\psi \in \mathfrak{D}(s_0) := \overline{C_0^\infty(\Omega)}^{\,\|\cdot\|_{s_0}}\,,$$

where

$$\|\cdot\|^2_{s_0} := \|\cdot\|^2_{\mathcal{H}^1(\Omega)} + \|(b - b_{\min})^{\frac{1}{2}} \cdot\|^2\,.$$

(Note that $\mathfrak{D}(s_0)$ is continuously and densely embedded in $\mathcal{H}^1_0(\Omega)$.) Since the form s_0 is densely defined, closed, symmetric and bounded from below [10, Sec. VII.1.1], it gives rise to a unique self-adjoint operator S_0 which is also bounded from below. Using the representation theorem [15, Chap. VI, Thm. 2.1], we have that

$$S_0\psi = \mathfrak{S}_0\psi\,, \qquad \psi \in \mathfrak{D}(S_0) = \left\{\psi \in \mathfrak{D}(s_0) \,|\, \mathfrak{S}_0\psi \in L^2(\Omega)\right\}.$$

Since a is bounded, the quadratic form

$$s_\mu[\psi] := s_0[\psi] + \mu(\psi, a\psi)\,, \qquad \psi \in \mathfrak{D}(s_\mu) := \mathfrak{D}(s_0)\,,$$

gives rise to a self-adjoint operator S_μ which is bounded from below and satisfies

$$S_\mu\psi = \mathfrak{S}_\mu\psi\,, \qquad \psi \in \mathfrak{D}(S_\mu) = \mathfrak{D}(S_0)\,. \tag{6.4}$$

Let $\{\gamma_n(\mu)\}_{n\in\mathbb{N}^*}$ be the non-decreasing sequence of numbers corresponding to the spectral problem of S_μ according to the Rayleigh-Ritz variational formula, *i.e.*

$$\gamma_n(\mu) := \inf\left\{ \sup_{\psi\in L} \frac{s_\mu[\psi]}{\|\psi\|^2} \;\middle|\; L \subseteq \mathfrak{D}(s_0) \;\&\; \dim(L) = n \right\}. \tag{6.5}$$

Proposition 6.1. *Assume* ⟨H1⟩. *Each* $\mu \mapsto \gamma_n(\mu)$ *is a continuous function with the uniform asymptotics*

$$\gamma_n(\mu) = \begin{cases} a_{\max}\,\mu + o(\mu) & \text{as} \quad \mu \to -\infty\,, \\ a_{\min}\,\mu + o(\mu) & \text{as} \quad \mu \to +\infty\,. \end{cases}$$

Proof. The continuity is immediate from the definition (6.5); indeed, the Lipschitz condition $|\gamma_n(\mu) - \gamma_n(\mu')| \leq \|a\|_\infty|\mu - \mu'|$ holds true. The asymptotics follow from the more general Theorem 6.4 in Appendix 6.6. □

We recall that $\gamma_n(\mu)$ represents either a discrete eigenvalue of S_μ or the threshold of its essential spectrum, *cf* [8, Sec. 4.5]; $\gamma_1(\mu)$ is the spectral threshold of S_μ, *i.e.*, $\gamma_1(\mu) = \inf\sigma(S_\mu)$.

Let $\gamma_\infty(\mu) := \inf\sigma_{\rm e}(S_\mu)$ denote the threshold of the essential spectrum of S_μ (if $\sigma_{\rm e}(S_\mu) = \varnothing$, we put $\gamma_\infty(\mu) := +\infty$). The following formula is a generalization (*cf* [10, Sec. X.5]) of a result due to Persson [19]:

$$\gamma_\infty(\mu) = \sup\left\{ \inf_{\psi\in C_0^\infty(\Omega\setminus K)} \frac{s_\mu[\psi]}{\|\psi\|^2} \;\middle|\; K \text{ compact subset of } \Omega \right\}. \tag{6.6}$$

Proposition 6.2. *Assume* ⟨H1⟩. $\mu \mapsto \gamma_\infty(\mu)$ *is a continuous function satisfying*

$$\gamma_\infty(\mu) \geq \begin{cases} \gamma_\infty(0) + a_{\max}\,\mu & \text{if} \quad \mu \leq 0\,, \\ \gamma_\infty(0) + a_{\min}\,\mu & \text{if} \quad \mu \geq 0\,. \end{cases}$$

Proof. The Lipschitz condition $|\gamma_\infty(\mu) - \gamma_\infty(\mu')| \leq \|a\|_\infty|\mu - \mu'|$ and the lower bounds to $\gamma_\infty(\mu)$ are immediate consequences of (6.6). □

6.2.2 The damped wave operator

Let us introduce the Hilbert space $\mathcal{H} := \mathfrak{D}(s_0) \times L^2(\Omega)$ of vectors $\Psi \equiv \{\psi_1, \psi_2\}$, where $\psi_1 \in \mathfrak{D}(s_0)$ and $\psi_2 \in L^2(\Omega)$, and let us equip it with the norm $\|\cdot\|_{\mathcal{H}}$ defined by

$$\|\Psi\|^2_{\mathcal{H}} \equiv \|\{\psi_1, \psi_2\}\|^2_{\mathcal{H}} := \|\psi_1\|^2_{s_0} + \|\psi_2\|^2\,.$$

It is clear that $\mathcal{H}$ is the completion of $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ w.r.t. $\|\cdot\|_{\mathcal{H}}$.

Writing $\psi_1 := \psi$ and $\psi_2 := \psi_t$, the problem (6.1) is equivalent to the first-order system

$$\begin{cases} \Psi_t = A_\alpha\Psi\,, \\ \Psi(0,\cdot) = \Phi \in \mathfrak{D}(A_\alpha)\,, \end{cases} \tag{6.7}$$

where A_α is the operator associated with (6.3); it is properly defined as follows:

$$\begin{aligned} A_\alpha\{\psi_1, \psi_2\} &:= \left\{\psi_2, -S_0\psi_1 - \alpha a\psi_2\right\}, \\ \{\psi_1, \psi_2\} \in \mathfrak{D}(A_\alpha) &:= \mathfrak{D}(S_0) \times \mathfrak{D}(s_0)\,. \end{aligned} \tag{6.8}$$

Proposition 6.3. *Assume* ⟨H1⟩. *The operator A_α is densely defined and closed. Furthermore, it is the generator of a C_0-semigroup on $\mathcal{H}$.*

Proof. The density and closedness follow due to the same properties for S_0, s_0 and the boundedness assumption on a. The C_0-semigroup property is proved in Appendix 6.7 (*cf* Theorem 6.5). □

We shall use the following sufficient condition (characterization of the essential spectrum due to Wolf [21]).

Lemma 6.1. *Assume* ⟨H1⟩. *If there exists a singular sequence of A_α corresponding to $\lambda \in \mathbb{C}$,* i.e.*, a sequence $\{\Psi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{D}(A_\alpha)$ with the properties*

$$\|\Psi_n\|_{\mathcal{H}} = 1\,, \qquad \Psi_n \xrightarrow[n\to\infty]{w} 0 \quad \textit{and} \quad (A_\alpha - \lambda 1)\Psi_n \xrightarrow[n\to\infty]{} 0 \quad \textit{in} \quad \mathcal{H}\,,$$

then $\lambda \in \sigma_{\mathrm{e}}(A_\alpha)$.

We recall that the existence of singular sequence provides also a necessary condition in the characterization of essential spectrum for self-adjoint operators.

6.3 Real spectrum of the damped wave operator

Our further approach is based on the following trivial, but essential, observation that links the spectral problems for the operators A_α and S_μ.

Lemma 6.2. *Assume* ⟨H1⟩. $\forall \mu \in \mathbb{R},\ \forall \alpha > 0$,

$$\begin{aligned} &\text{(i)} && -(\mu/\alpha)^2 \in \sigma_{\mathrm{p}}(S_\mu) && \iff && \mu/\alpha \in \sigma_{\mathrm{p}}(A_\alpha)\,, \\ &\text{(ii)} && -(\mu/\alpha)^2 \in \sigma_{\mathrm{e}}(S_\mu) && \implies && \mu/\alpha \in \sigma_{\mathrm{e}}(A_\alpha)\,. \end{aligned}$$

Proof. ad (i). If $-(\mu/\alpha)^2 \in \sigma_{\mathrm{p}}(S_\mu)$, then there exists a $\psi \in \mathfrak{D}(S_0)$ such that $S_\mu\psi = -(\mu/\alpha)^2\psi$ and $\Psi \equiv \{\psi_1, \psi_2\} := \{\psi, (\mu/\alpha)\psi\} \in \mathfrak{D}(A_\alpha)$ is easily checked to satisfy $A_\alpha\Psi = (\mu/\alpha)\Psi$. Conversely, if $\mu/\alpha \in \sigma_{\mathrm{p}}(A_\alpha)$, then there exist $\psi_1 \in \mathfrak{D}(S_0)$ and $\psi_2 \in \mathfrak{D}(s_0)$ satisfying the system $\psi_2 = (\mu/\alpha)\psi_1\,, -S_0\psi_1 - \alpha a\psi_2 = (\mu/\alpha)\psi_2\,$, which yields $S_\mu\psi_1 = -(\mu/\alpha)^2\psi_1$.

ad (ii). If $-(\mu/\alpha)^2 \in \sigma_{\mathrm{e}}(S_\mu)$, then there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{D}(S_0)$ for which $\|\psi_n\| = 1$, $\psi_n \xrightarrow{w} 0$ and $(S_\mu + (\mu/\alpha)^2 1)\psi_n \to 0$ in $L^2(\Omega)$ as $n \to \infty$. We shall show that the sequence $\{\Psi_n/\|\Psi_n\|_{\mathcal{H}}\}_{n\in\mathbb{N}}$, where $\Psi_n \equiv \{\psi_n^1, \psi_n^2\} := \{\psi_n, (\mu/\alpha)\psi_n\}$, is the singular sequence of Lemma 6.1 with $\lambda = \mu/\alpha$. First of all, we note that $\|\Psi_n\|_{\mathcal{H}} \geq \|\psi_n\|_{s_0} \geq \|\psi_n\| = 1$, so the sequence is well-defined, normalized in $\mathcal{H}$, and it is enough to check the weak and strong convergence of Lemma 6.1 for the sequence $\{\Psi_n\}_{n\in\mathbb{N}}$. Clearly,

$$(A_\alpha - (\mu/\alpha)1)\Psi_n = \{0, -(S_\mu + (\mu/\alpha)^2 1)\psi_n\} \xrightarrow[n\to\infty]{} \{0,0\} \quad \text{in} \quad \mathfrak{D}(s_0) \times L^2(\Omega).$$

It is also clear that $\psi_n^2 \xrightarrow{w} 0$ in $L^2(\Omega)$ as $n \to \infty$, so it remains to show that $\psi_n^1 \equiv \psi_n \xrightarrow{w} 0$ in $\mathfrak{D}(s_0)$ as $n \to \infty$. The latter can be seen by writing

$$(\phi, \psi_n)_{s_0} = \big(\phi, [S_\mu + (\mu/\alpha)^2 1]\psi_n\big) + \big(\phi, [1 - b_{\min} - \mu a - (\mu/\alpha)^2]\psi_n\big)$$

for every $\phi \in \mathfrak{D}(s_0)$. □

The importance of this result is that it makes it possible to look for real points in the spectrum of A_α by considering the much simpler (since self-adjoint) spectral problem for S_μ. (Although the latter continues to make sense for complex μ, in such a case, however, one is merely trading one non-self-adjoint problem for another.) In view of our definition of essential spectrum, it is clear that, in general, there might be a $\mu/\alpha \in \sigma_{\mathrm{e}}(A_\alpha)$ such that $-(\mu/\alpha)^2 \notin \sigma_{\mathrm{e}}(S_\mu)$, and that is why we do not have a proof for the converse implication in (ii) of Lemma 6.2.

Lemma 6.2 yields

Proposition 6.4. *Assume* ⟨H1⟩. *For any $\alpha > 0$,*

$$\begin{aligned} &\text{(i)} && 0 \in \sigma_{\mathrm{p}}(S_0) && \iff && 0 \in \sigma_{\mathrm{p}}(A_\alpha)\,, \\ &\text{(ii)} && 0 \in \sigma_{\mathrm{e}}(S_0) && \implies && 0 \in \sigma_{\mathrm{e}}(A_\alpha)\,, \\ &\text{(iii)} && \gamma_1(0) < 0 && \implies && \big(\sigma(A_\alpha) \cap \mathbb{R}^*_- \neq \varnothing \quad \& \quad \sigma(A_\alpha) \cap \mathbb{R}^*_+ \neq \varnothing\big)\,, \\ &\text{(iv)} && \gamma_\infty(0) < 0 && \implies && \big(\sigma_{\mathrm{e}}(A_\alpha) \cap \mathbb{R}^*_- \neq \varnothing \quad \& \quad \sigma_{\mathrm{e}}(A_\alpha) \cap \mathbb{R}^*_+ \neq \varnothing\big)\,. \end{aligned}$$

Proof. (i) and (ii) are direct consequences of Lemma 6.2. Since $\gamma_1(0) < 0$, it follows by Proposition 6.1 that there exist at least one negative and one positive μ at which the curve $\mu \mapsto \gamma_1(\mu) \in \sigma(S_\mu)$ intersects the parabola $\mu \mapsto -(\mu/\alpha)^2$. If $\gamma_\infty(0) < 0$, one applies the same argument to γ_∞, with help of Proposition 6.2. □

In (iii) of Proposition 6.4, since the spectral nature of $\gamma_1(\mu)$, *i.e.*, whether it belongs to the discrete or essential spectrum of S_μ, may vary with μ, we do not know of which spectral nature are the corresponding points in the real spectrum of A_α. This can be decided, however, if S_μ has no negative essential spectrum. Let us define

$$N_\mu := \#\big\{n \in \mathbb{N}^* \,\big|\, \gamma_n(\mu) < \min\{0, \gamma_\infty(\mu)\}\big\},$$

i.e., the number of negative eigenvalues of S_μ below its essential spectrum, counting multiplicities.

Proposition 6.5. *Assume* ⟨H1⟩. *If $\gamma_\infty(\mu) \geq 0$ for all $\mu \leq 0$ (respectively all $\mu \geq 0$), then there exist at least N_0 negative (respectively N_0 positive) eigenvalues of A_α for all $\alpha > 0$.*

Proof. The claim is trivial for $N_0 = 0$. Let us assume $N_0 \geq 1$ and $\gamma_\infty(\mu) \geq 0$ for all $\mu \leq 0$. Under the assumptions, $\gamma_n(0) \in \sigma_{\rm d}(S_0) \cap \mathbb{R}^*_-$, $n \in \{1, \ldots, N_0\}$, and one applies the argument of the proof of Proposition 6.4 to every curve $\mu \mapsto \gamma_n(\mu)$. The difference is now that $\gamma_n(\mu)$ cannot become $\gamma_\infty(\mu)$ for $\mu \leq 0$, without crossing the parabola $\mu \mapsto -(\mu/\alpha)^2$, so the intersection corresponds to an eigenvalue. The same argument holds for the case $\mu \geq 0$. □

Proposition 6.5 is a generalization of [11, Prop. 3.2], where the result was established for both bounded Ω and b; then $\gamma_\infty(\mu) = +\infty$ (*i.e.*, $\sigma_{\rm e}(S_\mu) = \varnothing$) and $N_0 < +\infty$. It is worth to notice that, in our more general setting, $\gamma_\infty(\mu)$ can be finite and N_0 infinite.

To state and prove the main result of the paper, we impose on the damping a the following indefinitness condition

⟨H2⟩ $\quad a_{\min} < 0$ *and* $a_{\max} > 0$.

Theorem 6.2. *Assume* ⟨H1⟩ *and* ⟨H2⟩. *For all sufficiently large α, there is at least one negative and one positive point in the spectrum of A_α. Moreover,*

(i) *if $\lim_{\mu \to -\infty} \gamma_\infty(\mu)/\mu < a_{\max}$ (respectively $\lim_{\mu \to +\infty} \gamma_\infty(\mu)/\mu > a_{\min}$), then there exists a positive increasing sequence $\{\alpha_j^-\}_{j=1}^\infty$ (respectively $\{\alpha_j^+\}_{j=1}^\infty$) such that: $\forall \alpha \in [\alpha_j^-, \alpha_{j+1}^-)$ there exist at least j negative eigenvalues of A_α (respectively $\forall \alpha \in [\alpha_j^+, \alpha_{j+1}^+)$ there exist at least j positive eigenvalues of A_α);*

(ii) *if $\gamma_\infty(\mu) > 0$ for all $\mu \leq 0$ (respectively for all $\mu \geq 0$), then there exists a positive increasing sequence $\{\alpha_j^-\}_{j=1}^\infty$ (respectively $\{\alpha_j^+\}_{j=1}^\infty$) such that: $\forall \alpha \in (\alpha_j^-, \alpha_{j+1}^-]$ there exist at least $2j+N_0$ negative eigenvalues of A_α (respectively $\forall \alpha \in (\alpha_j^+, \alpha_{j+1}^+]$ there exist at least $2j+N_0$ positive eigenvalues of A_α) and $\forall \alpha \in (0, \alpha_1^-]$ there exist at least N_0 negative eigenvalues of A_α (respectively $\forall \alpha \in (0, \alpha_1^+]$ there exist at least N_0 positive eigenvalues of A_α);*

(iii) *if $\gamma_\infty(\mu) < 0$ for some $\mu \leq 0$ (respectively for some $\mu \geq 0$), then there is at least one negative (respectively one positive) point in $\sigma_{\rm e}(A_\alpha)$ for all sufficiently large α;*

(iv) *if $\gamma_\infty(0) > 0$ and $\gamma_\infty(\mu) < 0$ for some $\mu \leq 0$ (respectively for some $\mu \geq 0$), then there are at least two negative (respectively two positive) points in $\sigma_{\rm e}(A_\alpha)$ for all sufficiently large α.*

Proof. We shall prove the claims for the case $\mu \geq 0$ (*i.e.*, the results about the positive spectrum of A_α), the case of $\mu \leq 0$ being similar.

If $\gamma_1(0) < 0$, then the first statement holds for all $\alpha > 0$ by Proposition 6.4. If $\gamma_1(0) > 0$, then the curve $\mu \mapsto \gamma_1(\mu)$ does not intersect the parabola $\mu \mapsto -(\mu/\alpha)^2$ for sufficiently small α, but it does (in fact, at least twice), in view of Proposition 6.1 and ⟨H2⟩, if α is large enough; the intersections are positive and the result follows by Lemma 6.2. If $\gamma_1(0) = 0$, then there will be at least one positive intersection by the same argument.

ad (i). By virtue of Proposition 6.1, the assumption ⟨H2⟩ and the condition about the asymptotic behaviour of the threshold of the essential spectrum, it follows that for any $n \in \mathbb{N}^*$ there is an $\mu_n > 0$ such that $\gamma_n(\mu) < \min\{0, \gamma_\infty(\mu)\}$ for all $\mu \geq \mu_n$, *i.e.*, $\gamma_n(\mu)$ is a negative discrete eigenvalue of S_μ for all $\mu \geq \mu_n$. Moreover, the sequence $\{\mu_n\}_{n=1}^\infty$ can be chosen in such a way that the sequence $\{\alpha_n^+\}_{n=1}^\infty$ defined by $\alpha_n^+ := \mu_n/\sqrt{-\gamma_n(\mu_n)}$ is increasing. However, by Proposition 6.1, each curve $\mu \mapsto \gamma_n(\mu)$ has at least one intersection with the parabola $\mu \mapsto -(\mu/\alpha)^2$ for all $\alpha \in [\alpha_n^+, \infty)$. Hence, by Lemma 6.2, A_α has at least n positive eigenvalues for all $\alpha \in [\alpha_n^+, \alpha_{n+1}^+)$.

ad (ii). The N_0 positive eigenvalues of A_α correspond, by Proposition 6.5, to N_0 negative eigenvalues of S_0, for all $\alpha > 0$. Let $n_0 > N_0$ be the lowest of all n satisfying $\gamma_n(0) > 0$ (it always exists since $\gamma_\infty(0) > 0$).

Let $n \in \{n_0, \dots, \infty\}$. Since $\gamma_n(0) > 0$, the curve $\mu \mapsto \gamma_n(\mu)$ does not intersect the parabola $\mu \mapsto -(\mu/\alpha)^2$ for α sufficiently small. It follows by Proposition 6.1 that, by increasing α, it is possible to make these two curves touch for some $\mu > 0$. Denote by $\tilde{\alpha}^+_{n-n_0+1}$ the point for which this happens for the first time. Then, as α increases past this value, there will always exist at least two intersections for positive μ. Hence, by Lemma 6.2, A_α has at least $2(n - n_0 + 1)$ positive eigenvalues for all $\alpha \in (\tilde{\alpha}^+_{n-n_0+1}, \infty)$. Since γ_n is below γ_{n+1}, it follows that the sequence $\{\tilde{\alpha}^+_j\}_{j=1}^\infty$ is not decreasing. In fact, there exists an increasing subsequence $\{\alpha^+_j\}_{j=1}^\infty$ because each negative $\gamma_n(\mu)$ represents an eigenvalue of *finite* multiplicity.

ad (iii). Given a $\mu \geq 0$ such that $\gamma_\infty(\mu) < 0$, we define $\alpha_0 := \mu/\sqrt{-\gamma_\infty(\mu)}$ and it follows by Proposition 6.2 that there is at least one positive intersection of $\mu \mapsto \gamma_\infty(\mu)$ with the parabola $\mu \mapsto -(\mu/\alpha)^2$ for every $\alpha \in [\alpha_0, \infty)$. The result then follows by Lemma 6.2.

ad (iv). Since $\gamma_\infty(0) > 0$, the curve $\mu \mapsto \gamma_\infty(\mu)$ does not intersect the parabola $\mu \mapsto -(\mu/\alpha)^2$ for α sufficiently small. However, since there is a $\mu > 0$ such that $\gamma_\infty(\mu) < 0$, it follows that, by increasing α, it is possible to make these two curves touch for some $\mu > 0$. Denote by α_0 the point for which this happens for the first time. Then, by Proposition 6.2, as α increases past this value, there will always exist at least two intersections for positive μ. Hence, by Lemma 6.2, A_α has at least two positive points in the its essential spectrum for all $\alpha \in (\alpha_0, \infty)$. □

Remarks 6.1. Assume ⟨H2⟩ and $\gamma_\infty(\mu) > 0$ for all $\mu \in \mathbb{R}$, and denote by $\lambda(\alpha)$ a real eigenvalue of A_α, as described in the proof of the part (ii) of Theorem 6.2. Following [11], it is possible to give some more properties of $\lambda(\alpha)$.

For instance, the following properties are clear from the proof of Proposition 6.5 and the part (ii) of Theorem 6.2 (*cf* also [11, proof of Prop. 3.8]): If $\lambda(\alpha) < 0$ (respectively $\lambda(\alpha) > 0$) corresponds to a negative eigenvalue of S_0, the function $\alpha \mapsto \lambda(\alpha)$ is decreasing (respectively increasing) and $\lambda(\alpha) \to -\infty$ (respectively $+\infty$) as $\alpha \to +\infty$. In the case of the two negative (respectively positive) eigenvalues, corresponding to some positive $\gamma_n(0)$, one of these eigenvalues is increasing (respectively decreasing) to zero, while the other is decreasing to $-\infty$ (respectively increasing to $+\infty$).

Also, since γ_1 remains below all other curves γ_n and the eigenfunction of S_μ corresponding to $\gamma_1(\mu) < 0$ can be chosen to be positive, we get the following characterization of the eigenfunctions of A_α corresponding to its extreme real eigenvalues (*cf* [11, Thm. 3.9]): If $N_0 > 0$, then the eigenfunction $\Psi \equiv \{\psi_1, \psi_2\}$ corresponding to the smallest negative (respectively largest positive) eigenvalue of A_α can be chosen in such a way that $\psi_1(x) > 0$ and $\psi_2(x) < 0$ (respectively $\psi_2(x) > 0$) for all $x \in \Omega$. If $N_0 = 0$ and A_α has negative (respectively positive) eigenvalues (this always happens for α sufficiently large), then the eigenfunctions $\Psi \equiv \{\psi_1, \psi_2\}$ corresponding to the largest or the smallest of these eigenvalues can be chosen in such a way that $\psi_1(x) > 0$ and $\psi_2(x) < 0$ (respectively $\psi_2(x) > 0$) for all $x \in \Omega$.

If both Ω and b are bounded, Theorem 6.2 reduces to [11, Thm. 3.6]. Actually, all the complexities of the present theorem are due to the possible presence of essential spectrum in our more general setting. The question of essential spectrum is discussed in more details in the following section.

6.4 More on the essential spectrum

All the spectral results of the previous section are based on proving an intersection of the functions γ_n or γ_∞ with a parabola. The results are qualitative since we have used just the continuity and asymptotic properties of the functions (Propositions 6.1 and 6.2). The purpose of this section is twofold. Firstly, we use known properties of γ_∞ in the case where a converges to a fixed value at infinity in order to state more precise results about the real essential spectrum of A_α. Secondly, on characteristic examples, we discuss the nature of the real spectrum of A_α as related to the form of the domain Ω or the behaviour of the potential b.

6.4.1 Asymptotically constant damping

If $\gamma_n(\mu)$ is a discrete eigenvalue of S_μ for all $\mu \in \mathbb{R}$ and a is not constant, then it is hard to say anything about the behaviour of the curve $\mu \mapsto \gamma_n(\mu)$, apart from small values of the parameter μ (via perturbation techniques) or large μ (*cf* Proposition 6.1). On the other hand, the behaviour of the curve $\mu \mapsto \gamma_\infty(\mu)$ is expected to be usually simple because, heuristically speaking, the essential spectrum comes from "what happens very far away" only. The above statement is made precise in context of the following result, which can be established by means of the decomposition principle [10, Chap. X].

Proposition 6.6. *Assume* ⟨H1⟩. *If* Ω *is unbounded and the limit*

$$a_\infty := \lim_{|x| \to \infty,\, x \in \Omega} a(x)$$

exists, then

$$\forall \mu \in \mathbb{R}, \qquad \sigma_{\mathrm{e}}(S_\mu) = \sigma_{\mathrm{e}}(S_0) + a_\infty \, \mu \, .$$

In particular, under the assumptions of Proposition 6.6, the curve $\mu \mapsto \gamma_\infty(\mu)$ is linear (if $\gamma_\infty(0) = +\infty$, then $\gamma_\infty(\mu) = +\infty$ for all $\mu \in \mathbb{R}$). Applications of this result to Theorem 6.2 are obvious. For instance, the sufficient conditions from the case (i) of Theorem 6.2 are satisfied if $a_\infty < a_{\max}$ (respectively $a_\infty > a_{\min}$). Also, it follows from the case (iii) of Theorem 6.2 that there is always a negative (respectively positive), point in $\sigma_{\mathrm{e}}(A_\alpha)$ provided $\sigma_{\mathrm{e}}(S_0) \neq \varnothing$ and $a_\infty > 0$ (respectively $a_\infty < 0$). Actually, a stronger result holds if the essential spectrum of S_0 is an infinite interval:

Proposition 6.7. *Assume the hypotheses of* Proposition 6.6. *Suppose also that* $\sigma_{\mathrm{e}}(S_0) = [\gamma_\infty(0), \infty)$ *and* $\delta_\alpha := \alpha^2 a_\infty^2 - 4\gamma_\infty(0) \geq 0$. *Then*

$$\left[\tfrac{1}{2} \left(-\alpha a_\infty - \sqrt{\delta_\alpha} \right) , \tfrac{1}{2} \left(-\alpha a_\infty + \sqrt{\delta_\alpha} \right) \right] \subseteq \sigma_{\mathrm{e}}(A_\alpha) \, .$$

Proof. Combining the assumption $\sigma_{\mathrm{e}}(S_0) = [\gamma_\infty(0), \infty)$, Proposition 6.6 and Lemma 6.2, we see that every point between the intersections (if they exist) of the parabola $\mu \mapsto -(\mu/\alpha)^2$ with the line $\mu \mapsto \gamma_\infty(0) + a_\infty \mu$ lies in the essential spectrum of A_α. The claim then follows by solving the corresponding quadratic equation (in particular, the condition $\delta_\alpha \geq 0$ ensures its solvability). □

6.4.2 Examples

In this subsection the significant features of Ω or b as regards $\sigma_{\mathrm{e}}(A_\alpha)$ are discussed. Since the spectrum of A_α is purely discrete whenever Ω is bounded, we restrict ourselves to unbounded Ω here. We also assume that the hypotheses of Proposition 6.6 hold true, in particular, a_∞ denotes the asymptotic value of the damping a.

Different types of domains

We adopt the following classification of domains in Euclidean space due to Glazman [13, § 49] (see also [10, Sec. X.6.1]): Ω is *quasi-conical* if it contains arbitrarily large balls; Ω is *quasi-cylindrical* if it is not quasi-conical but contains a sequence of identical disjoint balls; Ω is *quasi-bounded* if it is neither quasi-conical nor quasi-cylindrical. For simplicity, let us assume that $b = 0$, so that S_0 is just the Dirichlet Laplacian in Ω.

Quasi-bounded domains If the boundary of Ω is sufficiently smooth (*cf* [10, Sec. X.6.1] for precise conditions in terms of capacity), Molčanov's criterion [17] implies that the spectrum of S_0 is purely discrete. Then the same is true for the operator A_α, *i.e.*, $\sigma_{\mathrm{e}}(A_\alpha) = \varnothing$. If the hypothesis ⟨H2⟩ holds true, the part (ii) of Theorem 6.2 claims that there are always negative and positive eigenvalues of A_α for α sufficiently large. This situation is analogous to the bounded case [11].

Quasi-conical domains The spectrum of S_0 is purely essential and equals the interval $[0, \infty)$. By Propositions 6.6 and 6.1, $\sigma_{\mathrm{e}}(S_\mu) = [a_\infty \mu, \infty)$, and there are discrete eigenvalues below the essential spectrum for sufficiently large positive μ if $a_\infty > a_{\min}$ or sufficiently large negative μ if $a_\infty < a_{\max}$. By Proposition 6.7,

$$\left[-\tfrac{\alpha}{2} \left(|a_\infty| + a_\infty \right) , \tfrac{\alpha}{2} \left(|a_\infty| - a_\infty \right) \right] \subseteq \sigma_{\mathrm{e}}(A_\alpha) \, ,$$

i.e., A_α has a real essential spectrum for any $\alpha \geq 0$. Information about the real point spectrum of A_α is contained in the part (i) of Theorem 6.2.

Quasi-cylindrical domains Since the precise location of $\sigma_{\mathrm{e}}(S_0)$ is difficult in this situation (a detailed analysis of this problem may be found in Glazman's book [13]), we shall rather illustrate the nature of $\sigma(A_\alpha)$ in the particular case of tubes.

Let Ω be a tubular neighbourhood of radius $r > 0$ about an infinite curve in $\mathbb{R}^d$, with $d \geq 2$, and assume that it does not overlap itself and that the condition $\|\kappa_1\|_\infty \, r < 1$ holds true, where κ_1 denotes the first curvature of the reference curve. If the tube is asymptotically straight in the sense that κ_1 vanishes at infinity, it was shown in [7] that the essential spectrum of S_0 is the interval $[\nu_1, \infty)$, where $\nu_1 > 0$ denotes the first eigenvalue of the Dirichlet Laplacian in the $(d-1)$-dimensional ball of radius r. (Furthermore, there exists at least one discrete eigenvalue in $(0, \nu_1)$ whenever $\kappa_1 \neq 0$.) It follows that the assumptions of Proposition 6.7 are satisfied if $a_\infty \neq 0$ and α is sufficiently large. That is, depending on the sign of a_∞, A_α always has an open interval in its negative or positive essential spectrum for α sufficiently large. An information about the real point spectrum of A_α is contained in the part (i) or (ii) of Theorem 6.2.

The behaviour of the potential

For simplicity, let us assume that Ω is the whole space $\mathbb{R}^d$ (a special case of quasi-conical domains). Since the potential b is assumed to be locally bounded, the essential spectrum of S_0 will depend on the behaviour of b at infinity only. Assuming

$$\lim_{|x|\to\infty} b(x) = b_\infty \in [b_{\min}, +\infty]\,,$$

we distinguish two particular behaviours:

$\underline{b_\infty = +\infty}$ It is easy to see that $\gamma_\infty(0) = +\infty$ and the spectrum of S_0 is therefore purely discrete. Consequently, the spectrum of A_α is purely discrete as well and the situation is analogous with the bounded case [11].

$\underline{b_\infty < +\infty}$ The essential spectrum of S_0 is the interval $[b_\infty, \infty)$ and there might be discrete eigenvalues in $[b_{\min}, b_\infty)$ if $b_{\min} < b_\infty$. By Proposition 6.6, we have $\sigma_{\rm e}(S_\mu) = [b_\infty + a_\infty\mu, \infty)$, and this gives an information about the real essential (respectively real point) spectrum of A_α through Proposition 6.7 (respectively through the part (i) or (ii) of Theorem 6.2).

6.5 Remarks and open questions

In a similar way to what was done in [11], the results given here can be used to obtain information about semilinear wave equations of the form

$$u_{tt} + f(x, u, u_t) = \Delta u, \quad x \in \Omega. \tag{6.9}$$

In particular, the stability of stationary solutions of (6.9) can be related to that of solutions of the parabolic equation

$$u_t + f(x, u, 0) = \Delta u, \quad x \in \Omega. \tag{6.10}$$

A typical example of such a result would be the following, the proof of which follows in the same way as that of the corresponding result in [11] – note that equations (6.9) and (6.10) have the same set of stationary solutions.

Theorem 6.3. *Let w be a stationary solution of equation* (6.9)*. Then if w is linearly unstable when considered as a stationary solution of the parabolic equation* (6.10)*, it is also linearly unstable as a stationary solution of* (6.9)*.*

More specific results in the spirit of those in [11] may be given in the case of quasi-bounded domains, for instance.

Regarding stability issues, and as in the bounded case, it would be of interest to know if there are situations when the trivial solution of (6.1) remains (asymptotically) stable for small α when a is allowed to become negative. In the former case, the problem remains open in dimension higher than one, but there are several results for the one dimensional problem showing that this is still possible [2, 12]. However, even in the case of the real line, the situation for unbounded domains seems to be more difficult due to the presence of essential spectrum. Also, the proof of stability in the case of an interval used information about the asymptotic behaviour of the spectrum which is not available in the unbounded case.

Finally, let us remark that it is possible to adapt the approach of the present paper to damped wave equations with a more general elliptic operator instead of S_0 considered here.

APPENDICES

6.6 Eigenvalue asymptotics

In this appendix we prove the asymptotics of Proposition 6.1. We proceed in a greater generality by admitting local singularities of the function b and that a is bounded just from below. That is, the assumption ⟨H1⟩ is replaced by

⟨H1′⟩ $a, b \in L^1_{\rm loc}(\Omega)$ *and*

(a) $a_{\min} := \operatorname{ess\,inf} a > -\infty$,

(b) $b = b_1 + b_2$ *where* $b_1 \in L^1_{\rm loc}(\Omega)$, $b_{1,\min} := \operatorname{ess\,inf} b_1 > -\infty$ *and* $b_2 \in L^p(\Omega)$ *for some* $p \in [d, \infty]$.

Here we restrict ourselves to $\mu \geq 0$ (the case of $\mu \leq 0$ being included if $a \in L^\infty(\Omega)$) and define the realization S_μ of the differential expression (6.2) as the operator associated with the sesquilinear form

$$\begin{aligned} s_\mu(\phi,\psi) &:= (\nabla\phi, \nabla\psi) + \mu\,(\phi, a\psi) + (\phi, b\psi)\,, \\ \phi,\psi \in \mathfrak{D}(s_\mu) &:= \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{s_\mu}}\,, \\ \|\cdot\|_{s_\mu}^2 &:= \|\cdot\|_{\mathcal{H}^1(\Omega)}^2 + \left\| \left[\mu\,(a - a_{\min}) + b_1 - b_{1,\min} + b_2^+\right]^{\frac{1}{2}} \cdot \right\|^2, \end{aligned}$$

where b_2^+ denotes the positive part of b_2. The operator S_μ is self-adjoint and bounded from below because the form s_μ is densely defined, closed, symmetric and bounded from below [10, Sec. VII.1.1]. We also verify that

$$S_\mu\psi = \mathfrak{S}_\mu\psi\,, \qquad \psi \in \mathfrak{D}(S_\mu) = \left\{\psi \in \mathfrak{D}(s_\mu)\,|\,\mathfrak{S}_\mu\psi \in L^2(\Omega)\right\}. \tag{6.11}$$

It is clear that the operators verifying (6.4) or (6.11) coincide provided $a \in L^\infty(\Omega)$ and $b_2 = 0$.

We recall the definition (6.5) of the sequence $\{\gamma_n(\mu)\}_{n\in\mathbb{N}^*}$ and prove

Theorem 6.4. *Assume* ⟨H1′⟩. *One has*

$$\forall n \in \mathbb{N}^*, \qquad \lim_{\mu\to+\infty} \frac{\gamma_n(\mu)}{\mu} = a_{\min}\,.$$

Proof. Since $S_\mu \geq (\inf\sigma(S_0) + \mu\,a_{\min})1$, we have $\lim_{\mu\to+\infty}\gamma_n(\mu)/\mu \geq a_{\min}$ by the minimax principle. To obtain the opposite estimate, we are inspired by [1, proof of Thm. 2.2]. Let M_a denote the maximal operator of multiplication by a, *i.e.*, $M_a\psi := a\psi$, $\psi \in \mathfrak{D}(M_a) := \{\psi \in L^2(\Omega)|\ a\psi \in L^2(\Omega)\}$. The spectrum of M_a is purely essential and equal to the essential range of the generated function a. In particular, $a_{\min} \in \sigma_{\rm e}(M_a)$. By the spectral theorem, there exists a sequence $\{\psi_j\}_{j\in\mathbb{N}^*} \subseteq \mathfrak{D}(M_a)$ orthonormalized in $L^2(\Omega)$ such that $\|(M_a - a_{\min}1)\psi_j\| \to 0$ as $j \to \infty$. Since $C_0^\infty(\Omega)$ is dense in $\mathfrak{D}(M_a)$, it follows that there is also a sequence $\{\varphi_j\}_{j\in\mathbb{N}^*} \subseteq C_0^\infty(\Omega)$ satisfying

$$(\varphi_i,\varphi_j) - \delta_{ij} \longrightarrow 0 \qquad \text{and} \qquad \big(\varphi_i, (M_a - a_{\min}1)\varphi_j\big) \longrightarrow 0$$

as $i,j \to \infty$. Now, given $N \in \mathbb{N}^*$, choose $k = k(N) \in \mathbb{N}$ sufficiently large so that

$$A(N) - v_{\min}\,1 \leq N^{-1}1 \qquad \text{and} \qquad B(N) \geq \tfrac{1}{2}\,1\,,$$

where $A(N)$, respectively $B(N)$, is the $N \times N$ symmetric matrix with entries $(\varphi_{i+k}, a\varphi_{j+k})$, respectively $(\varphi_{i+k},\varphi_{j+k})$, for $i,j \in \{1,\ldots,N\}$. In view of (6.5) and since $\mathrm{span}\{\varphi_{j+k}\}_{j=1}^N$ is an N-dimensional subspace of $\mathfrak{D}(s_\mu)$, one has $\gamma_n(\mu) \leq c_n(\mu,N)$ for all $n \in \{1,\ldots,N\}$, where $\{c_n(\mu,N)\}_{n=1}^N$ are the eigenvalues (written in increasing order and repeated according to multiplicity) of the matrix $C(\mu,N) \equiv \big(C_{ij}(\mu,N)\big)_{i,j=1}^N$ defined by $C_{ij}(\mu,N) := s_\mu(\varphi_{i+k},\varphi_{j+k})$. However, it is easy to see that

$$C(\mu,N) \leq \mu\left(a_{\min} + N^{-1}\right)1 + d(N)\,1\,,$$

where $d(N)$ denotes the maximal eigenvalue of the matrix

$$\big((\nabla\varphi_{i+k}, \nabla\varphi_{j+k}) + (\varphi_{i+k}, b\varphi_{j+k})\big)_{i,j=1}^N\,.$$

Hence, for all $n \in \{1,\ldots,N\}$,

$$\lim_{\mu\to+\infty}\frac{\gamma_n(\mu)}{\mu} \leq a_{\min} + N^{-1}$$

with N being arbitrarily large. □

Remark 6.1. The asymptotic behaviour of Theorem 6.4 for both bounded Ω and a was established in [3, Corol. 2.6] (see also [4, proof of Lemma 2.1] and [5]).

6.7 The semigroup property

Unable to find a suitable reference, in this appendix we prove that the operator A_α is the infinitesimal generator of a C_0-semigroup on $\mathcal{H}$. Our strategy is to modify the approach of [22, Sec. XIV.3] (see also [18, Sec. 7.4]), where the property was shown for $\Omega = \mathbb{R}^d$, $b \in L^\infty(\Omega)$ and $a = 0$.

We will need some preliminaries. Firstly, we prove the following solvability result, which is well known if b and Ω were bounded.

Lemma 6.3. *Assume* ⟨H1⟩. *Let* $\eta \in (0, \eta_\alpha)$ *with* $\eta_\alpha := \frac{1}{2}\big(1 + \alpha|a_{\min}| + |b_{\min}|\big)^{-1}$. *For any* $\phi \in L^2(\Omega)$, *there is a unique function* $\psi \in \mathfrak{D}(S_0)$ *satisfying*

$$\big(1 + \eta\,\alpha a + \eta^2 S_0\big)\psi = \phi\,. \tag{6.12}$$

Proof. We are inspired by [16, Remark 2.3.2]. First of all, if $\eta \in (0, \eta_\alpha)$, then we can establish the *a priori* bound

$$\|\psi\|_{s_0} \le (\sqrt{2}\,\eta)^{-1}\,\|\phi\|\,. \tag{6.13}$$

Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be an exhaustion sequence of Ω (*i.e.* each Ω_n is a bounded open set such that $\Omega_n \subset\subset \Omega_{n+1}$ and $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$). Let $\psi_n \in \mathcal{H}_0^1(\Omega_n)$ be the solution of the boundary problem (6.12) restricted to Ω_n; the existence and uniqueness of such solutions is guaranteed, *e.g.*, by [16, Sec. II.3]. We extend ψ_n to the whole Ω by taking it to be zero outside Ω_n, and we retain the same notation ψ_n for this extension. All the ψ_n are elements of $\mathfrak{D}(s_0)$ and satisfy the *a priori* bound (6.13). Consequently, $\{\psi_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $\mathfrak{D}(s_0)$ and there exists a subsequence $\{\psi_{n_k}\}_{k\in\mathbb{N}}$ which converges weakly in $\mathfrak{D}(s_0)$ to some function ψ. This function is the desired solution. Indeed, the functions ψ_n satisfy the integral identities

$$(\varphi, \psi_n) + \eta\,(\varphi, \alpha a \psi_n) + \eta^2 (\varphi, S_0 \psi_n) = (\varphi, \phi) \tag{6.14}$$

for all $\varphi \in C_0^\infty(\Omega_n)$. In (6.14), we fix some $\varphi \in C_0^\infty(\Omega_n)$, and then take the limit in the subsequence $\{n_k\}_{k\in\mathbb{N}}$ chosen above, assuming that $n_k \ge n$. In the limit we obtain (6.14) for ψ with the function φ that we took. Since these φ form a dense set in $\mathfrak{D}(s_0)$, ψ satisfies the identity (6.14) for all $\varphi \in \mathfrak{D}(s_0)$, hence it is a generalized solution of the problem (6.12). By the uniqueness, which is guaranteed by the bound (6.13), the whole sequence $\{\psi_n\}_{n\in\mathbb{N}}$ will converge in this way to the solution $\psi \in \mathfrak{D}(s_0)$. The property $\psi \in \mathfrak{D}(S_0)$ follows from (6.12) and the facts that $\phi \in L^2(\Omega)$ and a is bounded. □

Now we can prove

Lemma 6.4. *Assume* ⟨H1⟩. *Let* $\Phi \equiv \{\phi_1, \phi_2\} \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ *and* $\eta \in (0, \eta_\alpha)$. *Then the equation*

$$\Psi - \eta\, A_\alpha \Psi = \Phi \tag{6.15}$$

has a unique solution $\Psi \equiv \{\psi_1, \psi_2\} \in \mathfrak{D}(A_\alpha)$. *Moreover,*

$$\|\Psi\|_{\mathcal{H}} \le (1 - \eta/\eta_\alpha)^{-1}\,\|\Phi\|_{\mathcal{H}}\,.$$

Proof. Let $\varphi_1, \varphi_2 \in \mathfrak{D}(S_0)$ be the respective solutions of

$$\begin{aligned}\big(1 + \eta\,\alpha a + \eta^2 S_0\big)\varphi_1 &= \phi_1\,,\\ \big(1 + \eta\,\alpha a + \eta^2 S_0\big)\varphi_2 &= \alpha a \phi_1 + \phi_2\,.\end{aligned}$$

Set

$$\psi_1 := \varphi_1 + \eta\varphi_2\,, \qquad \psi_2 := -(\alpha a + \eta\, S_0)\varphi_1 + \varphi_2\,.$$

It is easy to check that $\Psi := \{\psi_1, \psi_2\} \in \mathfrak{D}(S_0) \times L^2(\Omega)$ and it satisfies the equation (6.15), *i.e.*,

$$\psi_1 - \eta\,\psi_2 = \phi_1\,, \qquad \eta\, S_0\psi_1 + \big(1 + \eta\,\alpha a\big)\psi_2 = \phi_2\,.$$

Actually, it follows from the first of the last two equations that $\psi_2 \in \mathfrak{D}(s_0)$, so $\Psi \in \mathfrak{D}(A_\alpha)$ and it is the desired solution. Moreover, we have

$$\begin{aligned}\|\Phi\|_{\mathcal{H}}^2 &\equiv \|\phi_1\|_{s_0}^2 + \|\phi_2\|^2\\ &= \|\psi_1 - \eta\,\psi_2\|_{s_0}^2 + \|\eta\, S_0\psi_1 + \big(1 + \eta\,\alpha a\big)\psi_2\|^2\\ &\ge \|\psi_1\|_{s_0}^2 + \|\psi_2\|^2 + 2\eta\,(\psi_2, \alpha a \psi_2)\\ &\quad - 2\eta\,\mathrm{Re}\big(\psi_2, [-\Delta + b - b_{\min} + 1]\psi_1\big) + 2\eta\,\mathrm{Re}(\psi_2, S_0\psi_1)\\ &\ge \|\psi_1\|_{s_0}^2 + \|\psi_2\|^2 + 2\eta\,\alpha a_{\min}\,\|\psi_2\|^2 - \eta\,\big(\|\psi_2\|^2 + |1 - b_{\min}|\,\|\psi_1\|^2\big)\\ &\ge \big(1 - \eta\,\big[2 + 2\alpha|a_{\min}| + |b_{\min}|\big]\big)\,\|\Psi\|_{\mathcal{H}}^2\\ &\ge (1 - \eta/\eta_\alpha)\,\|\Psi\|_{\mathcal{H}}^2\end{aligned}$$

and $(1 - |x|) \ge (1 - |x|)^2$ for $|x| \le 1$. □

Lemma 6.4 shows that the range of the operator $1 - \eta A_\alpha$ contains $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ for all $\eta \in (0, \eta_\alpha)$. Since A_α is closed (*cf* Proposition 6.3), the range of $1 - \eta A_\alpha$ is all of $\mathcal{H}$ and we have

Corollary 6.1. *Assume* ⟨H1⟩. *For every* $\eta \in (0, \eta_\alpha)$ *and* $\Phi \in \mathcal{H}$ *the equation* (6.15) *has a unique solution* $\Psi \in \mathfrak{D}(A_\alpha)$ *and* $\|\Psi\|_{\mathcal{H}} \leq (1 - \eta/\eta_\alpha)^{-1} \|\Phi\|_{\mathcal{H}}$.

Now we are ready to prove

Theorem 6.5. *Assume* ⟨H1⟩. *The operator* A_α *is the infinitesimal generator of a* C_0*-semigroup* U_α *on* $\mathcal{H}$, *satisfying*

$$\forall t \in (0, \infty), \qquad \|U_\alpha(t)\| \leq e^{\omega_\alpha t} \tag{6.16}$$

with $\omega_\alpha := 2\big(1 + \alpha|a_{\min}| + |b_{\min}|\big)$.

Proof. The domain of A_α is clearly dense in $\mathcal{H}$ (*cf* Proposition 6.3). ¿From Corollary 6.1 it follows that $(z - A_\alpha)^{-1}$ exists for all $z \in (\omega_\alpha, \infty)$, $\omega_\alpha = \eta_\alpha^{-1}$, and satisfies

$$\big\|(z - A_\alpha)^{-1}\big\| \leq (z - \omega_\alpha)^{-1} \qquad \text{for} \quad z > \omega_\alpha \,.$$

The claim then follows by [18, Thm. 1.5.3]. □

Remark 6.2. Since a is bounded, the proof could also be done by showing that A_0 is an infinitesimal generator of a C_0-group U_0 and applying [18, Thm 1.1 of Chap. 3].

As an application of Theorem 6.5, one has the following

Corollary 6.2. *Assume* ⟨H1⟩. *Let* $\phi_1 \in \mathfrak{D}(S_0)$ *and* $\phi_2 \in \mathfrak{D}(s_0)$. *Then there exists a unique* $t \mapsto \psi(t, \cdot) \in C^0\big([0, \infty); \mathfrak{D}(S_0)\big) \cap C^1\big([0, \infty); \mathfrak{D}(s_0)\big) \cap C^2\big([0, \infty); L^2(\Omega)\big)$ *satisfying the initial value problem* (6.1).

Proof. Let U_α be the semigroup generated by A_α and set

$$\big\{\psi_1(t, x), \psi_2(t, x)\big\} := U_\alpha(t) \big\{\phi_1(x), \phi_2(x)\big\} \,.$$

Then

$$\frac{\partial}{\partial t}\{\psi_1(t, \cdot), \psi_2(t, \cdot)\} = A_\alpha\big\{\psi_1(t, \cdot), \psi_2(t, \cdot)\big\} = \big\{\psi_2(t, \cdot), -S_0\psi_1(t, \cdot) - \alpha a\psi_2(t, \cdot)\big\}$$

and ψ_1 is the desired solution. □

Acknowledgements

This work was partially supported by FCT/POCTI/FEDER, Portugal, and the AS CR project K1010104.

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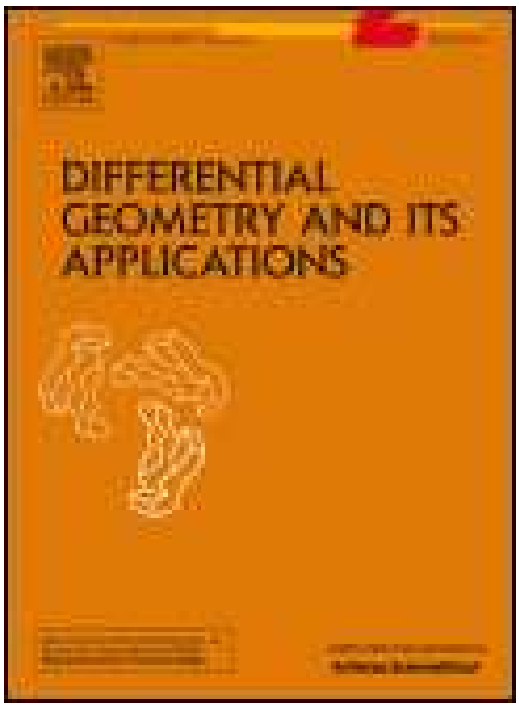
Part II

Quantum waveguides

Chapter 7

Geometrically induced discrete spectrum in curved tubes



Published in: Differential Geometry and its Applications **23** (2005), 95–105

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Geometrically induced discrete spectrum in curved tubes

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Abstract. The Dirichlet Laplacian in curved tubes of arbitrary cross-section rotating w.r.t. the Tang frame along infinite curves in Euclidean spaces of arbitrary dimension is investigated. If the reference curve is not straight and its curvatures vanish at infinity, we prove that the essential spectrum as a set coincides with the spectrum of the straight tube of the same cross-section and that the discrete spectrum is not empty.

7.1 Introduction

The relationships between the geometric properties of an Euclidean region and the spectrum of the associated Dirichlet Laplacian constitute one of the classical problems of *spectral geometry*, with important motivations coming both from classical and quantum physics. In this paper we consider this type of interactions in the case where the region is an infinite tube. In particular, we are interested in the influence of the curvature on the stability of the essential spectrum and the existence of discrete eigenvalues.

Let $s \mapsto \Gamma(s)$ be a unit-speed infinite curve in $\mathbb{R}^d$, $d \geq 2$. Assuming that the curve is C^d-smooth and possesses an appropriate C^1-smooth Frenet frame $\{e_1, \ldots, e_d\}$ (Assumption C), the i^{th} curvature κ_i of Γ, $i \in \{1, \ldots, d-1\}$, is a continuous function of the arc-length parameter $s \in \mathbb{R}$. Given a bounded open connected set ω in $\mathbb{R}^{d-1}$, we define the tube Ω of cross-section ω about Γ by

$$\Omega := \mathcal{L}(\mathbb{R} \times \omega)\,, \qquad \mathcal{L}(s, u_2, \ldots, u_d) := \Gamma(s) + u_\mu\, \mathcal{R}_{\mu\nu}(s)\, e_\nu(s)\,, \tag{7.1}$$

where μ, ν are summation indices taking values in $\{2, \ldots, d\}$ and $(\mathcal{R}_{\mu\nu})$ is a family of rotation matrices in $\mathbb{R}^{d-1}$ chosen in such a way that $(s, u_2, \ldots, u_d)$ are orthogonal "coordinates" in Ω (*cf* Section 7.2.2), *i.e.* ω rotates along Γ w.r.t. the Tang frame [15]. We make the hypotheses (Assumption D) that κ_1 is bounded, $\|\kappa_1\|_\infty \sup_{u\in\omega} |u| < 1$, and Ω does not overlap.

Our object of interest is the Dirichlet Laplacian associated with Ω, *i.e.*

$$-\Delta_D^\Omega \qquad \text{on} \qquad L^2(\Omega). \tag{7.2}$$

A physical motivation to study this operator for $d = 2, 3$ comes from the fact that it is (up to a physical constant) the quantum Hamiltonian of a free particle constrained to Ω, which is widely used to model the dynamics in mesoscopic systems called *quantum waveguides* [2, 12].

If Γ is a straight line (*i.e.* all $\kappa_i = 0$), then it is easy to see that the spectrum of (7.2) is purely absolutely continuous and equal to the interval $[\mu_1, \infty)$, where

$$\mu_1 \text{ is the first eigenvalue of the Dirichlet Laplacian in } \omega\,.$$

The purpose of the present paper is to prove that the essential spectrum of the Laplacian (7.2) is stable as a set under any curvature which vanishes at infinity, and that there is always a geometrically induced spectrum, *i.e.* the spectrum below μ_1, whenever the tube is non-trivially curved.

Theorem 7.1. *Let Ω be the infinite tube defined above.*

(i) *If* $\lim_{|s|\to\infty} \kappa_1(s) = 0$, *then* $\sigma_{\text{ess}}(-\Delta_D^\Omega) = [\mu_1, \infty)$;

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(ii) *If* $\kappa_1 \neq 0$, *then* $\inf\sigma(-\Delta_D^\Omega) < \mu_1$.

Consequently, if the tube is not straight but it is straight asymptotically, then $-\Delta_D^\Omega$ *has at least one eigenvalue of finite multiplicity below its essential spectrum,* i.e. $\sigma_{\rm disc}(-\Delta_D^\Omega) \neq \varnothing$.

Here the particularly interesting property is the existence of discrete spectrum, which is a non-trivial property for unbounded regions. From the physical point of view, one then deals with quantum *bound states* of the Hamiltonian (7.2), which are known to disturb the transport of the particle in the waveguide.

Spectral results of Theorem 7.1 were proved first by P. Exner and P. Šeba [5] in 1989 for planar strips (*i.e.* $d = 2$) under the additional assumptions that the strip width was sufficiently small and the curvature κ_1 was rapidly decaying at infinity (roughly speaking as $|s|^{-(\frac{3}{2}+\epsilon)}$). A few years later, J. Goldstone and R. L. Jaffe [7] proved the results without the restriction on the strip width, provided the curvature κ_1 had a compact support, and generalised it to the tubes of circular cross-sections in $\mathbb{R}^3$. References to other improvements can be found in the review article [2], where the second part of Theorem 7.1 was proved under stronger conditions for $d = 2, 3$ and circular cross-section. The first part was proved there just for compactly supported κ_1 (otherwise, under the additional hypothesis that also the first two derivatives of κ_1 vanished at infinity, the authors localised the threshold of the essential spectrum only). Let us also mention the paper [13] where a significant weakening of various regularity assumptions was achieved for $d = 2$. The first part of Theorem 7.1 was proved for $d = 2$ in the recent paper [10]. The present paper is devoted to a generalisation of the results to higher dimensions and cross-sections rotating along the curve w.r.t. the Tang frame.

Our strategy to prove Theorem 7.1 is explained briefly as follows. Introducing a diffeomorphism from Ω to the straight tube $\Omega_0 := \mathbb{R} \times \omega$ by means of the mapping $\mathcal{L} : \Omega_0 \to \Omega$, we transfer the (simple) Laplacian (7.2) on (complicated) Ω into a unitarily equivalent (complicated) operator H of the Laplace-Beltrami form on (simple) Ω_0, *cf* (7.15). This is the contents of the preliminary Section 7.2. The rest of the paper, Section 7.3, is devoted to the proof of Theorem 7.1. In Section 7.3.1, we employ a general characterisation of essential spectrum (Lemma 7.1) adopted from [1] in order to establish the first part of Theorem 7.1. The reader will notice that the characterisation is better than the classical Weyl criterion in the sense that it deals with quadratic forms instead of the associated operators themselves, *i.e.* we do not need to impose any condition on the derivatives of the coefficients of H in our case. The proof of the second part of Theorem 7.1 in Section 7.3.2 is based on the construction of an appropriate trial function for H inspired by the initial idea of [7].

Throughout this paper, we use the repeated indices convention with the range of Latin and Greek indices being $1, \ldots, d$ and $2, \ldots, d$, respectively. The partial derivative w.r.t. a coordinate x_i, $x \equiv (s, u_2, \ldots, u_d) \in \mathbb{R}^d$, is denoted by a comma with the index i.

7.2 Preliminaries

7.2.1 The reference curve

Given an integer $d \geq 2$, let $\Gamma : \mathbb{R} \to \mathbb{R}^d$ be a unit-speed C^d-smooth curve satisfying the following hypothesis.

Assumption C. Γ possesses a positively oriented Frenet frame $\{e_1, \ldots, e_d\}$ with the properties that

(i) $e_1 = \dot\Gamma$;

(ii) $\forall i \in \{1, \ldots, d\}, \quad e_i \in C^1(\mathbb{R}, \mathbb{R}^d)$;

(iii) $\forall i \in \{1, \ldots, d-1\}$, $\forall s \in \mathbb{R}, \quad \dot e_i(s)$ lies in the span of $e_1(s), \ldots, e_{i+1}(s)$.

Remark 7.1. We refer to [42, Sec. 1.2] for the notion of moving and Frenet frames. A sufficient condition to ensure the existence of the Frenet frame of Assumption C is to require that for all $s \in \mathbb{R}$, the vectors $\dot\Gamma(s), \Gamma^{(2)}(s), \ldots, \Gamma^{(d-1)}(s)$ are linearly independent, *cf* [42, Prop. 1.2.2]. This is always satisfied if $d = 2$. However, we do not assume *a priori* this non-degeneracy condition for $d \geq 3$ because it excludes the curves such that $\Gamma \restriction I$ lies in a lower-dimensional subspace of $\mathbb{R}^d$ for some open $I \subseteq \mathbb{R}$. We also refer to Remark 7.4 below for further discussions on Assumption C.

The properties of $\{e_1, \ldots, e_d\}$ summarised in Assumption C yield the Serret-Frenet formulae, *cf* [42, Sec. 1.3],

$$\dot e_i = \mathcal{K}_{ij}\, e_j\,, \tag{7.3}$$

where $\mathcal{K}_{ij}$ are coefficients of the skew-symmetric $d \times d$ matrix defined by

$$(\mathcal{K}_{ij}) := \begin{pmatrix} 0 & \kappa_1 & & 0 \\ -\kappa_1 & \ddots & \ddots & \\ & \ddots & \ddots & \kappa_{d-1} \\ 0 & & -\kappa_{d-1} & 0 \end{pmatrix}. \tag{7.4}$$

Here $\kappa_i : \mathbb{R} \to \mathbb{R}$ is called the i^{th} curvature of Γ. Under Assumption C, the curvatures are continuous functions of the arc-length parameter $s \in \mathbb{R}$.

7.2.2 The Tang frame

We introduce now another moving frame along Γ which better reflects the geometry of the curve and will be more convenient for our further purposes.

Let the $(d-1) \times (d-1)$ matrix $(\mathcal{R}_{\mu\nu})$ be defined by the system of differential equations

$$\dot{\mathcal{R}}_{\mu\nu} + \mathcal{R}_{\mu\rho}\, \mathcal{K}_{\rho\nu} = 0 \tag{7.5}$$

with the initial conditions that $(\mathcal{R}_{\mu\nu}(s_0))$ is a rotation matrix in $\mathbb{R}^{d-1}$ for some $s_0 \in \mathbb{R}$, *i.e.*,

$$\det(\mathcal{R}_{\mu\nu}(s_0)) = 1 \qquad \text{and} \qquad \mathcal{R}_{\mu\rho}(s_0)\, \mathcal{R}_{\nu\rho}(s_0) = \delta_{\mu\nu}. \tag{7.6}$$

Under our assumptions, the solution of (7.5) exists and is continuous by standard arguments in the theory of differential equations, *cf* [11, Sec. 4]. Furthermore, the conditions (7.6) are satisfied for *all* $s_0 \in \mathbb{R}$. Indeed, by means of Liouville's formula [11, Thm. 4.7.1] and $\mathrm{tr}(\mathcal{K}_{\mu\nu}) = 0$, one checks that $\det(\mathcal{R}_{\mu\nu}) = 1$ identically, while the validity of the second condition for all $s_0 \in \mathbb{R}$ is obtained via the skew-symmetry of $(\mathcal{K}_{ij})$:

$$(\mathcal{R}_{\mu\rho}\, \mathcal{R}_{\nu\rho})^{\cdot} = -\mathcal{R}_{\mu\rho}\, \mathcal{R}_{\nu\sigma}\, (\mathcal{K}_{\rho\sigma} + \mathcal{K}_{\sigma\rho}) = 0.$$

We set

$$(\mathcal{R}_{ij}) := \begin{pmatrix} 1 & 0 \\ 0 & (\mathcal{R}_{\mu\nu}) \end{pmatrix} \tag{7.7}$$

and define the moving frame $\{\tilde{e}_1, \dots, \tilde{e}_d\}$ along Γ by

$$\tilde{e}_i := \mathcal{R}_{ij}\, e_j. \tag{7.8}$$

Combining (7.3) with (7.5) and (7.4), one easily finds

$$\dot{\tilde{e}}_1 = \kappa_1\, e_2 \qquad \text{and} \qquad \dot{\tilde{e}}_\mu = -\kappa_1\, \mathcal{R}_{\mu 2}\, e_1. \tag{7.9}$$

We call the moving frame $\{\tilde{e}_1, \dots, \tilde{e}_d\}$ the *Tang frame* throughout this paper because it is a natural generalisation of the Tang frame known from the theory of three-dimensional waveguides [15]. Its advantage will be clear from the subsequent section.

7.2.3 Tubes

Given a bounded open connected set ω in $\mathbb{R}^{d-1}$, let Ω_0 denote the straight tube $\mathbb{R} \times \omega$. We define the curved tube Ω of the same cross-section ω about Γ as the image of the mapping, *cf* (7.1),

$$\mathcal{L} : \Omega_0 \to \mathbb{R}^d : \ \{(s,u) \mapsto \Gamma(s) + \tilde{e}_\mu(s)\, u_\mu\}, \tag{7.10}$$

i.e., $\Omega := \mathcal{L}(\Omega_0)$, where $u \equiv (u_2, \dots, u_d)$.

Our strategy to deal with the curved geometry of the tube is to identify Ω with the Riemannian manifold (Ω_0, g_{ij}), where (g_{ij}) is the metric tensor induced by $\mathcal{L}$, *i.e.* $g_{ij} := \mathcal{L}_{,i} \cdot \mathcal{L}_{,j}$, where "$\cdot$" denotes the inner product in $\mathbb{R}^d$. (In other words, we parameterise Ω globally by means of the "coordinates" (s,u).) To this aim, we need to impose a natural restriction on Ω in order to ensure that $\mathcal{L} : \Omega_0 \to \Omega$ is a C^1-diffeomorphism. Namely, defining

$$a := \sup_{u \in \omega} |u|, \tag{7.11}$$

where $|u| := \sqrt{u_\mu u_\mu}$, we make the hypothesis

Assumption D.

(i) $\kappa_1 \in L^\infty(\mathbb{R})$ and $a\,\|\kappa_1\|_\infty < 1$;

(ii) Ω does not overlap.

Using formulae (7.9), one easily finds

$$(g_{ij}) = \mathrm{diag}\,\big(h^2, 1, \ldots, 1\big) \qquad \text{with} \qquad h(s,u) := 1 - \kappa_1(s)\,\mathcal{R}_{\mu 2}(s)\,u_\mu\,. \tag{7.12}$$

By virtue of the inverse function theorem, the mapping $\mathcal{L} : \Omega_0 \to \Omega$ is a local C^1-diffeomorphism provided h does not vanish on Ω_0, which is guaranteed by the condition (i) of Assumption D because

$$0 < C_- \le h(s,u) \le C_+ < 2 \qquad \text{with} \quad C_\pm := 1 \pm a\,\|\kappa_1\|_\infty, \tag{7.13}$$

where we have used that $\mathcal{R}_{\mu 2}\,\mathcal{R}_{\mu 2} = 1$ by (7.6) and $\sqrt{u_\mu u_\mu} < a$ by (7.11). The mapping then becomes a global diffeomorphism if it is required to be injective in addition, *cf* the condition (ii) of Assumption D.

Remark 7.2. Formally, it is possible to consider (Ω_0, g_{ij}) as an abstract Riemannian manifold where only the curve Γ is embedded in $\mathbb{R}^d$. Then we do not need to assume the condition (ii) of Assumption D.

Note that the metric tensor (7.12) is diagonal due to our special choice of the "transverse" frame $\{\tilde{e}_2, \ldots, \tilde{e}_d\}$, which is the advantage of the Tang frame. At the same time, it should be stressed here that while the shape of the tube Ω is not influenced by a special choice of the rotation $(\mathcal{R}_{\mu\nu})$ provided ω is circular, this may not be longer true for a general cross-section. In this paper, we choose rotations determined by the Tang frame due to the technical simplicity.

We set $g := \det(g_{ij}) = h^2$, which defines through $d\mathrm{vol} := h(s,u)\,ds\,du$ the volume element of Ω; here du denotes the $(d-1)$-dimensional Lebesgue measure in ω.

Remark 7.3 (Low-dimensional examples)**.** When $d = 2$, the cross-section ω is just the interval $(-a, a)$, the curve Γ has only one curvature $\kappa_1 =: \kappa$, the rotation matrix $(\mathcal{R}_{\mu\nu})$ equals (the number) 1 and

$$h(s,u) = 1 - \kappa(s)\,u.$$

When $d = 3$, it is convenient to make the Ansatz

$$(\mathcal{R}_{\mu\nu}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where θ is a real-valued differentiable function. Then it is easy to see that (7.5) reduces to the differential equation $\dot{\theta} = \tau$, where τ is the torsion of Γ, *i.e.* one put $\kappa := \kappa_1$ and $\tau := \kappa_2$. Choosing θ as an integral of τ, we can write

$$h(s,u) = 1 - \kappa(s)\,(\cos\theta(s)\,u_2 + \sin\theta(s)\,u_3)\,.$$

Remark 7.4 (On Assumption C)**.** As pointed out by P. Exner [3], the existence of the Frenet frame required in Assumption C is rather a technical hypothesis only. Indeed, what we actually need is that $\mathcal{L} : \Omega_0 \to \Omega$, with $\mathcal{L}$ given by (7.10), is a C^1-diffeomorphism, and this is possible to ensure in certain situations even if Assumption C does not hold. To see this, let ω be circular and consider a curve Γ possessing the required Frenet frame on $(-\infty, 0)$ and $(0, \infty)$, $e_1 \in C^1(\{0\})$, but $e_\mu \notin C^0(\{0\})$ in the sense that $e_\mu(0+) = S_{\mu\nu} e_\nu(0-)$ for each $\mu \in \{2, \ldots, d\}$, where $(S_{\mu\nu}) \neq 1$ is a constant matrix satisfying relations analogous to (7.6), *i.e.* the transverse frames $\{e_2(0+), \ldots, e_d(0+)\}$ and $\{e_2(0-), \ldots, e_d(0-)\}$ are rotated to each other (see [14, Chap. 1, pp. 34] for an example of such a curve in $\mathbb{R}^3$). Since the rotation matrix $(R_{\mu\nu})$ is determined uniquely up to a multiplication by a constant rotation matrix, the Tang frame defined by (7.8) can be chosen to be continuous at zero by the requirement $R_{\mu\nu}(0+) S_{\nu\rho} = R_{\mu\rho}(0-)$. The C^1-continuity at zero then follows by (7.9) together with the fact that necessarily $\kappa_1(0) = 0$.

7.2.4 The Laplacian

Our strategy to investigate the Laplacian (7.2) is to express it in the coordinates determined by (7.10). More specifically, using the mapping (7.10), we identify the Hilbert space $L^2(\Omega)$ with $L^2(\Omega_0, d\mathrm{vol})$ and consider on the latter the sesquilinear form

$$Q(\psi, \phi) := \int_{\Omega_0} \overline{\psi_{,i}}\, g^{ij}\, \phi_{,j}\; d\mathrm{vol}\,, \qquad \psi, \phi \in \mathrm{Dom}\, Q := W_0^{1,2}(\Omega_0, d\mathrm{vol}), \tag{7.14}$$

where g^{ij} denotes the coefficients of the inverse of the metric tensor (7.12). The form Q is clearly densely defined, non-negative, symmetric and closed on its domain. Consequently, there exists a non-negative self-adjoint operator H associated with Q which satisfies $\mathrm{Dom}\, H \subset \mathrm{Dom}\, Q$. We have

$$\mathrm{Dom}\, H = \left\{ \psi \in W_0^{1,2}(\Omega_0, d\mathrm{vol}) \;\middle|\; \partial_i g^{\frac{1}{2}} g^{ij} \partial_j \psi \in L^2(\Omega_0, d\mathrm{vol}) \right\},$$
$$\forall \psi \in \mathrm{Dom}\, H, \quad H\psi = -g^{-\frac{1}{2}} \partial_i \big(g^{\frac{1}{2}} g^{ij} \partial_j \psi \big). \tag{7.15}$$

Actually, (7.15) is a general expression for the Laplace-Beltrami operator in a manifold equipped with a metric (g_{ij}). Using the particular form (7.12) of our metric tensor, we can write

$$H = -\frac{1}{h}\,\partial_1\,\frac{1}{h}\,\partial_1 - \partial_\mu\partial_\mu + \frac{\kappa_1\,\mathcal{R}_{\mu 2}}{h}\,\partial_\mu \tag{7.16}$$

in the form sense.

The norm and the inner product in the Hilbert space $L^2(\Omega_0, d\mathrm{vol})$ will be denoted by $\|\cdot\|_g$ and $(\cdot,\cdot)_g$, respectively. The usual notation without the subscript "g" will be reserved for the similar objects in $L^2(\Omega_0)$.

Remark 7.5 (Unitarily equivalent operator)**.** Assuming that the reference curve Γ is C^{d+1}-smooth, it is possible to "rewrite" H into a Schrödinger-type operator acting on the Hilbert space $L^2(\Omega_0)$, without the additional weight $g^{\frac{1}{2}}$ in the measure of integration. Indeed, defining $\hat{H} := UHU^{-1}$, where $U : \psi \mapsto g^{\frac{1}{4}}\psi$ is a unitary transformation from $L^2(\Omega_0, d\mathrm{vol})$ to $L^2(\Omega_0)$, we get

$$\hat{H} = -g^{-\frac{1}{4}} \partial_i g^{\frac{1}{2}} g^{ij} \partial_j g^{-\frac{1}{4}} \qquad \text{on} \quad L^2(\Omega_0)$$

in the form sense. Commuting then $g^{-\frac{1}{4}}$ with the gradient components, we can write

$$\hat{H} = -\partial_1\,\frac{1}{h^2}\,\partial_1 + \partial_\mu\partial_\mu + V \tag{7.17}$$

in the form sense, where

$$V := -\frac{5}{4}\,\frac{(h_{,1})^2}{h^4} + \frac{1}{2}\,\frac{h_{,11}}{h^3} - \frac{1}{4}\,\frac{h_{,\mu}\,h_{,\mu}}{h^2} + \frac{1}{2}\,\frac{h_{,\mu\mu}}{h} \tag{7.18}$$
$$= -\frac{1}{4}\,\frac{\kappa_1^2}{h^2} + \frac{1}{2}\,\frac{h_{,11}}{h^3} - \frac{5}{4}\,\frac{(h_{,1})^2}{h^4}\,. \tag{7.19}$$

Actually, (7.17) with (7.18) is a general formula valid for any C^1-smooth metric of the form

$$(g_{ij}) = \mathrm{diag}(h^2, 1, \dots, 1)\,.$$

(Note that the required regularity is indeed sufficient if the formula for the potential (7.18) is understood in the weak sense of forms.) In our special case when h is given by (7.12), we find easily that $h_{,\mu}(\cdot, u) = -\kappa_1\,\mathcal{R}_{\mu 2}$, $h_{,\mu\nu} = 0$, and (7.19) follows at once. Moreover, (7.5) gives

$$h_{,1}(\cdot, u) = u_\mu\,\mathcal{R}_{\mu\alpha}\big(\dot{\mathcal{K}}_{\alpha 1} - \mathcal{K}_{\alpha\beta}\mathcal{K}_{\beta 1}\big)\,,$$
$$h_{,11}(\cdot, u) = u_\mu\,\mathcal{R}_{\mu\alpha}\big(\ddot{\mathcal{K}}_{\alpha 1} - \dot{\mathcal{K}}_{\alpha\beta}\mathcal{K}_{\beta 1} - 2\mathcal{K}_{\alpha\beta}\dot{\mathcal{K}}_{\beta 1} + \mathcal{K}_{\alpha\beta}\mathcal{K}_{\beta\gamma}\mathcal{K}_{\gamma 1}\big)\,.$$

We shall neither need nor use the unitarily equivalent operator from the above remark, however, for motivation purposes, it is interesting to notice that the potential V becomes attractive if a is sufficiently small and the curvatures, together with some of their derivatives, vanish at infinity. Since the latter also implies that h tends to 1 as $a \to 0$, it is easy to see that $\hat{H}$ has always discrete eigenvalues below its essential spectrum for a small enough. In this paper, we prove this property under an asymptotic condition which involves the curvature κ_1 only (*cf* (7.22) below) and without any restriction on a (except for the natural one in Assumption D). We also note that various techniques from the theory of Schrödinger operators can be applied to $\hat{H}$, *cf* [2].

7.2.5 Straight tubes

If the tube is straight in the sense that each $\kappa_i = 0$, then the Laplacian (7.2) coincides with the decoupled operator

$$H_0 := \overline{-\Delta^{\mathbb{R}} \otimes 1 + 1 \otimes (-\Delta_D^\omega)} \qquad \text{on} \quad L^2(\mathbb{R}) \otimes L^2(\omega), \tag{7.20}$$

where 1 denotes the identity operator on the corresponding spaces and the bar stands for the closure. The operators $-\Delta^{\mathbb{R}}$ and $-\Delta_D^\omega$ denote the usual Laplacian on $L^2(\mathbb{R})$ and the Dirichlet Laplacian on $L^2(\omega)$, respectively. Alternatively, H_0 can be introduced as the operator associated with the form Q_0 defined by (7.14), where now the metric tensor is the identity matrix (δ_{ij}) and $d\mathrm{vol} = ds\,du$ is the Lebesgue measure in $\mathbb{R} \times \omega$.

The operator $-\Delta_D^\omega$ has a purely discrete spectrum consisting of eigenvalues $\mu_1 < \mu_2 \leq \dots \mu_n < \dots$; the corresponding eigenfunctions are denoted as $\mathcal{J}_n$ and we normalise them in such a way that $\|\mathcal{J}_n\|_{L^2(\omega)} = 1$. The lowest eigenvalue μ_1 is, of course, positive, simple and the eigenfunction $\mathcal{J}_1$ can be chosen positive.

In view of the decomposition (7.20),

$$\sigma(H_0) = \sigma_{\mathrm{ess}}(H_0) = [\mu_1, \infty) \tag{7.21}$$

and the spectrum is absolutely continuous.

7.3 Proofs

7.3.1 The essential spectrum

We prove that the essential spectrum of a curved tube Ω coincides with the one of Ω_0 provided the former is straight asymptotically in the sense that

$$\lim_{|s|\to\infty} \kappa_1(s) = 0. \tag{7.22}$$

Our method is based on the following characterisation of the essential spectrum of H.

Lemma 7.1. *$\lambda \in \sigma_{\mathrm{ess}}(H)$ if and only if there exists $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathrm{Dom}\, Q$ such that*

(i) $\forall n \in \mathbb{N}, \quad \|\psi_n\|_g = 1,$

(ii) $\forall n \in \mathbb{N}, \quad \mathrm{supp}\, \psi_n \subset \{(s,u) \in \Omega_0 \mid |s| \geq n\},$

(iii) $(H-\lambda)\psi_n \xrightarrow[n\to\infty]{} 0 \quad$ *in* $(\mathrm{Dom}\, Q)^*$.

Here $(\mathrm{Dom}\, Q)^*$ denotes the dual of the space $\mathrm{Dom}\, Q$ defined in (7.14). We note that $H+1 : \mathrm{Dom}\, Q \to (\mathrm{Dom}\, Q)^*$ is an isomorphism and

$$\|\psi\|_{-1,g} := \|\psi\|_{(\mathrm{Dom}\, Q)^*} = \sup_{\phi\in(\mathrm{Dom}\, Q)\setminus\{0\}} \frac{|(\phi,\psi)_g|}{\|\phi\|_{1,g}} \tag{7.23}$$

with

$$\|\phi\|_{1,g} := \|\phi\|_{\mathrm{Dom}\, Q} = \sqrt{Q[\phi] + \|\phi\|_g^2}\,.$$

The proof of the above lemma is quite similar to the proof of Lemma 4.2 in [1]. It is based on a general characterisation of essential spectrum, [11, Lemma 4.1], which is better than the Weyl criterion in the sense that the former requires to find a sequence from the form domain of H only (*cf* the statement of Lemma 7.1 and the required property (iii) with the Weyl criterion [63, Thm. 7.24]). The second property (ii) reflects the fact that the essential spectrum is determined by the geometry at infinity only.

Remark 7.6. Since the metric (g_{ij}) is uniformly elliptic due to (7.13), the norms in the spaces $L^2(\Omega_0, d\mathrm{vol})$ and $W_0^{1,2}(\Omega_0, d\mathrm{vol})$ are equivalent with those of $L^2(\Omega_0)$ and $W_0^{1,2}(\Omega_0)$, respectively, and the respective spaces can be identified as sets. In particular,

$$C_-\|\psi\|^2 \leq \|\psi\|_g^2 \leq C_+\|\psi\|^2,$$

and similarly for $\|\cdot\|_{1,g}$ and $\|\cdot\|_{-1,g}$.

Proof of Theorem 7.1, part (i). Let $\lambda \in \sigma_{\mathrm{ess}}(H_0) \equiv [\mu_1, \infty)$. By Lemma 7.1, there exists a sequence $\{\tilde\psi_n\}_{n\in\mathbb{N}} \subset \mathrm{Dom}\, Q_0$ such that it satisfies the properties (i)–(iii) of the Lemma for $g_{ij} = \delta_{ij}$, $H = H_0$ and $Q = Q_0$. We will show that the sequence $\{\psi_n\}_{n\in\mathbb{N}}$ defined by $\psi_n := \tilde\psi_n/\|\tilde\psi_n\|_g$ for every $n \in \mathbb{N}$ satisfies the properties (i)–(iii) of Lemma 7.1 for g_{ij}, H and Q, *i.e.* $\sigma_{\mathrm{ess}}(H_0) \subseteq \sigma_{\mathrm{ess}}(H)$. First of all, notice that ψ_n is well defined and belongs to $\mathrm{Dom}\, Q$ for every $n \in \mathbb{N}$ due to Remark 7.6. Moreover, writing

$$\tilde\psi_n = (1+H_0)^{-1}(H_0-\lambda)\tilde\psi_n + (1+H_0)^{-1}(\lambda+1)\tilde\psi_n\,,$$

we see that the sequence $\{\tilde\psi_n\}_{n\in\mathbb{N}}$ is bounded in $\mathrm{Dom}\, Q_0$, and therefore $\{\psi_n\}_{n\in\mathbb{N}}$ is bounded in $\mathrm{Dom}\, Q$ by Remark 7.6. Since the conditions (i) and (ii) hold trivially true for $\{\psi_n\}_{n\in\mathbb{N}}$, it remains to check the third one. By the definition of H, *cf* (7.14), we can write

$$\big(\phi, (H-\lambda)\psi_n\big)_g = \big(\phi, (H_0-\lambda)\psi_n\big) + \big(\phi_{,i}, (g^{\frac12}g^{ij} - \delta^{ij})\psi_{n,j}\big) - \lambda\big(\phi, (g^{\frac12}-1)\psi_n\big)$$

for every $\phi \in \mathrm{Dom}\, Q$. The Minkowski inequality, the formula (7.23), the fact that $(H_0 - \lambda)\tilde{\psi}_n \to 0$ in $(\mathrm{Dom}\, Q_0)^*$ as $n \to \infty$, and a repeated use of Remark 7.6 yield that it is enough to show that

$$\sup_{\phi \in (\mathrm{Dom}\, Q)\setminus\{0\}} \frac{\left|\left(\phi_{,i}, (g^{ij} - g^{-\frac{1}{2}}\delta^{ij})\psi_{n,j}\right)_g\right| + \lambda\left|\left(\phi, (1 - g^{-\frac{1}{2}})\psi_n\right)_g\right|}{\|\phi\|_{1,g}} \xrightarrow[n\to\infty]{} 0.$$

However, the latter is easily established by means of the Schwarz inequality, the estimates $\|\phi_{,i}\|_g, \|\phi\|_g \le \|\phi\|_{1,g}$, the fact that $\{\psi_n\}_{n\in\mathbb{N}}$ is bounded in $\mathrm{Dom}\, Q$, and the expression for the metric (7.12) together with (7.22) and the property (ii) of Lemma 7.1.

One proves that $\sigma_{\mathrm{ess}}(H) \subseteq \sigma_{\mathrm{ess}}(H_0)$ in the same way. □

Remark 7.7. It is clear from the previous proof that a stronger result than the first part of Theorem 7.1 can be proved. If h and $\tilde{h}$ are two positive functions (determining through (7.12) two tube metrics (g_{ij}) and $(\tilde{g}_{ij})$, respectively) such that $\sup_{u\in\omega} |h(s,u) - \tilde{h}(s,u)| \to 0$ as $|s| \to \infty$, then the essential spectra of the corresponding operators H and $\tilde{H}$ (given by (7.15) with (g_{ij}) and $(\tilde{g}_{ij})$, respectively) coincide as sets.

Let us finally notice that a detailed study of the *nature* of the essential spectrum in curved tubes has been performed in [9]; in particular, the absence of singular continuous spectrum is proved there under suitable assumptions about the decay of curvature at infinity.

7.3.2 The geometrically induced spectrum

In this section we show that $\inf \sigma(H) < \mu_1$ whenever $\kappa_1 \neq 0$, *i.e.* there is always a spectrum below the energy μ_1 in non-trivially curved tubes Ω. We call it geometrically induced spectrum because it does not exist for the straight tube Ω_0, *cf* (7.21). Furthermore, it follows by the part (i) of Theorem 7.1 that this geometrically induced spectrum is discrete if we suppose (7.22) in addition.

Our proof is based on the variational strategy of finding a trial function Ψ from the form domain of H such that

$$Q_1[\Psi] := Q[\Psi] - \mu_1 \|\Psi\|_g^2 < 0. \tag{7.24}$$

The construction of such a Ψ follows the initial idea of [7] and the subsequent improvements of [13] and [14, Thm. 2.1].

Proof of Theorem 7.1, part (ii). Let $\{\Psi_n\}_{n\in\mathbb{N}} \subset \mathrm{Dom}\, Q$ and $\Phi \in \mathrm{Dom}\, Q$. Defining $\Psi_{n,\varepsilon} := \Psi_n + \varepsilon\,\Phi$ for every $(n,\varepsilon) \in \mathbb{N}\times\mathbb{R}$, we can write

$$Q_1[\Psi_{n,\varepsilon}] = Q_1[\Psi_n] + 2\varepsilon\, Q_1(\Phi, \Psi_n) + \varepsilon^2 Q_1[\Phi].$$

Our strategy will be to choose $\{\Psi_n\}_{n\in\mathbb{N}}$ and Φ so that

$$\lim_{n\to\infty} Q_1[\Psi_n] = 0 \qquad \text{and} \qquad \lim_{n\to\infty} Q_1(\Phi, \Psi_n) \neq 0. \tag{7.25}$$

Then we can choose a sufficiently large $n \in \mathbb{N}$ and a sufficiently small $\varepsilon \in \mathbb{R}$ with a suitable sign so that $Q_1[\Psi_{n,\varepsilon}] < 0$, which proves the claim.

We put $\Psi_n := \varphi_n \otimes \mathcal{J}_1$, where $\mathcal{J}_1$ is the first eigenfunction of $-\Delta_D^\omega$, *cf* Section 7.2.5, and $\{\varphi_n\}_{n\in\mathbb{N}}$ is a mollifier of 1 in $W^{1,2}(\mathbb{R})$, *i.e.* a family of functions φ_n from $W^{1,2}(\mathbb{R})$ satisfying:

(i) $\forall n \in \mathbb{N}, \quad 0 \le \varphi_n \le 1,$

(ii) $\varphi_n(s) \xrightarrow[n\to\infty]{} 1 \quad$ for a.e. $s \in \mathbb{R}$,

(iii) $\|\dot{\varphi}_n\|_{L^2(\mathbb{R})} \xrightarrow[n\to\infty]{} 0.$

(Probably the simplest example of such a family is given by the continuous even φ_n's such that they are equal to 1 on $[0,n)$, with a constant derivative on $[n, 2n+1)$, and equal to 0 on $[2n+1, \infty)$.) Using the expression (7.16) for the Laplacian and the fact that $(\partial_\mu\partial_\mu + \mu_1)\mathcal{J}_1 = 0$, we obtain immediately that

$$Q_1[\Psi_n] = \left(\Psi_{n,1}, h^{-1}\Psi_{n,1}\right) + \left(\Psi_n, \kappa_1 \mathcal{R}_{\mu 2}\Psi_{n,\mu}\right).$$

The second term at the r.h.s. is equal to zero by an integration by parts, while the first (positive) one can be estimated from above by $C_-^{-1}\|\Psi_{n,1}\|^2$ due to (7.13). Since $\|\Psi_{n,1}\| = \|\dot{\varphi}_n\|_{L^2(\mathbb{R})}$ by the normalisation of $\mathcal{J}_1$, we verify the first property of (7.25).

The second property is checked if we take, for instance,

$$\Phi(s,u) := \phi(s)\,\mathcal{R}_{\mu 2}(s)\,u_\mu\,\mathcal{J}_1(u) \;\in \mathrm{Dom}\,Q,$$

where $\phi \in W^{1,2}(\mathbb{R}) \setminus \{0\}$ is a non-negative function with a compact support contained in an interval where κ_1 is not zero and does not change sign (such an interval surely exists because $\kappa_1 \neq 0$ is continuous). Indeed, in the same way as above, we find

$$Q_1(\Phi,\Psi_n) = \big(\Phi_{,1}, h^{-1}\Psi_{n,1}\big) + \big(\Phi, \kappa_1 \mathcal{R}_{\mu 2}\Psi_{n,\mu}\big)\,,$$

where the first term at the r.h.s. tends to zero as $n \to \infty$ because its absolute value can be estimated by $C_-^{-1}\|\Phi_{,1}\|\|\Psi_{n,1}\|$, while the second one is equal to

$$-\tfrac{1}{2}\big(\phi\,\mathcal{J}_1, \kappa_1\mathcal{R}_{\mu 2}\mathcal{R}_{\mu 2}\,\varphi_n\,\mathcal{J}_1\big) = -\tfrac{1}{2}\,(\phi, \kappa_1\varphi_n)_{L^2(\mathbb{R})}$$

by an integration by parts; the last identity then holds due to (7.6) and the normalisation of $\mathcal{J}_1$. Summing up, we conclude that

$$\lim_{n\to\infty} Q_1(\Phi,\Psi_n) = -\frac{1}{2}\int_{\mathrm{supp}\,\phi} \phi(s)\,\kappa_1(s)\,ds \neq 0$$

by the dominated convergence theorem. □

Remark 7.8. Suppose Assumptions C and D. If the tube Ω is non-trivially curved and asymptotically straight, it follows by Theorem 7.1 that $\sigma_{\mathrm{disc}}(-\Delta_D^\Omega) \subset [0,\mu_1)$ and it is not empty. Furthermore, it can be shown by standard arguments (see, *e.g.*, [29, Sec. 8.12]) that the minimum eigenvalue, *i.e.* $\inf\sigma(-\Delta_D^\Omega)$, is simple and has a positive eigenfunction. One also has $\inf\sigma(-\Delta_D^\Omega) > 0$. (Actually, a stronger lower bound to the spectral threshold has been derived in [4].)

Acknowledgements

The authors wish to thank Pavel Exner for useful discussions and Remark 7.4. This work was partially supported by FCT/POCTI/FEDER, Portugal, and GA AS CR grant IAA 1048101.

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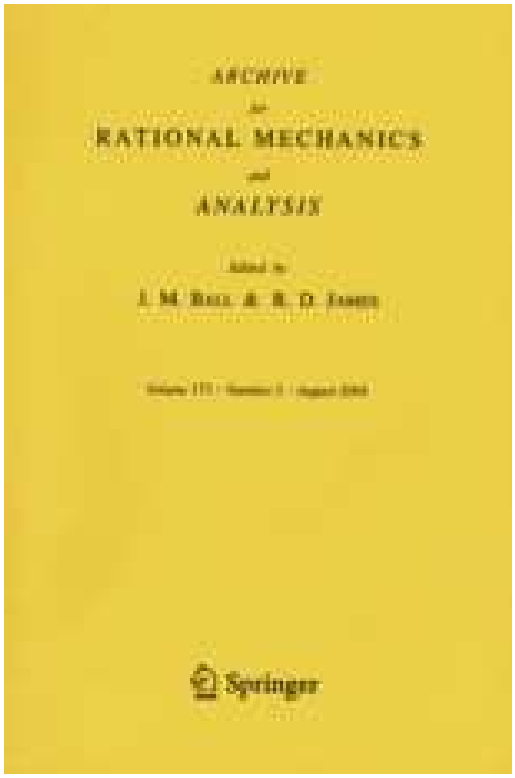
Chapter 8

A Hardy inequality in twisted waveguides



Published in: Archive for Rational Mechanics and Analysis **188** (2008), 245–264

Joint work with: Tomas Ekholm, Hynek Kovařík

A Hardy inequality in twisted waveguides

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Abstract. We show that twisting of an infinite straight three-dimensional tube with non-circular cross-section gives rise to a Hardy-type inequality for the associated Dirichlet Laplacian. As an application we prove certain stability of the spectrum of the Dirichlet Laplacian in locally and mildly bent tubes. Namely, it is known that any local bending, no matter how small, generates eigenvalues below the essential spectrum of the Laplacian in the tubes with arbitrary cross-sections rotated along a reference curve in an appropriate way. In the present paper we show that for any other rotation some critical strength of the bending is needed in order to induce a non-empty discrete spectrum.

8.1 Introduction

The Dirichlet Laplacian in infinite tubular domains has been intensively studied as a model for the Hamiltonian of a non-relativistic particle in *quantum waveguides*; we refer to [6, 16, 13] for the physical background and references. Among a variety of results established so far, let us point out the papers [9, 10, 17, 6, 15, 5] where the existence of *bound states* generated by a local *bending* of a straight waveguide is proved. This is an interesting phenomenon for several reasons. From the physical point of view, one deals with a geometrically induced effect of purely quantum origin, with important consequences for the transport in curved *nanostructures*. Mathematically, the tubes represent a class of *quasi-cylindrical* domains for which the spectral results of this type are non-trivial.

More specifically, it has been proved in the references mentioned above that the Dirichlet Laplacian in non-self-intersecting tubular neighborhoods of the form

$$\{x \in \mathbb{R}^d \mid \mathrm{dist}(x, \Gamma) < a\}\,, \qquad d \geq 2\,, \tag{8.1}$$

where a is a positive number and Γ is an infinite curve of non-trivial curvature vanishing at infinity, always possesses discrete eigenvalues. On the other hand, the essential spectrum coincides as a set with the spectrum of the straight tube of radius a. In other words, the spectrum of the Laplacian is unstable under bending. The bound states may be generated also by other local deformations of straight waveguides, *e.g.*, by a adding a "bump" [4, 2, 7].

On the other hand, the first two authors of this paper have shown recently in [7] (see also [1]) that a presence of an appropriate local magnetic field in a 2-dimensional waveguide leads to the existence of a Hardy-type inequality for the corresponding Hamiltonian. Consequently, the spectrum of the magnetic Schrödinger operator becomes stable as a set against sufficiently weak perturbations of the type considered above.

In this paper we show that in tubes with non-circular cross-sections the same stability effect can be achieved by a purely geometrical deformation which preserves the shape of the cross-section: *twisting*. We restrict to $d = 3$ and replace the definition (8.1) by a tube obtained by translating an arbitrary cross-section along a reference curve Γ according to a smooth *moving frame* of Γ (*i.e.* the triad of the tangent and any two normal vectors perpendicular to each other), see Figure 1.3. We say that the tube is *twisted* provided (i) the cross-section is not rotationally symmetric (*cf* (8.8) below) and (ii) the projection of the derivative of one normal vector of the moving frame to the other one is not zero. The second condition can be expressed solely in terms of the difference between the second curvature (also called torsion) of Γ and the derivative of the angle between the normal vectors of the chosen moving frame and a Frenet frame of Γ (*cf* (8.14) below); the latter determines certain rotations of the cross-section along the curve. That is, twisting and bending may be viewed as two independent deformations of a straight tube. In order to describe the main results of the paper, we distinguish two particular types of twisting.

First, when Γ is a straight line, then of course the curvatures are zero and the twisting comes only from rotations of a non-circular cross-section along the line. In this situation, we establish Theorem 8.3 containing a Hardy-type inequality for the Dirichlet Laplacian in a straight locally twisted tube. Roughly speaking, this tells us that a local twisting stabilizes the transport in straight tubes with non-circular cross-sections.

Second, when Γ is curved, the torsion is in general non-zero and we show that it plays the same role as the twisting due to the rotations of a non-circular cross-section in the twisted straight case. More specifically, we use Theorem 8.3 to establish Theorem 8.1 saying that the spectrum of the Dirichlet Laplacian in a twisted, mildly and locally bent tubes coincides with the spectrum of a straight tube, which is purely essential. This fact has important consequences. For it has been proved in [5] that any non-trivial curvature vanishing at infinity generates eigenvalues below the essential spectrum, provided the cross-section is translated along Γ according to the so-called *Tang frame* (*cf* (8.13) below). We also refer to [12] for analogous results in mildly curved tubes. But the choice of the Tang frame for the moving frame giving rise to the tube means that the rotation of the cross-section compensates the torsion. Our Theorem 8.1 shows that this special rotation is the only possible choice for which the discrete eigenvalues appear for any non-zero curvature of Γ; any other rotation of the cross-section will eliminate the discrete eigenvalues if the curvature is not strong enough. In the curved case, we also establish Theorem 8.2 extending the result of Theorem 8.1 to the case when also the torsion is mild.

After submission of this paper, two other works related to the present topic appeared. First, Grushin has in [12] a result similar to our Theorem 8.1; namely, using a perturbation technique developed in [11], he proves that there are no discrete eigenvalues in tubes which are simultaneously mildly curved and mildly twisted. Second, by private communication we learned about the result of Bouchitté, Mascarenhas and Trabucho, [3], who demonstrate the repulsive effect of twisting in bounded tubes by deriving the asymptotics of the eigenvalues as the thickness of the tube cross-section goes to zero; in the limit they discover an effective potential which has a positive part if and only if the twisting is present in our setting. We would like to stress that, apart from the different method we use, the importance of our results lies in the fact that the non-existence of discrete spectrum follows as a consequence of a stronger property: the Hardy-type inequality of Theorem 8.3, and that we are not restricted to thin tubes.

The organization of the paper is as follows. In the following Section 8.2, we present our main results; namely, the Hardy-type inequality (Theorem 8.3) and the stability result concerning the spectrum in twisted mildly bent tubes (Theorems 8.1 and 8.2). The Hardy-type inequality and its local version (Theorems 8.3 and 8.4, respectively) are proved in Section 8.3. In order to deal with the Laplacian in a twisted bent tube, we have to develop certain geometric preliminaries; this is done in Section 8.4. Theorems 8.1 and 8.2 are proved at the end of Section 8.4. In the Appendix, we state a sufficient condition which guarantees that a twisted bent tube does not intersect.

The summation convention is adopted throughout the paper and, if not otherwise stated, the range of Latin and Greek indices is assumed to be $1,2,3$ and $2,3$, respectively. The indices θ and τ are reserved for a function and a vector, respectively, and are excluded from the summation convention. If U is an open set, we denote by $-\Delta_D^U$ the Dirichlet Laplacian in U, *i.e.* the self-adjoint operator associated in $L^2(U)$ with the quadratic form Q_D^U defined by $Q_D^U[\psi] := \int_U |\nabla\psi|^2$, $\psi \in \mathfrak{D}(Q_D^U) := \mathcal{H}_0^1(U)$.

8.2 Main results

8.2.1 Twisted bent tubes

The tubes we consider in the present paper are determined by a *reference curve* Γ, a *cross-section* ω and an *angle* function θ determining a moving frame of Γ. We restrict ourselves to curves characterized by their curvature functions.

Let κ_1 and κ_2 be C^1-smooth functions on $\mathbb{R}$ satisfying

$$\kappa_1 > 0 \quad \text{on} \quad I \qquad \text{and} \qquad \kappa_1, \kappa_2 = 0 \quad \text{on} \quad \mathbb{R} \setminus I\,, \tag{8.2}$$

where I is some fixed bounded open interval. Then there exists a unit-speed (*i.e.* parameterised by arc length) C^3-smooth curve $\Gamma : \mathbb{R} \to \mathbb{R}^3$ whose first and second curvature functions are κ_1 and κ_2, respectively; Γ is uniquely determined up to congruent transformations. More precisely, the restriction $\Gamma \restriction I$ can be reconstructed from κ_1 and κ_2 by means of a standard procedure (*cf* [42, Thm. 1.3.6]) and it possesses a uniquely determined C^2-smooth distinguished Frenet frame $\{e_1, e_2, e_3\}$. Since κ_1 and κ_2 vanish outside I, the complement of $\Gamma \restriction I$ is formed by two straight semi-infinite lines and we can extend the triad $\{e_1, e_2, e_3\}$ to a global C^2-smooth Frenet frame of Γ. The latter, also denoted by $\{e_1, e_2, e_3\}$, satisfies the Serret-Frenet equations (*cf* [42, Sec. 1.3]) on Γ:

$$\dot{e}_i = \mathcal{K}_{ij}\, e_j\,, \qquad i \in \{1,2,3\}\,, \tag{8.3}$$

where the matrix-valued function $(\mathcal{K}_{ij})$ has the skew-symmetric form

$$(\mathcal{K}_{ij}) = \begin{pmatrix} 0 & \kappa_1 & 0 \\ -\kappa_1 & 0 & \kappa_2 \\ 0 & -\kappa_2 & 0 \end{pmatrix}. \tag{8.4}$$

Equation (8.3) can be viewed as the governing equations defining the global Frenet frame and therefore the curve through $e_1 = \dot{\Gamma}$. The components e_1, e_2 and e_3 are the tangent, normal and binormal vectors of Γ, respectively, and κ_2 is sometimes called the torsion (then κ_1 is simply called the curvature).

Given a C^1_0-smooth function θ on $\mathbb{R}$, we define the matrix valued function

$$(\mathcal{R}^\theta_{\mu\nu}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{8.5}$$

(recall the convention for indices from the end of Introduction). Then the triad $\{e_1, \mathcal{R}^\theta_{2\nu} e_\nu, \mathcal{R}^\theta_{3\nu} e_\nu\}$ defines a C^1-smooth moving frame of Γ having normal vectors rotated by the angle $\theta(s)$ with respect to the Frenet frame at $s \in \mathbb{R}$. Later on, a stronger regularity of θ will be required, namely,

$$\ddot{\theta} \in L^\infty(\mathbb{R})\,. \tag{8.6}$$

Let ω be a bounded open connected set in $\mathbb{R}^2$ and introduce the quantity

$$a := \sup_{t\in\omega} |t|\,. \tag{8.7}$$

We assume that ω is not rotationally invariant with respect to the origin, *i.e.*,

$$\exists \alpha \in (0, 2\pi)\,, \qquad \big\{ \big(t_\mu \mathcal{R}^\alpha_{\mu 2}, t_\mu \mathcal{R}^\alpha_{\mu 3}\big) \mid (t_2, t_3) \in \omega \big\} \neq \omega\,. \tag{8.8}$$

We define a *twisted bent tube* Ω about Γ as the image

$$\Omega := \mathcal{L}(\mathbb{R} \times \omega)\,, \tag{8.9}$$

where $\mathcal{L}$ is the mapping from $\mathbb{R} \times \omega$ to $\mathbb{R}^3$ defined by

$$\mathcal{L}(s,t) := \Gamma(s) + t_\mu\, \mathcal{R}^\theta_{\mu\nu}(s)\, e_\nu(s)\,. \tag{8.10}$$

We make the natural hypotheses that

$$a\, \|\kappa_1\|_\infty < 1 \qquad \text{and} \qquad \mathcal{L} \text{ is injective}\,, \tag{8.11}$$

so that Ω has indeed the geometrical meaning of a non-self-intersecting tube; sufficient conditions ensuring the injectivity of $\mathcal{L}$ are derived in the Appendix.

Our object of interest is the Dirichlet Laplacian in the tube, $-\Delta^\Omega_D$. In the simplest case when the tube is straight (*i.e.* $I = \varnothing$) and the cross-section ω is not rotated with respect to a Frenet frame of the reference straight line (*i.e.* $\dot{\theta} = 0$), it is easy to locate the spectrum:

$$\operatorname{spec}(-\Delta^{\mathbb{R}\times\omega}_D) = [E_1, \infty)\,, \tag{8.12}$$

where E_1 is the lowest eigenvalue of the Dirichlet Laplacian in ω.

Sufficient conditions for the existence of a discrete spectrum of $-\Delta^\Omega_D$ were recently obtained in [5, 12]. In particular, it is known from [5] that if the cross-section ω is rotated appropriately, namely in such a way that

$$\dot{\theta} = \kappa_2\,, \tag{8.13}$$

then any non-trivial bending (*i.e.* $I \neq \varnothing$) generates eigenvalues below E_1, while the essential spectrum is unchanged.

As one of the main results of the present paper we show that condition (8.13) is necessary for the existence of discrete spectrum in *mildly* bent tubes with non-circular cross-sections:

Theorem 8.1. *Given C^1_0-curvature functions* (8.2)*, a bounded open connected set $\omega \subset \mathbb{R}^2$ satisfying non-symmetricity condition* (8.8) *and a C^1_0-smooth angle function θ satisfying* (8.6)*, let Ω be the tube as above satisfying* (8.11)*. If*

$$\kappa_2 - \dot{\theta} \neq 0\,, \tag{8.14}$$

then there exists a positive number ε such that

$$\|\kappa_1\|_\infty + \|\dot{\kappa}_1\|_\infty \leq \varepsilon \quad \Longrightarrow \quad \operatorname{spec}(-\Delta^\Omega_D) = [E_1, \infty)\,.$$

Here ε depends on κ_2, $\dot{\theta}$ and ω.

An explicit lower bound for the constant ε is given by the estimates made in Section 8.4.3 when proving Theorem 8.1; we also refer to Proposition 8.1 in the Appendix for a sufficient conditions ensuring the validity of (8.11).

Theorem 8.1 tells us that twisting, induced either by torsion or by a rotation different from (8.13), acts against the attractive interaction induced by bending. Its proof is based on a Hardy-type inequality in straight tubes presented in the following Section 8.2.2. The latter provides other variants of Theorem 8.1, *e.g.*, in the situation when also the torsion is mild:

Theorem 8.2. *Under the hypotheses of Theorem 8.1, with* (8.14) *being replaced by*

$$\dot\theta \neq 0\,, \tag{8.15}$$

there exists a positive number ε such that

$$\|\kappa_1\|_\infty + \|\dot\kappa_1\|_\infty + \|\kappa_2\|_\infty \leq \varepsilon \quad \Longrightarrow \quad \mathrm{spec}\,(-\Delta_D^\Omega) = [E_1, \infty)\,.$$

Here ε depends on $\dot\theta$, ω and I.

We refer the reader to Section 8.5 for more comments on Theorems 8.1 and 8.2.

8.2.2 Twisted straight tubes

The proof of Theorems 8.1 and 8.2 is based on the fact that a twist of a straight tube leads to a Hardy-type inequality for the corresponding Dirichlet Laplacian. This is the central idea of the present paper which is of independent interest.

By the *straight tube* we mean the product set $\mathbb{R}\times\omega$. To any radial vector $t \equiv (t_2, t_3) \in \mathbb{R}^2$, we associate the normal vector $\tau(t) := (t_3, -t_2)$, introduce the angular-derivative operator

$$\partial_\tau := t_3\,\partial_2 - t_2\,\partial_3 \tag{8.16}$$

and use the same symbol for the differential expression $1\otimes\partial_\tau$ on $\mathbb{R}\times\omega$.

Given a bounded function $\sigma : \mathbb{R}\to\mathbb{R}$, we denote by the same letter the function $\sigma\otimes 1$ on $\mathbb{R}\times\omega$ and consider the self-adjoint operator L_σ in $L^2(\mathbb{R}\times\omega)$ associated with the Dirichlet quadratic form

$$l_\sigma[\psi] := \|\partial_1\psi - \sigma\,\partial_\tau\psi\|^2 + \|\partial_2\psi\|^2 + \|\partial_3\psi\|^2\,, \tag{8.17}$$

with $\psi \in \mathfrak{D}(l_\sigma) := \mathcal{H}_0^1(\mathbb{R}\times\omega)$, where $\|\cdot\|$ denotes the norm in $L^2(\mathbb{R}\times\omega)$. Notice that the spectrum of L_σ does not start below E_1 due to the basic inequality

$$\|\nabla\varphi\|^2_{L^2(\omega)} \;\geq\; E_1\,\|\varphi\|^2_{L^2(\omega)}\,, \qquad \forall\varphi\in\mathcal{H}_0^1(\omega)\,. \tag{8.18}$$

The connection between L_σ and a *twisted straight tube* is based on the fact that for $\sigma = \dot\theta$, L_σ is unitarily equivalent to the Dirichlet Laplacian acting in a tube given by (8.9) for the choice $\Gamma(s) = (s, 0, 0)$, after passing to the coordinates determined by (8.10). This can be verified by a straightforward calculation.

If $\sigma = 0$, L_0 is just the Dirichlet Laplacian in $\mathbb{R}\times\omega$, its spectrum is given by (8.12) and there is no Hardy inequality associated with the shifted operator $L_0 - E_1$. The latter means that given any multiplication operator V generated by a non-zero, non-positive function from $C_0^\infty(\mathbb{R}\times\omega)$, the operator $L_0 - E_1 + V$ has a negative eigenvalue. This is also true for non-trivial σ in the case of circular ω centered in the origin of $\mathbb{R}^2$, since then the angular-derivative term in (8.17) vanishes for the test functions of the form $\varphi\otimes\mathcal{J}$ on $\mathbb{R}\times\omega$, where $\mathcal{J}$ is an eigenfunction of the Dirichlet Laplacian corresponding to E_1. However, in all other situations there is always a Hardy-type inequality:

Theorem 8.3. *Let ω be a bounded open connected subset of $\mathbb{R}^2$ satisfying the non-symmetricity condition* (8.8)*. Let σ be a compactly supported continuous function with bounded derivatives and suppose that σ is not identically zero. Then, for all $\psi\in\mathcal{H}_0^1(\mathbb{R}\times\omega)$ and any s_0 such that $\sigma(s_0)\neq 0$ we have*

$$l_\sigma[\psi] - E_1\,\|\psi\|^2 \;\geq\; c_h \int_{\mathbb{R}\times\omega} \frac{|\psi(s,t)|^2}{1+(s-s_0)^2}\;ds\,dt\,, \tag{8.19}$$

where c_h is a positive constant independent of ψ but depending on s_0, σ and ω.

It is possible to find an explicit lower bound for the constant c_h; we give an estimate in (8.28). The particular kind of Hardy weight in (8.19) is due to the classical one-dimensional Hardy inequality (8.27) used in the proof of Theorem 8.3.

The assumption that σ has a compact support ensures that the essential spectrum of L_σ coincides with (8.12). As a consequence of the Hardy-type inequality (8.19), we get that the presence of a non-trivial σ in (8.17) represents a repulsive interaction in the sense that there is no other spectrum for all small potential-type perturbations having $\mathcal{O}(s^{-2})$ decay at infinity.

As explained above, the special choice $\sigma = \dot\theta$ leads to a direct geometric interpretation of L_σ in connection with the twisted straight tubes. As another application of Theorem 8.3, we shall apply it to the twisted bent tubes, namely, with the choice $\sigma = \kappa_2 - \dot\theta$ to prove Theorem 8.1 and with $\sigma = \dot\theta$ to prove Theorem 8.2 (*cf* Section 8.4.3). Here the main idea is to pass to the curvilinear coordinates induced by (8.10) in which the Laplacian $-\Delta_D^\Omega$ becomes L_σ plus an explicit (second-order) perturbation. The quadratic form of the perturbation is not of definite sign but it vanishes either if the C^1-norm of κ_1 tends to zero for $\sigma = \kappa_2 - \dot\theta$ or if both the C^1-norm of κ_1 and the supremum norm of κ_2 tend to zero for $\sigma = \dot\theta$. Hence the proofs of Theorems 8.1 and 8.2 reduce to an algebraic comparison of quadratic forms, the main trouble being the second order of the perturbation.

8.3 Hardy inequality for twisted straight tubes

In this section, we establish Theorem 8.3 in two steps. After certain preliminaries, we first derive a "local" Hardy inequality (Theorem 8.4). Then the local result is "smeared out" by means of a classical one-dimensional Hardy inequality.

8.3.1 Preliminaries

Definition 8.1. To any $\omega \subset \mathbb{R}^2$, we associate the number

$$\lambda := \inf \frac{\|\nabla\varphi\|^2_{L^2(\omega)} - E_1 \|\varphi\|^2_{L^2(\omega)} + \left\|\partial_\tau\varphi\right\|^2_{L^2(\omega)}}{\|\varphi\|^2_{L^2(\omega)}},$$

where the infimum is taken over all non-zero functions from $\mathcal{H}^1_0(\omega)$.

It is clear from (8.18) that λ is a non-negative quantity. Our Hardy inequality is based on the fact that λ is always positive for non-circular cross-sections.

Lemma 8.1. *If ω satisfies* (8.8)*, then $\lambda > 0$.*

Proof. The quadratic form b defined on $L^2(\omega)$ by

$$b[\varphi] := \|\nabla\varphi\|^2_{L^2(\omega)} - E_1 \|\varphi\|^2_{L^2(\omega)} + \left\|\partial_\tau\varphi\right\|^2_{L^2(\omega)}, \quad \varphi \in \mathfrak{D}(b) := \mathcal{H}^1_0(\omega),$$

is non-negative (*cf* (8.18)), densely defined and closed; the last two statements follow from the boundedness of τ and from the fact that they hold true for the quadratic form defining the Dirichlet Laplacian in ω. Consequently, b gives rise to a self-adjoint operator B. Moreover, since $B \geq -\Delta_D^\omega - E_1$, and the spectrum of $-\Delta_D^\omega$ is purely discrete, the minimax principle implies that B has a purely discrete spectrum, too. λ is clearly the lowest eigenvalue of B. Assume that $\lambda = 0$. Then, firstly, the ground state φ of B and $-\Delta_D^\omega$ coincide, hence φ is analytic and positive in ω; secondly, we have $\partial_\tau\varphi = 0$. This implies that the angular derivative of φ is zero. Together with our assumption on ω we can conclude that there is a point in ω where φ vanishes. This contradicts the positivity of φ. □

Next we need a specific lower bound for the spectrum of the Schrödinger operator in a bounded one-dimensional interval with Neumann boundary conditions and a characteristic function of a subinterval as the potential.

Lemma 8.2. *Let Λ be a bounded open interval of $\mathbb{R}$. Then for any open subinterval $\Lambda' \subset \Lambda$ and any $f \in \mathcal{H}^1(\Lambda)$, the following inequality holds:*

$$\|f\|^2_{L^2(\Lambda)} \leq c\,(\Lambda, \Lambda') \left(\|f\|^2_{L^2(\Lambda')} + \|f'\|^2_{L^2(\Lambda)}\right),$$

where $c(\Lambda, \Lambda') := \max\left\{2 + 16\,(|\Lambda|/|\Lambda'|)^2, 4\,|\Lambda|^2\right\}$.

Proof. Without loss of generality, we may suppose that $\Lambda' := (-b/2, b/2)$ with some positive b. Define a function g on Λ by

$$g(x) := \begin{cases} 2\,|x|/b & \text{for } |x| \le b/2\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

Let f be any function from $\mathcal{H}^1(\Lambda)$. Then $(fg)(0) = 0$ and the Cauchy-Schwarz inequality gives

$$|f(x)g(x)|^2 \le |x| \int_0^x |(fg)'|^2 \le |\Lambda|\,\|(fg)'\|^2_{L^2(\Lambda)} \tag{8.20}$$

for any $x \in \Lambda$. Now we write $f = fg + f(1-g)$ to get

$$\|f\|^2_{L^2(\Lambda)} \le 2\,\|fg\|^2_{L^2(\Lambda)} + 2\,\|f(1-g)\|^2_{L^2(\Lambda)} = 2\,\|fg\|^2_{L^2(\Lambda)} + 2\,\|f\|^2_{L^2(\Lambda')}\,.$$

Using the estimate (8.20) and the fact that $|g'| = 2\,|\Lambda'|^{-1}$ on Λ', we obtain the statement of the lemma. □

8.3.2 A local Hardy inequality

Since σ is continuous and has compact support there are closed intervals A_j such that

$$\operatorname{supp}(\sigma) = \bigcup_{j\in K} A_j \quad \text{and} \quad |A_i \cap A_j| = 0\,,\ i \ne j\,, \tag{8.21}$$

where $K \subseteq \mathbb{N}$ is an index set. The main result of this subsection is the following local type of Hardy inequality:

Theorem 8.4. *Let the assumptions of Theorem 8.3 hold. For every $j \in K$ there is a positive constant a_j depending on $\sigma \restriction A_j$ such that for all $\psi \in \mathcal{H}^1_0(\mathbb{R}\times\omega)$,*

$$\int_{A_j\times\omega} \left(|\partial_2\psi|^2 + |\partial_3\psi|^2 + |\partial_1\psi - \sigma\,\partial_\tau\psi|^2 - E_1\,|\psi|^2\right) \;\ge\; a_j\,\lambda \int_{A_j\times\omega} |\sigma\,\psi|^2\,, \tag{8.22}$$

where λ is the positive constant from Definition 8.1 depending only on the geometry of ω.

To prove Theorem 8.4, it will be useful to introduce the following quantities:

Definition 8.2. For any $M \subseteq \mathbb{R}$ and $\psi \in \mathcal{H}^1_0(\mathbb{R}\times\omega)$, we define

$$\begin{aligned} I_1^M &:= \|\chi_M \nabla'\psi\|^2 - E_1\,\|\chi_M\psi\|^2\,, & I_3^M &:= \|\chi_M\sigma\,\partial_\tau\psi\|^2\,, \\ I_2^M &:= \|\chi_M\partial_1\psi\|^2\,, & I_{2,3}^M &:= -2\,\Re\,(\partial_1\psi, \chi_M\sigma\,\partial_\tau\psi)\,, \end{aligned}$$

where χ_M denotes the characteristic function of the set $M\times\omega$, ∇' denotes the gradient operator in the "transverse" coordinates (t_2, t_3) and $(\cdot,\cdot)$ is the inner product generated by $\|\cdot\|$.

Note that I_1^M is non-negative due to (8.18) and that we have

$$l_\sigma[\psi] - E_1\,\|\psi\|^2 = I_1^{\mathbb{R}} + I_2^{\mathbb{R}} + I_3^{\operatorname{supp}(\sigma)} + I_{2,3}^{\operatorname{supp}(\sigma)}\,. \tag{8.23}$$

Let A be the union of any (finite or infinite) sub-collection of the intervals A_j.

The following lemma enables us to estimate the mixed term $I^A_{2,3}$.

Lemma 8.3. *Let the assumptions of Theorem 8.3 be satisfied. Then for each positive numbers α and β, there exists a constant $\gamma_{\alpha,\beta}$ depending also on $\sigma \restriction A$ such that for any $\psi \in \mathcal{H}^1_0(\mathbb{R}\times\omega)$,*

$$|I^A_{2,3}| \;\le\; \gamma_{\alpha,\beta}\,I_1^A + \alpha\,I_2^B + \beta\,I_3^A\,,$$

where $B := (\inf A, \sup A)$.

Proof. It suffices to prove the result for real-valued functions ψ from the dense subspace $C_0^\infty(\mathbb{R}\times\omega)$. We employ the decomposition

$$\psi(s,t) = \mathcal{J}(t)\,\phi(s,t)\,, \qquad (s,t) \in \mathbb{R}\times\omega\,, \tag{8.24}$$

where $\mathcal{J}$ is a positive eigenfunction of the Dirichlet Laplacian on $L^2(\omega)$ corresponding to E_1 (we shall denote by the same symbol the function $1\otimes\mathcal{J}$ on $\mathbb{R}\times\omega$), and ϕ is a real-valued function from $C_0^\infty(\mathbb{R}\times\omega)$, actually introduced by (8.24). Then

$$\begin{aligned} I_1^A &= \|\chi_A\mathcal{J}\nabla'\phi\|^2\,, & I_3^A &= \|\chi_A\sigma(\mathcal{J}\partial_\tau\phi + \phi\partial_\tau\mathcal{J})\|^2\,, \\ I_2^A &= \|\chi_A\mathcal{J}\partial_1\phi\|^2\,, & I_{2,3}^A &= -2\left(\mathcal{J}\partial_1\phi, \chi_A\sigma\,(\mathcal{J}\partial_\tau\phi + \phi\partial_\tau\mathcal{J})\right), \end{aligned}$$

where we have integrated by parts to establish the identity for I_1^A. Using

$$|\sigma\,\partial_\tau\phi|^2 \le c_1\,|\nabla'\phi|^2\,, \qquad \text{with} \quad c_1 := \|\sigma\restriction A\|_\infty^2\,a^2\,,$$

and applying the Cauchy-Schwarz inequality and the Cauchy inequality with $\alpha>0$, the first term in the sum of $I_{2,3}^A$ can be estimated as follows:

$$\left|2\left(\mathcal{J}\partial_1\phi, \chi_A\sigma\,\mathcal{J}\,\partial_\tau\phi\right)\right| \le 2\,\sqrt{c_1}\sqrt{I_1^A}\sqrt{I_2^A} \le \frac{2\,c_1}{\alpha}\,I_1^A + \frac{\alpha}{2}\,I_2^A\,. \tag{8.25}$$

In order to estimate the second term, we first combine integrations by parts to get

$$\left|2\left(\mathcal{J}\partial_1\phi, \chi_A\sigma\,\phi\,\partial_\tau\mathcal{J}\right)\right| = \left|\left(\phi, \chi_A\dot\sigma\,\mathcal{J}^2\,\partial_\tau\phi\right)\right|.$$

Using

$$|\dot\sigma\,\partial_\tau\phi|^2 \le c_2\,|\nabla'\phi|^2\,, \qquad \text{with} \quad c_2 := \|\dot\sigma\restriction A\|_\infty^2\,a^2\,,$$

and the Cauchy-Schwarz inequality, we have

$$\left|\left(\phi, \chi_A\dot\sigma\,\mathcal{J}^2\,\partial_\tau\phi\right)\right|^2 \le c_2\,I_1^A\,\|\chi_A\mathcal{J}\phi\|^2\,,$$

Obviously, we can find an open interval $A'\subset A$ such that there exists a certain positive number σ_0, for which

$$\sigma(s)\ge\sigma_0, \qquad \forall\, s\in A'\,.$$

Lemma 8.2 tells us that

$$\begin{aligned}\|\chi_A\,\mathcal{J}\phi\|^2 \le \|\chi_B\,\mathcal{J}\phi\|^2 &\le c(B,A')\left(I_2^B + \|\chi_{A'}\,\mathcal{J}\phi\|^2\right)\\ &\le c(B,A')\left(I_2^B + \sigma_0^{-2}\|\chi_{A'}\,\sigma\mathcal{J}\phi\|^2\right).\end{aligned}$$

Moreover, for each fixed value of $s\in\mathbb{R}$ we have $\sigma(s)\,\mathcal{J}\,\phi(s,\cdot)\in\mathcal{H}_0^1(\omega)$, and therefore we can apply Lemma 8.1 to obtain

$$\|\chi_{A'}\,\sigma\mathcal{J}\phi\|^2 \le \|\chi_A\sigma\mathcal{J}\phi\|^2 \le \lambda^{-1}\left(I_3^A + \|\sigma\|_\infty^2\,I_1^A\right).$$

Writing $c_3 := c_2\,c(B,A')\lambda^{-1}\sigma_0^{-2}$, we conclude that

$$\begin{aligned}\left|\left(\phi, \chi_A\,\dot\sigma\,\mathcal{J}^2\,\partial_\tau\phi\right)\right|^2 &\le c_3\,I_1^A\left(\|\sigma\|_\infty^2\,I_1^A + \lambda\,\sigma_0^2\,I_2^B + I_3^A\right)\\ &\le \left(\tilde\gamma_{\alpha,\beta}\,I_1^A + \frac{\alpha}{2}\,I_2^B + \beta\,I_3^A\right)^2\end{aligned} \tag{8.26}$$

for any $\beta>0$ and $\tilde\gamma_{\alpha,\beta} := \max\{\sqrt{c_3}\,\|\sigma\|_\infty\,,\, c_3(2\beta)^{-1}\,,\, c_3\,\lambda\,\sigma_0^2\,\alpha^{-1}\}$. Finally, combining (8.25) with (8.26), the estimate for $|I_{2,3}^A|$ follows by setting $\gamma_{\alpha,\beta} := \tilde\gamma_{\alpha,\beta} + 2\,c_1\alpha^{-1}$. □

Now we are in a position to establish Theorem 8.4.

of Theorem 8.4. We take $A=A_j$, $\alpha=1$, $\beta<1$ and keep in mind that $\gamma_{1,\beta}$ in Lemma 8.3 depends on j. We define $\gamma(\beta,j) := \max\{1/2, \gamma_{1,\beta}\}$. Lemma 8.3 then gives

$$\begin{aligned}&\int_{A_j\times\omega}\left(|\nabla'\psi|^2 + |\partial_1\psi - \sigma\,\partial_\tau\psi|^2 - E_1\,|\psi|^2\right)\\ &\qquad\ge \frac{1}{2}\,I_1^{A_j} + \left(1-\frac{1}{2\gamma(\beta,j)}\right)\left(I_2^{A_j} + I_3^{A_j} - |I_{2,3}^{A_j}|\right) + \frac{1-\beta}{2\gamma(\beta,j)}\,I_3^{A_j}\end{aligned}$$

Since $I_2^{A_j} + I_3^{A_j} - |I_{2,3}^{A_j}| \ge 0$, we get from Lemma 8.1 that

$$\begin{aligned}&\int_{A_j\times\omega}\left(|\nabla'\psi|^2 + |\partial_1\psi - \sigma\,\partial_\tau\psi|^2 - E_1|\psi|^2\right)\\ &\qquad\ge\ a_j\left(\|\sigma\restriction A_j\|_\infty^2\,I_1^{A_j} + I_3^{A_j}\right)\ \ge\ a_j\,\lambda\int_{A_j\times\omega}|\sigma\,\psi|^2\,,\end{aligned}$$

where

$$a_j = \frac{1}{2}\,\min\left\{\frac{1}{\|\sigma\restriction A_j\|_\infty^2}\,,\,\frac{1-\beta}{\gamma(\beta,j)}\right\}.$$

□

Remark 8.1. Note that the Hardy weight on the right hand side of (8.22) cannot be made arbitrarily large by increasing σ, since the constant a_j is proportional to $\|\sigma \restriction A_j\|_\infty^{-2}$ if the latter is large enough. We want to point out that this degree of decay of a_j is optimal if the axes of rotation intersects ω. Assume there exists an $\alpha < 2$, such that a_j is proportional to $\|\sigma \restriction A_j\|_\infty^{-\alpha}$ when $\|\sigma \restriction A_j\|_\infty \to \infty$. Consider a test function ψ of the form $\psi(s,t) := g(s)f(t)$, where $g \in \mathcal{H}^1(\mathbb{R})$ is supported inside A_j and $f \in \mathcal{H}_0^1(\omega)$ is radially symmetric with respect to the intersection of ω with the axes of rotation. Then $\partial_\tau \psi = 0$ on $A_j \times \omega$ and therefore the left hand side of (8.22) is for this test function independent of σ. Take $\sigma = n\,\tilde{\sigma}$ with $\tilde{\sigma}$ being a fixed function. The right hand side of (8.22) then tends to infinity as $n \to \infty$ which contradicts the inequality.

8.3.3 Proof of Theorem 8.3

For applications, it is convenient to replace the Hardy inequality of Theorem 8.4 with a compactly supported Hardy weight by a global one. To do so, we recall the following version of the one-dimensional Hardy inequality:

$$\int_{\mathbb{R}} \frac{|v(x)|^2}{x^2}\,dx \le 4 \int_{\mathbb{R}} |v'(x)|^2\,dx \tag{8.27}$$

for all $v \in C_0^\infty(\mathbb{R})$ with $v(0) = 0$. Inequality (8.27) extends by continuity to all $v \in \mathcal{H}^1(\mathbb{R})$ with $v(0) = 0$.

Without loss of generality we can assume that $s_0 = 0$. Let $J = [-b, b]$, with some positive number b, be an interval where $|\sigma| > 0$. Let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be defined by

$$\tilde{f}(s) := \begin{cases} 1 & \text{for } |s| \ge b\,, \\ |s|/b & \text{for } |s| < b\,, \end{cases}$$

and put $f := \tilde{f} \otimes 1$ on $\mathbb{R} \times \omega$. For any $\psi \in C_0^\infty(\mathbb{R} \times \omega)$, let us write $\psi = f\psi + (1-f)\psi$. Applying (8.27) to the function $s \mapsto (f\psi)(s,t)$ with t fixed, we arrive at

$$\begin{aligned} \int_{\mathbb{R}\times\omega} \frac{|\psi(s,t)|^2}{1+s^2}\,ds\,dt &\le 2 \int_{\mathbb{R}\times\omega} \frac{|\tilde{f}(s)\psi(s,t)|^2}{s^2}\,ds\,dt + 2\int_{J\times\omega} |(1-f)\psi|^2 \\ &\le 16\,\|(\partial_1 f)\psi\|^2 + 16\,\|f\partial_1\psi\|^2 + 2\,\|\chi_J(1-f)\psi\|^2 \\ &\le \left(\frac{16}{b^2} + 2\right)\|\chi_J\psi\|^2 + 16\,\|\partial_1\psi\|^2, \end{aligned}$$

where χ_J denotes the characteristic function of the set $J \times \omega$. Theorem 8.4 then implies that there exists a positive constant c_0 depending on σ such that

$$\|\chi_J\psi\|^2 \le \left(c_0\,\lambda \min_J |\sigma|\right)^{-1}\left(l_\sigma[\psi] - E_1\|\psi\|^2\right).$$

To estimate the second term we let $A = \operatorname{supp}(\sigma)$ and rewrite the inequality of Lemma 8.3 for $\beta = 1$ as

$$\gamma_\alpha^{-1}\,|I_{2,3}^A| \le I_1^A + \alpha\,\gamma_\alpha^{-1}\,I_2^B + \gamma_\alpha^{-1}\,I_3^A\,,$$

where $\gamma_\alpha := \max\{1, \gamma_{\alpha,1}\}$ and $\alpha \in (0,1)$. Substituting this inequality into (8.23), writing $I_{2,3}^A = \gamma_\alpha^{-1}\,I_{2,3}^A + (1-\gamma_\alpha^{-1})\,I_{2,3}^A$ and employing $I_2^A + I_3^A + I_{2,3}^A \ge 0$, we obtain

$$I_2^B = \|\chi_B\,\partial_1\psi\|^2 \le \gamma_\alpha\,(1-\alpha)^{-1}\left(l_\sigma[\psi] - E_1\|\psi\|^2\right).$$

On the complement of $B \times \omega$ we have a trivial estimate

$$\|\chi_{\mathbb{R}\setminus B}\,\partial_1\psi\|^2 \le l_\sigma[\psi] - E_1\|\psi\|^2\,.$$

Summing up, the density of $C_0^\infty(\mathbb{R}\times\omega)$ in $\mathcal{H}_0^1(\mathbb{R}\times\omega)$ implies Theorem 8.3 with

$$c_h \ge \left[\frac{16 + 2\,b^2}{b^2\,c_0\,\lambda\,\min_J|\sigma|^2} + 16\,\left(\frac{\gamma_\alpha}{1-\alpha} + 1\right)\right]^{-1}. \tag{8.28}$$

8.4 Twisted bent tubes

Here we develop a geometric background to study the Laplacian in bent and twisted tubes, and transform the former into a unitarily equivalent Schrödinger-type operator in a straight tube. At the end of this section, we also perform proofs of Theorems 8.1 and 8.2 using Theorem 8.3. We refer to Section 8.2.1 for definitions of basic geometric objects used throughout the paper.

While we are mainly interested in the curves determined by curvature functions of type (8.2), we stress that the formulae of Sections 8.4.1 and 8.4.2 are valid for arbitrary curves (it is only important to assume the existence of an appropriate Frenet frame for the reference curve of the tube, *cf* [5]).

8.4.1 Metric tensor

Assuming (8.11) and using the inverse function theorem, we see that the mapping $\mathcal{L}$ introduced in (8.10) induces a C^1-smooth diffeomorphism between the straight tube $\mathbb{R}\times\omega$ and the image Ω. This enables us to identify Ω with the Riemannian manifold $(\mathbb{R}\times\omega, G_{ij})$, where (G_{ij}) is the metric tensor induced by the embedding $\mathcal{L}$, *i.e.*

$$G_{ij} := (\partial_i\mathcal{L})\cdot(\partial_j\mathcal{L})\,,$$

with the *dot* being the scalar product in $\mathbb{R}^3$.

Using (8.3) and the orthogonality conditions $\mathcal{R}^\theta_{\mu\rho}\mathcal{R}^\theta_{\nu\rho}=\delta_{\mu\nu}$, we find

$$(G_{ij}) = \begin{pmatrix} h^2+h_\mu h_\mu & h_2 & h_3 \\ h_2 & 1 & 0 \\ h_3 & 0 & 1 \end{pmatrix}, \tag{8.29}$$

where

$$\begin{aligned} h(s,t) &:= 1-[t_2\cos\theta(s)+t_3\sin\theta(s)]\,\kappa_1(s)\,,\\ h_2(s,t) &:= -t_3\,[\kappa_2(s)-\dot\theta(s)]\,,\\ h_3(s,t) &:= t_2\,[\kappa_2(s)-\dot\theta(s)]\,. \end{aligned}$$

Furthermore,

$$G := \det(G_{ij}) = h^2\,,$$

which defines the volume element of $(\mathbb{R}\times\omega, G_{ij})$ by setting

$$d\mathrm{vol} := h(s,t)\,ds\,dt.$$

Here and in the sequel $dt\equiv dt_2\,dt_3$ denotes the 2-dimensional Lebesgue measure in ω.

The metric is uniformly bounded and elliptic in view of the first of the assumptions in (8.11); in particular, (8.7) yields

$$0<1-a\,\|\kappa_1\|_\infty \;\le\; h \;\le\; 1+a\,\|\kappa_1\|_\infty<\infty\,. \tag{8.30}$$

It can be directly checked that the inverse (G^{ij}) of the metric tensor (8.29) is given by

$$(G^{ij}) = \frac{1}{h^2}\begin{pmatrix} 1 & -h_2 & -h_3 \\ -h_2 & h^2+h_2^2 & h_2h_3 \\ -h_3 & h_3h_2 & h^2+h_3^2 \end{pmatrix}. \tag{8.31}$$

It is worth noticing that one has the decomposition

$$(G^{ij}) = \mathrm{diag}(0,1,1) + (S^{ij})\,, \tag{8.32}$$

where the matrix (S^{ij}) is positive semi-definite.

8.4.2 The Laplacian

Recalling the diffeomorphism between $\mathbb{R}\times\omega$ and Ω given by $\mathcal{L}$, we identify the Hilbert space $L^2(\Omega)$ with $L^2(\mathbb{R}\times\omega, d\mathrm{vol})$. Furthermore, the Dirichlet Laplacian $-\Delta_D^\Omega$ is unitarily equivalent to the self-adjoint operator $\tilde Q$ associated on $L^2(\mathbb{R}\times\omega, d\mathrm{vol})$ with the quadratic form

$$\tilde q[\psi] := \int_{\mathbb{R}\times\omega} (\overline{\partial_i\psi})\,G^{ij}\,(\partial_j\psi)\;d\mathrm{vol}\,, \qquad \psi\in\mathfrak{D}(\tilde q) := \mathcal{H}^1_0(\mathbb{R}\times\omega, d\mathrm{vol})\,. \tag{8.33}$$

We can write $\tilde Q = -G^{-1/2}\partial_i G^{1/2}G^{ij}\partial_j$ in the form sense, which is a general expression for the Laplace-Beltrami operator on a manifold equipped with a metric (G_{ij}).

Now we transform $\tilde Q$ into a unitarily equivalent operator Q acting in the Hilbert space $L^2(\mathbb{R}\times\omega)$, without the additional weight $G^{1/2}$ in the measure of integration. This is achieved by means of the unitary operator

$$\mathcal{U} : L^2(\mathbb{R}\times\omega, d\mathrm{vol}) \to L^2(\mathbb{R}\times\omega) : \big\{\psi\mapsto G^{1/4}\psi\big\}\,.$$

Defining $Q := \mathcal{U}\,\tilde Q\,\mathcal{U}^{-1}$, it is clear that Q is the operator associated with the quadratic form

$$q[\psi] := \tilde q[G^{-1/4}\psi]\,, \qquad \psi\in\mathfrak{D}(q) := \mathcal{H}^1_0(\mathbb{R}\times\omega)\,.$$

It is straightforward to check that

$$q[\psi] = \big(\partial_i\psi, G^{ij}\partial_j\psi\big) + \big(\psi, (\partial_iF)G^{ij}(\partial_jF)\,\psi\big) + 2\,\Re\,\big(\partial_i\psi, G^{ij}(\partial_jF)\,\psi\big)\,, \tag{8.34}$$

where

$$F := \log(G^{1/4})\,.$$

8.4.3 Proof of Theorems 8.1 and 8.2

Let us first prove Theorem 8.1. Putting $\sigma := \kappa_2 - \dot\theta$, we observe that l_σ is equal to q after letting $k := \|\kappa_1\|_\infty + \|\dot\kappa_1\|_\infty$ equal to zero in the latter form. Hence, the proof of Theorem 8.1 reduces to a comparison of these quadratic forms and the usage of Theorem 8.3. Let (G_0^{ij}) be the matrix (8.31) after letting $\kappa_1 = 0$, *i.e.* with h being replaced by 1 while h_2 and h_3 being unchanged; then $l_\sigma[\psi] = (\partial_i\psi, G_0^{ij}\partial_j\psi)$.

We strengthen the first of the hypotheses (8.11) to

$$\|\kappa_1\|_\infty \le 1/(2a)\,,$$

so that we have a uniform positive lower bound to h, namely $h \ge 1/2$ due to (8.30). It is straightforward to check that we have on $\mathbb{R}\times\omega$ the following pointwise inequalities:

$$\max_{i,j\in\{1,2,3\}} |G^{ij} - G_0^{ij}| \le C_1\, k\, \chi_I\,,$$
$$\max_{i\in\{1,2,3\}} |\partial_i F| \le C_2\, k\, \chi_I\,,$$

where χ_I denotes the characteristic function of the set $I\times\omega$ and

$$C_1 := 6\,a\,\big(1 + a\,\|\kappa_2 - \dot\theta\|_\infty\big)^2\,, \qquad C_2 := 1 + a\,\big(1 + \|\dot\theta\|_\infty\big)\,.$$

At the same time,

$$C_3^{-1}\,1 \le (G_0^{ij}) \le C_3\,1\,,$$

in the matrix-inequality sense on $\mathbb{R}\times\omega$, where 1 denotes the identity matrix and

$$C_3 := 1 + a\,\|\kappa_2 - \dot\theta\|_\infty + a^2\,\|\kappa_2 - \dot\theta\|_\infty^2\,.$$

Consequently, we have the following matrix inequality on $\mathbb{R}\times\omega$:

$$(1 - C_4\,k\,\chi_I)(G_0^{ij}) \le (G^{ij}) \le (1 + C_4\,k\,\chi_I)(G_0^{ij})\,,$$

where $C_4 := 3C_1C_3$. Finally, if we assume that $k \le 1$ we have

$$|(\partial_i F)G^{ij}(\partial_j F)| \le C_5^2\,k^2\,\chi_I\,, \tag{8.35}$$

where $C_5 := C_2\sqrt{3C_3(1+C_4)}$.

Let ψ be any function from $\mathcal{H}_0^1(\mathbb{R}\times\omega)$. First we estimate the term of indefinite sign on the right hand side of (8.34) as follows:

$$\begin{aligned} 2\,\big|\Re\,\big(\partial_i\psi, G^{ij}(\partial_j F)\,\psi\big)\big| &\le 2\,C_5\,k\,\Big(\chi_I\big[(\overline{\partial_i\psi})G^{ij}(\partial_j\psi)\big]^{1/2}, \chi_I|\psi|\Big) \\ &\le C_5^2\,k\,\|\chi_I\psi\|^2 + k\,\big(\partial_i\psi, \chi_I\,G^{ij}\partial_j\psi\big)\,. \end{aligned}$$

Here the first inequality is established by applying the Cauchy-Schwarz inequality to the inner product induced by (G^{ij}) and using (8.35). The second inequality follows by the Cauchy-Schwarz inequality in the Hilbert space $L^2(\mathbb{R}\times\omega)$ and by an elementary Cauchy inequality. Consequently,

$$q[\psi] \ge \big(\partial_i\psi, (1 - C_6\,k\,\chi_I)\,G_0^{ij}\partial_j\psi\big) - C_7\,k\,\|\chi_I\psi\|^2\,, \tag{8.36}$$

where $C_6 := 1 + C_4$ and $C_7 := 2C_5^2$.

Assume $k < C_6^{-1}$, using the decomposition of the type (8.32) for the matrix (G_0^{ij}), neglecting the positive contribution coming from the corresponding matrix (S_0^{ij}), using the Fubini theorem and applying (8.18) to the function $\varphi := \int_{\mathbb{R}} \sqrt{1 - C_6\,k\,\chi_I(s)}\,\psi(s,\cdot)\,ds$, we may estimate (8.36) as follows:

$$q[\psi] - E_1\|\psi\|^2 \ge (1 - C_6\,k)\big(l_\sigma[\psi] - E_1\|\psi\|^2\big) - (C_6E_1 + C_7)\,k\,\|\chi_I\psi\|^2\,.$$

Applying Theorem 8.3 to the right hand side of the previous inequality, we have

$$q[\psi] - E_1\|\psi\|^2 \ge \int_{\mathbb{R}\times\omega}\left(\frac{c_h\,(1 - C_6\,k)}{1 + (s - s_0)^2} - (C_6E_1 + C_7)\,k\,\chi_I(s)\right)|\psi(s,t)|^2\,ds\,dt,$$

where c_h is the Hardy constant of Theorem 8.3. This proves that the threshold of the spectrum of Q (and therefore of $-\Delta_D^\Omega$) is greater than or equal to E_1 for sufficiently small k.

In order to show that the whole interval $[E_1, \infty)$ belongs to the spectrum, it is enough to construct an appropriate Weyl sequence in the infinite straight ends of Ω. This concludes the proof of Theorem 8.1.

The proof of Theorem 8.2 is exactly the same, provided one chooses $\sigma := \dot\theta$ and $k := \|\kappa_1\|_\infty + \|\dot\kappa_1\|_\infty + \|\kappa_2\|_\infty$. Indeed, all the above estimates are valid with (G_0^{ij}) being now the matrix (8.31) after letting both κ_1 and κ_2 equal to zero, and with C_1 and C_3 being replaced by

$$C_1 := 6\,a\left(1 + a\,\|\kappa_2\|_\infty + a\,\|\dot\theta\|_\infty\right)^2, \qquad C_3 := \max\left\{2, 1 + 2\,a^2\,\|\dot\theta\|_\infty^2\right\},$$

respectively. Here C_1 can be further estimated by a constant independent of κ_2 provided one restricts, *e.g.*, to $\|\kappa_2\|_\infty < 1/a$.

8.5 Discussion

We established Hardy-type inequalities for twisted 3-dimensional tubes. As an application we showed that the discrete eigenvalues of the Dirichlet Laplacian in mildly and locally bent tubes can be eliminated by an appropriate twisting. However, we would like to point out that for $\sigma = \dot\theta$, Theorems 8.3 and 8.4 can be used to prove certain stability of transport in straight twisted tubes also against other types of perturbations. For example against a local enlargement of the straight tube, mentioned in Introduction, or in principle against any potential perturbation which decays at least as $\mathcal{O}(s^{-2})$ at infinity, where s is the longitudinal coordinate of the straight tube. The required decay at infinity is related to the decay of the Hardy weight in Theorem 8.3, which is inverse quadratic and cannot be improved for locally twisted tubes.

For straight twisted tubes, the Hardy weight in the local inequality (8.22) of Theorem 8.4 is given in terms of the the function $\dot\theta$ and the constant λ. Roughly speaking, the first tells us how fast the cross-section rotates, while the latter "measures" how much the cross-section differs from a disc. The actual value of λ depends of course on the geometry of ω.

The example of bent twisted tubes is of particular interest, since it shows the important role of the torsion. Namely, Theorem 8.1 tells us that, whenever $\dot\theta \neq \kappa_2$, the discrete eigenvalues in mildly curved tubes can be eliminated by torsion only. Note that Theorem 8.1 also provides a better lower bound to the spectrum in mildly bent tubes than that derived in [8].

Theorems 8.1 and 8.2 were proved for tubes about curves determined by (8.2). This restriction was made in order to construct the tube uniquely from given curvature functions by means of a uniquely determined Frenet frame. However, Theorems 8.1 and 8.2 will also hold for more general classes of tubes, namely, for those constructed about curves possessing the distinguished Frenet frame and with curvatures decaying as $\mathcal{O}(s^{-2})$ at infinity, where s is the arc-length parameter of the curve.

At least from the mathematical point of view, it would be interesting to extend Theorem 8.1 to higher dimensions. Here the main difficulty is that σ in the form analogous to (8.17) will be in general a tensor depending also on the transverse variables t. Nevertheless, a higher dimensional analogue of Theorem 8.2 is easy to derive along the same lines as in the present paper, provided one restricts to rotations of the cross-section just in one hyperplane.

Summing up, the twisting represents a repulsive geometric perturbation in the sense that it eliminates the discrete eigenvalues in mildly curved waveguides. Regarding the transport itself, an interesting open question is whether this also happens to the singular spectrum possibly contained in the essential spectrum. It would also be of a considerable interest to see what effect the twisting has on possible resonances, which might be induced by bending or by potential perturbations of the waveguide.

As mentioned in Introduction, the original motivation for our problem was quantum-mechanical, for the one-particle Hamiltonian in a tube with Dirichlet boundary conditions is a reasonable model for quantum waveguides [6, 16]. It is challenging to demonstrate the repulsive effect of twisting in other areas of physics too, specifically in electromagnetic waveguides, acoustic waveguides or in water-flow pipes. While the effect of twisting could be easier to observe in experiments with classical systems, theoretically the opposite is true, and the more complicated equations of motion and/or boundary conditions lead to completely different mathematical problems.

APPENDIX

8.6 Injectivity of the tube mapping

Let us conclude the paper by finding geometric conditions which guarantee the basic hypotheses (8.11).

The first condition of (8.11) ensures that the mapping $\mathcal{L}$ is an immersion due to (8.30). The second, injectivity condition requires to impose some global hypotheses about the geometry of the curve. Our approach is based on the following lemma:

Lemma 8.4. *Let* Γ *be determined by the curvature functions* (8.2). *Then for every* $i \in \{1,2,3\}$ *and all* $s_1, s_2 \in \mathbb{R}$,

$$\big|e_i(s_2) - e_i(s_1)\big| \le 2\,k_i\,\min\big\{|s_2 - s_1|, |I|\big\}\,,$$

where

$$k_i := \begin{cases} \|\kappa_1\|_\infty & \text{if} \quad i = 1\,, \\ \|\kappa_1\|_\infty + \|\kappa_2\|_\infty & \text{if} \quad i = 2\,, \\ \|\kappa_2\|_\infty & \text{if} \quad i = 3\,. \end{cases}$$

Proof. It follows from the Serret-Frenet equations (8.3) and (8.2) that

$$\big|e_i(s_2) - e_i(s_1)\big| \le 2\left|\int_{s_1}^{s_2} |\dot e_i|\right| \le 2\,k_i\left|\int_{s_1}^{s_2} \chi_I\right|,$$

which immediately establishes the assertion. □

As a consequence of Lemma 8.4, we get the inequality

$$e_i(s_2)\cdot e_i(s_1) \;\ge\; 1 - 2\,|I|^2\,k_i^2\,, \qquad i \in \{1,2,3\}\,. \tag{8.37}$$

In particular, since e_1 is the tangent vector of Γ, we obtain that the curve is not self-intersecting provided $|I|\,\|\kappa_1\|_\infty < 1$. A stronger sufficient condition ensures the injectivity of $\mathcal{L}$:

Proposition 8.1. *Let* Γ *be determined by the curvature functions* (8.2). *Then the hypotheses* (8.11) *hold true provided*

$$\max\big\{4\,|I|^2\,\|\kappa_1\|_\infty^2\,,\; 4\,a\,\big(\|\kappa_1\|_\infty + \|\kappa_2\|_\infty\big)\big\} \;<\; 1\,.$$

Proof. The idea is to observe that it is enough to show that the mapping Γ_t from $\mathbb{R}$ to $\mathbb{R}^3$ defined by

$$\Gamma_t(s) := \Gamma(s) + t_\mu\,\mathcal{R}_{\mu\nu}(s)\,e_\nu(s)$$

is injective for any *fixed* $t \in \mathbb{R}^2$ such that $|t| < a$ and *arbitrary* matrix-valued function $(\mathcal{R}_{\mu\nu}) : \mathbb{R} \to \mathsf{SO}(2)$. Let us assume that there exist $s_1 < s_2$ such that $\Gamma_t(s_1) = \Gamma_t(s_2)$. Then

$$\begin{aligned} 0 \;&= \Gamma(s_2) - \Gamma(s_1) \\ &\quad + t_\mu\,\big\{\big[\mathcal{R}_{\mu\nu}(s_2) - \mathcal{R}_{\mu\nu}(s_1)\big]\,e_\nu(s_1) + \mathcal{R}_{\mu\nu}(s_2)\,\big[e_\nu(s_2) - e_\nu(s_1)\big]\big\}\,. \end{aligned}$$

Taking the inner product of both sides of the vector identity with the tangent vector $e_1(s_1)$ and writing the difference $\Gamma(s_2) - \Gamma(s_1)$ as an integral, we arrive at the following scalar identity

$$0 = \int_{s_1}^{s_2} e_1(s_1)\cdot e_1(\xi)\,d\xi + t_\mu\,\mathcal{R}_{\mu\nu}(s_2)\,\big[e_\nu(s_2) - e_\nu(s_1)\big]\cdot e_1(s_1)\,.$$

Applying Lemma 8.4 together with the first inequality of (8.37), recalling the orthogonality of $(\mathcal{R}_{\mu\nu})$ and using obvious estimates, we obtain

$$0 \ge (s_2 - s_1)\,\big(1 - 2\,|I|^2\,k_1^2 - 2\,a\,k_2\big)\,.$$

This provides a contradiction for all curves satisfying the inequality of Proposition, unless $s_1 = s_2$. □

Remark 8.2. The ideas of this Appendix do not restrict to the special class of tubes about curves determined by (8.2). Indeed, assuming only the existence of an appropriate Frenet frame for the reference curve (*cf* [5]), more general sufficient conditions, involving integrals of curvatures, could be derived.

Acknowledgement

The authors are grateful to Timo Weidl for pointing out the presented problem to them. The work has partially been supported by the Czech Academy of Sciences and its Grant Agency within the projects IRP AV0Z10480505 and A100480501, by the project LC06002 of the Ministry of Education, Youth and Sports of the Czech Republic, and by DAAD within the project D-CZ 5/05-06. T.E. has partially been supported by the ESF European programme SPECT and D.K. has partially been supported by FCT/POCTI/FEDER, Portugal.

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Errata

1. The decomposition (8.21) is not valid in general. The correct statement is that $\mathrm{supp}(\sigma)$ is a closure of a countable union of *open* intervals. Anyway, this does not change anything on the arguments used afterwards.

2. We were not too cautious when defining what we call a "non-circular" or "non-symmetric" assumption about the cross-section ω. The condition (8.8) should be understood with the convention that we identify sets which possibly differ on a set of capacity zero. This is fairly treated in Chapter 14.

Chapter 9

A Hardy inequality in a twisted Dirichlet-Neumann waveguide

Published in: Mathematische Nachrichten **281** (2008), 1159–1168

Joint work with: Hynek Kovařík

A Hardy inequality in a twisted Dirichlet-Neumann waveguide

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Dedicated to Pavel Exner on the occasion of his 60th birthday

Abstract. We consider the Laplacian in a straight strip, subject to a combination of Dirichlet and Neumann boundary conditions. We show that a switch of the respective boundary conditions leads to a Hardy inequality for the Laplacian. As a byproduct of our method, we obtain a simple proof of a theorem of Dittrich and Kříž [5].

9.1 Introduction

The connection between spectral properties of the Laplacian in a waveguide-type domain, the domain geometry and various boundary conditions has been intensively studied in the last years, *cf* [6, 14, 12] and references therein. Particular attention has been paid to the geometrically induced discrete spectrum of the Dirichlet Laplacian in curved tubes of uniform cross-section [9, 10, 16, 6, 4] or in straight tubes with a local deformation of the boundary [3, 2]. Roughly speaking, it has been shown that a suitable bending or a local enlargement of a straight waveguide represents an effectively attractive perturbation and leads thus to the presence of eigenvalues below the essential spectrum of the Laplacian.

On the other hand, recently it has been observed in [8] that a local rotation of a non-circular cross-section of a three-dimensional straight tube creates a kind of repulsive perturbation. Namely, this type of deformation, called *twist*, gives rise to a Hardy inequality for the Dirichlet Laplacian. This avoids, up to some extent, the existence of discrete spectrum in the presence of an additional attractive perturbation, the bending or local enlargement being two examples. We refer to [8] for more details and possible higher-dimensional extensions.

The purpose of the present note is to demonstrate an analogous effect of twist in a two-dimensional waveguide with combined Dirichlet and Neumann boundary conditions. In this case the twist is represented by a switch of the boundary conditions at a given point, *cf* Figure 9.1. More precisely, given a real number ε and a positive number a, let $-\Delta_\varepsilon$ be the Laplacian in the strip $\mathbb{R}\times(-a,a)$, subject to Dirichlet boundary conditions on $(-\infty,-\varepsilon)\times\{-a\}\cup(\varepsilon,+\infty)\times\{a\}$ and Neumann boundary conditions on $(-\varepsilon,+\infty)\times\{-a\}\cup(-\infty,\varepsilon)\times\{a\}$, *cf* Figure 9.2. It is easy to locate the essential spectrum:

$$\sigma_{\rm ess}(-\Delta_\varepsilon)=[\pi^2/(4a)^2,\infty)\qquad\text{for every}\quad\varepsilon\in\mathbb{R}\,. \tag{9.1}$$

A simple Dirichlet-Neumann bracketing argument shows that there is no other spectrum for all non-positive ε, while discrete eigenvalues appear for sufficiently large positive ε. Our main result says that for ε equal to zero the operator $-\Delta_0$ satisfies the following Hardy type inequality in the sense of quadratic forms:

$$-\Delta_0-\left(\frac{\pi}{4a}\right)^2\;\geq\;\rho(\cdot)\,, \tag{9.2}$$

where $\rho:\mathbb{R}\times(-a,a)\to\mathbb{R}$ is a positive function.

We would like to emphasize that in the situation where the boundary conditions are not exchanged – *i.e.* the Laplacian in $\mathbb{R}\times(-a,a)$ with uniform Dirichlet boundary conditions on one connected part of the boundary and Neumann boundary conditions on the other one, *cf* the upper waveguide in Figure 9.1 – the essential spectrum coincides with the essential spectrum of our waveguide, but the inequality (9.2) fails to hold for any non-trivial $\rho\geq 0$. The latter can be shown by a simple test-function argument. In other words, the switch of the boundary conditions creates a kind of repulsive perturbation represented by the function ρ. This leads to a certain stability of the spectrum similar to the one observed in [8]. In particular, it follows from (9.2) that the discrete spectrum remains empty after perturbing $-\Delta_0$ by a sufficiently small attractive perturbation.

One example of attractive perturbation is changing the boundary conditions by increasing the parameter ε, *cf* Figure 9.2. Due to the switch of the boundary conditions, the discrete eigenvalues do not appear for any positive ε, but only when ε exceeds certain critical value $\varepsilon_c>0$. This effect was already observed by Dittrich and Kříž in [5]. Their result is obtained by a tedious decomposition of the Laplacian into the "transverse basis"

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Figure 9.1: We consider the lower waveguide as a *twist* perturbation of the upper one, the twist being defined as a switch of Dirichlet (thick lines) to Neumann (thin lines) boundary conditions at one point, and *vice versa*.

and this also provides an estimate on the critical value ε_c for which the eigenvalues emerge from the essential spectrum:

$$0.16\,a < \varepsilon_c < 0.68\,a\,. \tag{9.3}$$

Since the proof of our Hardy inequality (9.2) can be easily carried over to the case when ε is positive and small enough, we get as a byproduct of our method an alternative estimate on ε_c, too. The latter is worse than the one presented in [5], but on the other hand much simpler to obtain.

Finally, let us mention that Hardy inequalities for Schrödinger operators in two dimensions can be achieved by adding an appropriate local magnetic field to the system, too. This was first observed in [13] and later modified in [7] for Schrödinger operators in waveguides, *cf* also [1]. Curved waveguides in a homogeneous magnetic field have been recently studied in [15].

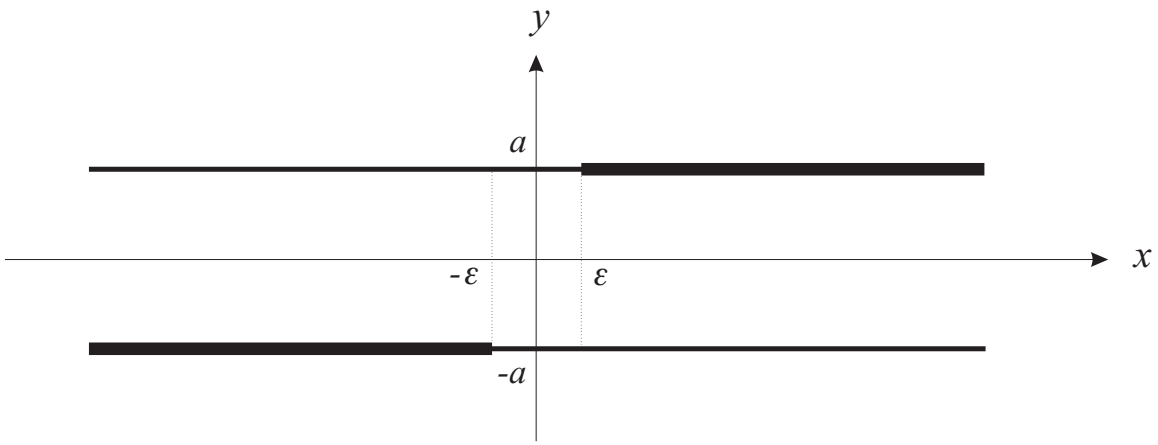

Figure 9.2: The geometry of our waveguide. The Dirichlet and Neumann boundary conditions are denoted by thick and thin lines, respectively.

9.2 Main results and ideas

The Laplacian $-\Delta_\varepsilon$ is defined as the unique self-adjoint operator associated with the closure of the quadratic form Q_ε defined in $L^2\big(\mathbb{R}\times(-a,a)\big)$ by

$$Q_\varepsilon[\psi] := \int_{\mathbb{R}\times(-a,a)} \big(|\partial_1\psi(x,y)|^2 + \partial_2\psi(x,y)|^2\big)\,dx\,dy \tag{9.4}$$

and by the domain $\mathfrak{D}(Q_\varepsilon)$ which consists of restrictions to $\mathbb{R}\times(-a,a)$ of infinitely smooth functions with compact support in $\mathbb{R}^2$ and vanishing on the part of the boundary where the Dirichlet boundary conditions are imposed (*cf* [5] for more details). We are interested in the shifted quadratic form $\tilde{Q}_\varepsilon$ defined on the form domain $\mathfrak{D}(Q_\varepsilon)$ by the prescription

$$\tilde{Q}_\varepsilon[\psi] := Q_\varepsilon[\psi] - \Big(\frac{\pi}{4a}\Big)^2 \int_{\mathbb{R}\times(-a,a)} |\psi(x,y)|^2\,dx\,dy\,. \tag{9.5}$$

If ε is negative, so that the opposite Dirichlet boundary conditions overlap, one can estimate the second term in (9.4) by the lowest eigenvalue of the Laplacian in the cross-section of length $2a$, subject to Dirichlet-Dirichlet or Dirichlet-Neumann boundary conditions. Neglecting the first term in (9.4), this immediately yields

$$-\Delta_\varepsilon - \left(\frac{\pi}{4a}\right)^2 \;\geq\; 3\left(\frac{\pi}{4a}\right)^2 \chi_{(\varepsilon,-\varepsilon)\times(-a,a)}(\cdot) \qquad \text{if} \qquad \varepsilon < 0 \tag{9.6}$$

in the sense of quadratic forms. Here χ_M denotes the characteristic function of a set M. The right hand side provides a non-negative Hardy weight in this case.

Of course, the trivial estimate leading to (9.6) is not useful for non-negative ε, in which case other methods have to be used. In this paper we get:

Theorem 9.1. *Given a real number ε and a positive number a, let $-\Delta_\varepsilon$ be the Laplacian in the strip $\mathbb{R}\times(-a,a)$, subject to Dirichlet boundary conditions on $(-\infty,-\varepsilon)\times\{-a\}\cup(\varepsilon,+\infty)\times\{a\}$ and Neumann boundary conditions on $(-\varepsilon,+\infty)\times\{-a\}\cup(-\infty,\varepsilon)\times\{a\}$.*

(i) There exists a positive constant c such that the inequality

$$-\Delta_0 - \left(\frac{\pi}{4a}\right)^2 \;\geq\; c\,\chi_\omega(\cdot) \tag{9.7}$$

holds in the sense of quadratic forms. Here $\omega \supseteq (-a,a)\times(-a,a)$ and

$$c \geq s_1 \left(\frac{\pi}{4a}\right)^2,$$

where s_1 is the smallest root of the equation

$$\sqrt{1-s}\;\tanh\left(\frac{\pi\sqrt{1-s}}{2\sqrt{2}}\right) = \sqrt{1/2+s}\;\tan\left(\frac{\pi\sqrt{1/2+s}}{2\sqrt{2}}\right). \tag{9.8}$$

(ii) There exists a positive constant $\varepsilon_c \geq t_1\,a$ such that

$$\sigma(-\Delta_\varepsilon) = \left[\pi^2/(4a)^2, \infty\right)$$

for all $\varepsilon \leq \varepsilon_c$. Here t_1 is the smallest positive root of the equation

$$\tanh\left(\frac{\pi\,(1-t)}{2\sqrt{2}}\right) = \sqrt{1/2}\;\tan\left(\frac{\pi\,(1+t)}{4}\right). \tag{9.9}$$

The first result, *i.e.* the Hardy inequality for $-\Delta_0$, is new. On the other hand, a positive lower bound on ε_c has already been established in [5], *cf* (9.3). In [5] the authors also find the numerical value $\varepsilon_c \approx 0.52\,a$. We have $s_1 \approx 0.039$ and $t_1 \approx 0.061$, and these numbers cannot be much improved by our method (*cf* the end of Section 9.4 for more details).

Although the effect which causes (9.7) is very similar to the twist studied in [8], the methods used in the respective proofs are completely different. The reason is that in our case the twist represents a singular deformation in the sense that it is discontinuous and occurs at one point only. Our main idea to prove Theorem 9.1 is to introduce rotated Cartesian coordinates in which one can employ the repulsive interaction due to the proximity of opposite Dirichlet boundary conditions, *cf* Figure 9.3. This is done in Section 9.3 where the initial problem is reduced to an ordinary differential equation. The latter is then investigated in Section 9.4 by standard methods for one-dimensional Schrödinger operators.

Note that Theorem 9.1 contains a weaker version of inequality (9.2), namely with a compactly supported Hardy weight. However, (9.2) can be easily deduced from it:

Corollary 9.1. *Inequality (9.2) holds true with the function ρ given by*

$$\rho(x,y) := \frac{c_h}{1+x^2}\,, \qquad c_h := \frac{c}{16c+2+16/a^2}\,,$$

where c is the constant from Theorem 9.1.

A short proof of Corollary 9.1, based on the classical one-dimensional Hardy inequality, is given in the concluding Section 9.5.

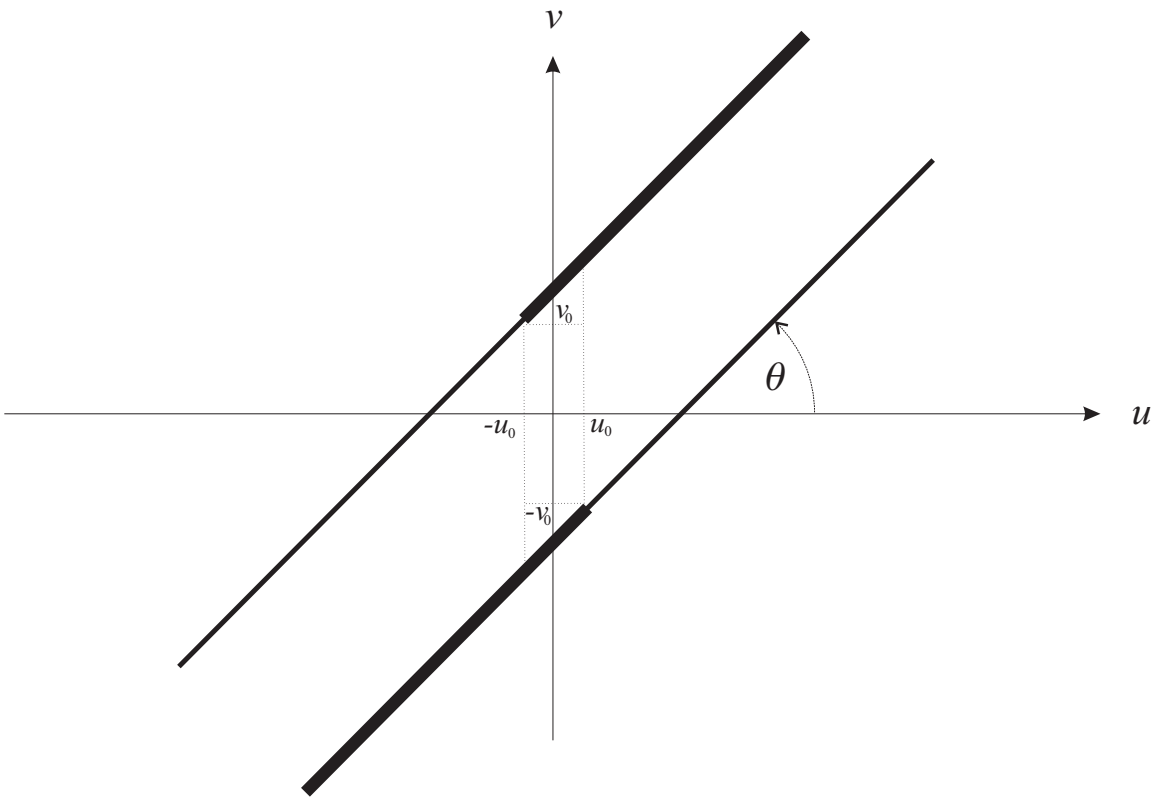

Figure 9.3: Rotating the Cartesian coordinate system by an appropriate angle θ, one can employ the repulsive interaction due to the proximity of opposite Dirichlet boundary conditions (thick lines).

9.3 Reduction to a one-dimensional problem

Hereafter we consider non-negative ε only. Let $(x, y) \in \mathbb{R}\times(-a, a)$. We introduce rotated Cartesian coordinates (u, v) by the change of variables

$$(x, y) = f(u, v) := \big(u\cos\theta + v\sin\theta, -u\sin\theta + v\cos\theta\big)\,, \tag{9.10}$$

where $\theta \in (0, \pi/2)$. Clearly, the mapping $f : \Omega \to \mathbb{R} \times (-a, a)$ is a diffeomorphism with the preimage

$$\begin{aligned}\Omega := f^{-1}(\mathbb{R}\times I) &= \big\{(u, v) \in \mathbb{R}^2 \mid u_-(v) < u < u_+(v)\big\}\\ &= \big\{(u, v) \in \mathbb{R}^2 \mid v_-(u) < v < v_+(u)\big\}\,,\end{aligned}$$

where

$$u_\pm(v) := \frac{\pm a + v\cos\theta}{\sin\theta}\,, \qquad v_\pm(u) := \frac{\pm a + u\sin\theta}{\cos\theta}\,.$$

Introducing the (unitary) change of trial function $\psi \mapsto \psi \circ f := \phi$ into the functional (9.4), we find

$$Q_\varepsilon[\phi\circ f^{-1}] = \int_\Omega \big(|\partial_1\phi(u, v)|^2 + |\partial_2\phi(u, v)|^2\big)\, du\, dv\,. \tag{9.11}$$

From the formulae

$$\phi\big(u, v_\pm(u)\big) = \psi\left(\frac{u \pm a\sin\theta}{\cos\theta}, \pm a\right), \qquad \phi\big(u_\pm(v), v\big) = \psi\left(\frac{v \pm a\cos\theta}{\sin\theta}, \mp a\right),$$

we observe the two following properties, respectively. First, $v \mapsto \phi(u, v)$ with u fixed satisfies Dirichlet boundary conditions at both boundary points $v_\pm(u)$ if, and only if,

$$|u| < u_0 := a\sin\theta - \varepsilon\cos\theta\,; \tag{9.12}$$

otherwise it satisfies a combination of Dirichlet and Neumann boundary conditions. (Recall that the boundary value of a function from the form domain as well as the value of its derivative are not affected by imposing

Neumann boundary conditions.) Second, $u \mapsto \phi(u,v)$ with v fixed satisfies a combination of Dirichlet and Neumann boundary conditions, if, and only if,

$$|v| > v_0 := a\cos\theta + \varepsilon\sin\theta\,; \tag{9.13}$$

otherwise it satisfies Neumann boundary conditions (*i.e.* none). While v_0 is positive by definition, we need to assume that

$$\varepsilon < a\,\tan\theta \tag{9.14}$$

in order to ensure the positivity of u_0.

We proceed by estimating the form (9.11) as follows. We estimate the second term in (9.11) by the lowest eigenvalue of the Laplacian in the cross-section of length $v_+(u) - v_-(u) = 2a/\cos\theta$, subject to the boundary conditions of the type that $v \mapsto \phi(u,v)$ satisfies. We also estimate the first term in (9.11) by the lowest eigenvalue of the Laplacian in the cross-section of length $u_+(v) - u_-(v) = 2a/\sin\theta$, subject to the boundary conditions of the type that $u \mapsto \phi(u,v)$ satisfies, but only in the subset of Ω where $|u| > u_0$ and $|v| > v_0$. Finally, we notice that the sum of the two estimating eigenvalues in the subset of Ω where $|u| > u_0$ and $|v| > v_0$ gives just $\pi^2/(4a)^2$ by virtue of the identity $\sin^2 + \cos^2 = 1$. That is,

$$\tilde{Q}_\varepsilon[\phi\circ f^{-1}] \geq \int_{\Omega_1\cup\Omega_2} |\partial_1\phi|^2 + q_+ \int_{\Omega_1} |\phi|^2 - q_- \int_{\Omega_2} |\phi|^2\,, \tag{9.15}$$

where

$$\Omega_1 := \{(u,v)\in\Omega \,|\, |u| < u_0\}\,, \qquad \Omega_2 := \{(u,v)\in\Omega \,|\, |v| < v_0,\ |u| > u_0\}\,,$$

and

$$q_+ := \left(\frac{\pi}{4a}\right)^2 (4\cos^2\theta - 1)\,, \qquad q_- := \left(\frac{\pi}{4a}\right)^2 \sin^2\theta\,. \tag{9.16}$$

Hereafter we further restrict the angle θ by the requirement

$$\theta \in (0, \pi/3)\,, \tag{9.17}$$

so that the term q_+ is positive.

We use the intermediate bound (9.15) as the starting point of the reduction to a one-dimensional problem. Let us introduce the disjoint sets

$$\Omega_1' := \{(u,v)\in\Omega \,|\, |u| < u_0,\ |v| > v_0\}\,, \qquad \Omega_2' := \{(u,v)\in\Omega \,|\, |v| < v_0\}\,,$$

and note that the inclusions $\Omega_1' \subset \Omega_1$ and $\Omega_2' \subset \overline{\Omega_1\cup\Omega_2}$ hold. Consequently, under the assumption (9.17), (9.15) implies the cruder bound

$$\tilde{Q}_\varepsilon[\phi\circ f^{-1}] \geq q_+ \int_{\Omega_1'} |\phi(u,v)|^2\,du\,dv + \int_{\Omega_2'} \lambda(v)\,|\phi(u,v)|^2\,du\,dv\,, \tag{9.18}$$

where $\lambda(v) \in (q_-, q_+)$ is the lowest eigenvalue of the one-dimensional Neumann Schrödinger operator with the step-like potential

$$V(u,v) := q_+\ \chi_{(-u_0,u_0)}(u) - q_-\ \chi_{(u_-(v),-u_0)\cup(u_0,u_+(v))}(u)\,.$$

More precisely,

$$\lambda(v) := \inf_{\varphi} \frac{\int_{u_-(v)}^{u_+(v)} \left[|\varphi'(u)|^2 + V(u,v)\,|\varphi(u)|^2\right] du}{\int_{u_-(v)}^{u_+(v)} |\varphi(u)|^2\,du}\,, \tag{9.19}$$

where the infimum is taken over all non-zero functions from the Sobolev space $W^{1,2}\big(u_-(v), u_+(v)\big)$.

The formula (9.18) together with (9.19) transfers the initial two-dimensional problem into the study of an ordinary differential equation. That is, it remains to investigate the function $v \mapsto \lambda(v)$.

9.4 Study of the one-dimensional problem

First of all, we observe that $v \mapsto \lambda(v)$ is an even function with values in the open interval (q_-, q_+) due to (9.17). Furthermore, its minimum is attained at the boundary points $v = \pm v_0$:

Lemma 9.1. *One has* $\inf_{v\in(-v_0,v_0)} \lambda(v) = \lambda(v_0)\,.$

Proof. Let h, l and δ be positive numbers such that $\delta < 1$. For any real c, we consider the one-dimensional Schrödinger operator

$$H_c := -\Delta + h\,\chi_{(c,c+\delta l)} \qquad \text{in} \qquad L^2\big((0,l)\big)\,,$$

subject to Neumann boundary conditions. (H_c is introduced in a standard way through the associated quadratic form defined in $W^{1,2}((0,l))$.) Let us show that

$$\forall c \in (0, l-\delta l), \qquad \inf\sigma(H_c) \geq \inf\sigma(H_0)\,, \tag{9.20}$$

which is equivalent to the statement of the Lemma. (Indeed, making an obvious change of integration variable in (9.19), we can reconsider the variational problem for $\lambda(v)$ on a fixed interval $(0,l)$ with $l := u_+(v) - u_-(v) = 2a/\sin\theta$. Then $\lambda(v) + q_-$ coincides with the lowest eigenvalue of H_c with $h := q_+ + q_-$, $\delta := (u_0/a)\sin\theta$ and $c := -u_0 - u_-(v) \geq -u_0 - u_-(v_0) = 0$.)

The reader is advised to consult Figure 9.4 for the following construction. Given $c \in (0, l-\delta l)$, we find $\alpha_1, \alpha_2 \in (0,1)$ such that

$$\alpha_1 + \alpha_2 = 1 \qquad \text{and} \qquad \frac{\alpha_1}{\alpha_2} = \frac{c}{l-(c+\delta l)}\,.$$

We also define parameters $\delta_1, \delta_2 \in (0,\delta)$ by the equations

$$\delta_1 + \delta_2 = \delta \qquad \text{and} \qquad \frac{\delta_1}{\delta_2} = \frac{\alpha_1}{\alpha_2}\,.$$

It follows that $\alpha_1 l = c + \delta_1 l$. Let $t^* := \alpha_1 l \in (0,l)$.

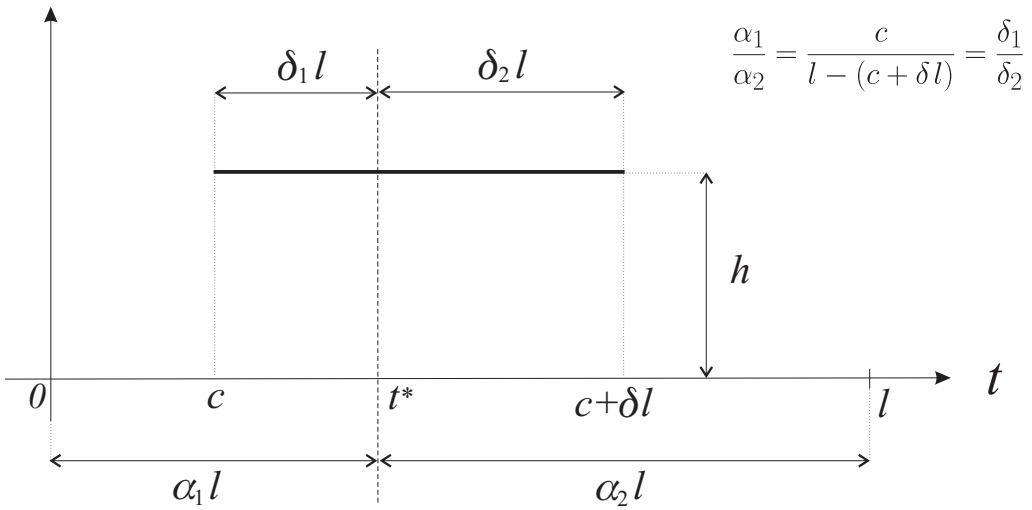

Figure 9.4: The construction used in the proof of Lemma 9.1.

The minimax principle yields

$$\inf\sigma(H_c) \geq \inf\sigma(H_c^N)\,, \tag{9.21}$$

where H_c^N is the operator obtained from H_c by imposing an additional Neumann boundary condition at the point t^*. To arrive at (9.21) one uses the fact that the form domain of H_c is contained in the form domain of H_c^N. The operator H_c^N is a direct sum of two operators, which are unitarily equivalent to

$$\begin{aligned} T_1 &:= -\Delta + h\,\chi_{(0,\delta_1 l)} && \text{in} \qquad L^2\big((0,\alpha_1 l)\big)\,,\\ T_2 &:= -\Delta + h\,\chi_{(0,\delta_2 l)} && \text{in} \qquad L^2\big((0,\alpha_2 l)\big)\,, \end{aligned}$$

respectively, both subject to Neumann boundary conditions. Hence,

$$\sigma(H_c^N) = \sigma(T_1) \cup \sigma(T_2)\,. \tag{9.22}$$

Obvious changes of variable show that that T_1 and T_2 are unitarily equivalent to the operators

$$\begin{aligned} \hat{T}_1 &:= -(\delta/\delta_1)^2\,\Delta + h\,\chi_{(0,\delta l)} && \text{in} \qquad L^2\big((0,l)\big)\,,\\ \hat{T}_2 &:= -(\delta/\delta_2)^2\,\Delta + h\,\chi_{(0,\delta l)} && \text{in} \qquad L^2\big((0,l)\big)\,, \end{aligned}$$

respectively, both subject to Neumann boundary conditions. Consequently,

$$T_1 \geq H_0 \qquad \text{and} \qquad T_2 \geq H_0$$

in the sense of quadratic forms. This together with (9.22) and (9.21) implies

$$\inf \sigma(H_c) \geq \min \left\{ \inf \sigma(T_1), \inf \sigma(T_2) \right\} \geq \inf \sigma(H_0) \,.$$

□

As a consequence of (9.18) and the above Lemma, we therefore obtain

$$\tilde{Q}_\varepsilon[\phi \circ f^{-1}] \geq \lambda(v_0) \int_{\Omega_1' \cup \Omega_2'} |\phi(u,v)|^2 \, du \, dv \,. \tag{9.23}$$

We now turn to a more quantitative study of $\lambda(v_0)$. The eigenvalue problem associated with (9.19) can be solved explicitly in the intervals where the potential V is constant. The problem on the whole interval is then solved by matching these solutions smoothly in the discontinuity points of V. Doing this, one easily finds that $\lambda(v_0)$ coincides with the smallest root $\lambda \in (q_-, q_+)$ of the equation

$$g_1(\lambda, \varepsilon, \theta) = g_2(\lambda, \varepsilon, \theta) \,, \tag{9.24}$$

where

$$g_1(\lambda, \varepsilon, \theta) := \sqrt{q_+ - \lambda} \, \tanh\left(2u_0 \sqrt{q_+ - \lambda}\right),$$
$$g_2(\lambda, \varepsilon, \theta) := \sqrt{q_- + \lambda} \, \tan\left(2v_0 \cot\theta \, \sqrt{q_- + \lambda}\right).$$

Recall that q_+, q_- and u_0, v_0 are introduced in (9.16) and (9.12)–(9.13), respectively. Of course, g_2 is not defined for all the values of the parameters $\lambda, \varepsilon, \theta$, and we should rather multiply (9.24) by $\cos\left(2v_0 \cot\theta \sqrt{q_- + \lambda}\right)$, but the resulting (regular) equation cannot be satisfied if the cosine equals zero, so we can leave (9.24) in the present form.

Let us first consider the case $\varepsilon = 0$. A necessary condition to guarantee the eligibility of our method to prove Theorem 9.1 is that $\lambda(v_0)$ is positive for certain angle θ satisfying (9.17). A numerical study of (9.24) shows that $\lambda(v_0)$ achieves its maximum, given approximately by $0.040\,\pi^2/(4a)^2$, for the angle $\theta \approx 0.774$. Observing that the optimal angle is close to $\pi/4 \approx 0.785$, let us fix henceforth:

$$\theta = \pi/4 \,. \tag{9.25}$$

Since $\lambda \mapsto g_1(\lambda, 0, \pi/4)$ is decreasing and continuous, $\lambda \mapsto g_2(\lambda, 0, \pi/4)$ is increasing and continuous, and at $\lambda = 0$ we have

$$\frac{g_1(0,0,\pi/4)}{g_2(0,0,\pi/4)} = \sqrt{2} \, \tanh\left(\sqrt{2}\,\pi/4\right) > 1 \,, \tag{9.26}$$

it follows that $\lambda(v_0)$ is indeed positive for the choice (9.25). As for the numerical value, it is straightforward to check that (9.24) reduces to (9.8) and we find that the smallest root s_1 of the latter equals approximately 0.039. Summing up, (9.23) implies

$$\tilde{Q}_\varepsilon[\phi \circ f^{-1}] \geq s_1 \left(\frac{\pi}{4a}\right)^2 \int_{\Omega_1' \cup \Omega_2'} |\phi(u,v)|^2 \, du \, dv \,,$$

provided the angle θ is chosen according to (9.25). In order to establish (i) of Theorem 9.1, it remains to realize that

$$\overline{f\left(\Omega_1' \cup \Omega_2'\right)} \supset (-a,a) \times (-a,a) \,,$$

where f is given by (9.10).

In the case of positive ε, we put λ equal to zero in (9.24) and look for the smallest positive ε satisfying the equation (9.24). This root satisfies the restriction (9.14) because $\varepsilon \mapsto g_1(0, \varepsilon, \pi/4)$ is decreasing and continuous, $\varepsilon \mapsto g_2(0, \varepsilon, \pi/4)$ is increasing and continuous, $g_1(0, a, \pi/4) = 0$, $g_2(0, \varepsilon, \pi/4)$ tends to $+\infty$ as $\varepsilon \to a$, and we have (9.26) for $\varepsilon = 0$. It is straightforward to check that (9.24) reduces to (9.9) for the choice (9.25) and the smallest positive root t_1 of the latter equals approximately 0.061. This proves that the spectrum of $-\Delta_\varepsilon$ does not start below $\pi^2/(4a)^2$ for all positive ε less than or equal to this critical value, while the opposite inclusion of (ii) of Theorem 9.1 follows from (9.1). Again, a more detailed numerical study of (9.24) shows that the best result reachable by the present method gives $\varepsilon_c \approx 0.063\,a$ with the optimal angle $\theta \approx 0.759$.

This concludes the proof of Theorem 9.1.

9.5 Proof of Corollary 9.1

The idea behind the proof of Corollary 9.1 is to employ the "longitudinal kinetic energy" in order to "smear out" a local Hardy weight over the whole strip. This is precisely expressed in the following lemma (*cf* also [17, Sec. 3.3] or [46, proof of Lem. 2]):

Lemma 9.2. *For any $\psi \in W^{1,2}\big(\mathbb{R}\times(-a,a)\big)$ and any positive number b, one has*

$$\int_{\mathbb{R}\times(-a,a)} |\partial_1\psi|^2 + \left(\frac{1}{8}+\frac{1}{b^2}\right)\int_{(-b,b)\times(-a,a)} |\psi|^2 \;\geq\; \frac{1}{16}\int_{\mathbb{R}\times(-a,a)} \frac{|\psi(x,y)|^2}{1+x^2}\,dx\,dy\,.$$

Proof. The lemma is based on the classical one-dimensional Hardy inequality

$$\int_{\mathbb{R}} |v'(x)|^2\,dx \geq \frac{1}{4}\int_{\mathbb{R}} \frac{|v(x)|^2}{x^2}\,dx \tag{9.27}$$

valid for any $v \in W^{1,2}(\mathbb{R})$ with $v(0)=0$. Let $\xi : \mathbb{R}\times(-a,a)\to\mathbb{R}$ be defined by $\xi(x,y) := |x|/b$ if $x<b$ and equal to 1 elsewhere. Writing $\psi = \xi\psi + (1-\xi)\psi$ and applying (9.27) to the function $x\mapsto(\xi\psi)(x,y)$ with help of Fubini's theorem, we arrive at

$$\begin{aligned}
\int \frac{|\psi(x,y)|^2}{1+x^2}\,dx\,dy &\leq 2\int \frac{|\xi\psi|^2(x,y)}{x^2}\,dx\,dy + 2\int |(1-\xi)\psi|^2 \\
&\leq 16\int |\xi|^2|\partial_1\psi|^2 + 16\int |\partial_1\xi|^2|\psi|^2 + 2\int |(1-\xi)\psi|^2 \\
&\leq 16\int |\partial_1\psi|^2 + 16\left(\frac{1}{8}+\frac{1}{b^2}\right)\int \chi_{(-b,b)\times(-a,a)}\,|\psi|^2\,,
\end{aligned}$$

where the integration sign indicates the integration over $\mathbb{R}\times(-a,a)$. □

Now Corollary 9.1 follows by rather elementary algebraic manipulations. The local Hardy inequality (9.7) is equivalent to

$$A+B \geq c\int_\omega |\psi|^2\,, \quad \text{with} \quad A := \int_{\mathbb{R}\times(-a,a)} |\partial_1\psi|^2\,,\;\; B := \int_{\mathbb{R}\times(-a,a)} |\partial_2\psi|^2 - \Big(\frac{\pi}{4a}\Big)^2 \int_{\mathbb{R}\times(-a,a)} |\psi|^2\,,$$

for any $\psi \in \overline{\mathfrak{D}(Q_\varepsilon)} \subset W^{1,2}\big(\mathbb{R}\times(-a,a)\big)$. Note that B is non-negative due to the boundary conditions that ψ satisfies. We write

$$A+B = \epsilon(A+B) + (1-\epsilon)(A+B) \geq \epsilon A + (1-\epsilon)(A+B) \qquad \text{with} \qquad \epsilon\in(0,1)\,,$$

and estimate the two terms on the right hand side of the inequality by means of Lemma 9.2 and the above equivalent form (9.7), respectively. Consequently,

$$A+B \geq \frac{\epsilon}{16}\int_{\mathbb{R}\times(-a,a)} \frac{|\psi(x,y)|^2}{1+x^2}\,dx\,dy + \left[c - \epsilon\left(c+\frac{1}{8}+\frac{1}{a^2}\right)\right]\int_{(-a,a)^2} |\psi|^2\,.$$

Choosing now ϵ so that the square bracket equals zero, we arrive at the claim of Corollary 9.1.

Acknowledgement

The authors are grateful to the referee for valuable comments. The work has partially been supported by the Czech Academy of Sciences and its Grant Agency within the projects IRP AV0Z10480505 and A100480501, by the project LC06002 of the Ministry of Education, Youth and Sports of the Czech Republic, and by DAAD within the project D-CZ 5/05-06. The second author has partially been supported by FCT/POCTI/FEDER, Portugal.

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Chapter 10

On the spectrum of curved quantum waveguides

Published in: Publications of the Research Institute for Mathematical Sciences **41** (2005), 757–791

Joint work with: Jan Kříž

On the Spectrum of Curved Quantum Waveguides

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Abstract. The spectrum of the Laplace operator in a curved strip of constant width built along an infinite plane curve, subject to three different types of boundary conditions (Dirichlet, Neumann and a combination of these ones, respectively), is investigated. We prove that the essential spectrum as a set is stable under any curvature of the reference curve which vanishes at infinity and find various sufficient conditions which guarantee the existence of geometrically induced discrete spectrum. Furthermore, we derive a lower bound on the gap between the essential spectrum and the spectral threshold for locally curved strips. The paper is also intended as an overview of some new and old results on spectral properties of curved quantum waveguides.

10.1 Introduction

Let Ω be a region in $\mathbb{R}^n$, $n \geq 1$, with sufficiently regular boundary $\partial\Omega$, and consider the corresponding Laplace operator $-\Delta$ on $L^2(\Omega)$ with mixed Dirichlet-Neumann boundary conditions. If Ω is bounded, then it is well known that the spectrum of the Laplacian is purely discrete, and properties of the eigenvalues have been intensively studied. On the other hand, it is easy to see that the spectrum is $[0, \infty)$, *i.e.* purely essential, if Ω is unbounded and sufficiently extended (say, conical) at infinity. Although it was shown already by F. Rellich in 1948, [52], that there exist unbounded regions whose spectrum contains discrete eigenvalues (or it is even purely discrete!), the spectral theory for the eigenvalues has attracted much less attention than in the bounded case.

However, recent advent of mesoscopic physics has given a fresh impetus to study the (discrete) spectrum of the Laplacian in unbounded regions. For, let us recall that the quantum Hamiltonian H of a free spin-less particle of effective mass m^* constrained to a spatial region Ω, *i.e.* $H = -\hbar^2/(2m^*)\Delta$ on $L^2(\Omega)$, represents a reasonable model for the dynamics in various semiconductor structures devised and produced in the laboratory nowadays. Here it is mostly natural to consider the Dirichlet boundary conditions on $\partial\Omega$ corresponding to a large chemical potential barrier, however, other situations modelling the impenetrable walls of Ω may be relevant as well (see *e.g.* [48]) and can in principle model different types of interphase in a solid. We refer to [14, 49, 40] for the physical background and references. An important category of these systems is represented by so-called *quantum waveguides* which are modelled by infinitely stretched tubular regions in $\mathbb{R}^n$ with $n = 2, 3$.

The simplest situation occurs if Ω is an infinite plane strip, *i.e.*, a (tubular neighbourhood of constant width along an infinite curve in $\mathbb{R}^2$. In 1989, P. Exner and P. Šeba demonstrated the existence of discrete spectrum for curved Dirichlet strips which were asymptotically straight and sufficiently thin, [31]. Numerous subsequent studies improved their result and generalised it to space tubes, [39, 53, 14]. For more information and other spectral and scattering properties, see the review paper [14] and references therein. An important improvement was made by J. Goldstone and R. L. Jaffe in 1992, [39]; the authors introduced a variational argument which enables them to demonstrate the existence of discrete eigenvalues without the restriction on the width of the strip. A recent article [45] deals with a more general situation when the strip is not constructed in $\mathbb{R}^2$ but in a two-dimensional Riemannian manifold. As examples of strips with mixed Dirichlet-Neumann boundary conditions, let us mention the works [12, 13]. The evidently more complicated case of Dirichlet layers, *i.e.*, Ω is a tubular neighbourhood about a complete non-compact surface in $\mathbb{R}^3$, was investigated in [15, 16, 28, 8].

A common property of the Dirichlet systems cited above is that a bending of a straight strip or layer generates discrete eigenvalues below the essential spectrum, *i.e.*, geometrically induced quantum *bound states*, which are known to disturb the particle transport. The result is also interesting from the semiclassical point of view because there are no classical closed trajectories in the tubes in question, apart from a zero measure set of initial conditions in the phase space. Hence, this is a pure quantum effect of geometrical origin. The spectral results become richer if one considers more complicated boundary conditions; here the problem is interesting even for some straight waveguides, [12].

Apart from the curved quantum waveguides, the discrete spectrum can be also generated by a local deformation of the boundary $\partial\Omega$ of straight tubes and layers, [5, 4, 36], via introducing an obstacle, [17, 10, 1],

or impurities modelled by a Dirac interaction, [19, 27, 29, 30], coupling several waveguides by a window, [33, 34, 35], *etc.* The spectrum of periodically and randomly curved waveguides was investigated in [57, 55] and [42], respectively. Finally, let us mention systems where $\Omega = \mathbb{R}^n$, $n = 2, 3$, and the quantum waveguide is introduced by means of a magnetic field, [21, 26, 22], or a strong Dirac interaction supported by an infinite curve or surface, [20, 37, 24, 25, 18, 23].

10.2 Scope of the Paper

The main aim of the present paper is to study the geometrically induced (discrete) spectrum of the quantum Hamiltonian H^ι of a free non-relativistic particle living in the infinite planar curved strip (the index ι will distinguish different boundary conditions considered here). Removing physical constants, H^ι is identified with the Laplacian $-\Delta$ on $L^2(\Omega)$ where Ω is a (one-sided) tubular neighbourhood of a fixed width $d > 0$ along an infinite plane curve Γ of curvature k, see Figure 10.1. The boundary $\partial\Omega$ consists of two parallel connected curves which are supposed to be of class C^2. We compare three different types of boundary conditions representing impenetrable walls of the strip in the sense that there is no probability current through the boundary. In particular, we consider the recently widely investigated *Dirichlet* boundary condition ($\iota := D$), the *Neumann* boundary condition ($\iota := N$) and the simplest *combination* of the both just mentioned ($\iota := DN$): the Dirichlet boundary condition imposed on one connected component of $\partial\Omega$ and the Neumann condition on the other one.

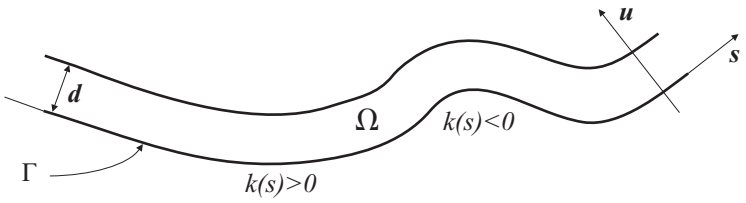

Figure 10.1: Configuration space Ω defined as a strip over an infinite curve Γ in $\mathbb{R}^2$.

If the reference curve Γ is a *straight* line, then it is rather a textbook exercise to analyse the operator H^ι by means of a separation of variables and conclude that its spectrum is purely absolutely continuous and equals the interval $[E_1^\iota, \infty)$, where the non-negative value E_1^ι is determined by the respective boundary conditions, see (10.10). However, the spectral problem for H^ι becomes always difficult whenever Γ is *curved*, and two basic questions arise in this context:

1. Which geometry preserves the essential spectrum $[E_1^\iota, \infty)$?
2. Which geometry produces a spectrum below E_1^ι?

These questions represent ultimate concern of this paper. We try to make a survey of known answers and contribute to the problem by our own results. Furthermore, if the spectrum below E_1^ι exists, we establish various estimates of the spectral threshold $\inf \sigma(H^\iota)$. It should be stressed here that the existence of *discrete* spectrum, *i.e.* the issue mentioned in Introduction, is proved whenever the considered geometry is in accordance with *both* the above questions (because then the spectrum below E_1^ι consists of isolated eigenvalues of finite multiplicity only).

Concerning the first question, we show that the essential spectrum of a curved strip coincides with the spectrum of the straight one provided the reference curve Γ is straight asymptotically in the sense that its curvature vanishes at infinity, *cf* Theorem 10.1. Although this sufficient condition is very natural and in perfect accordance with the intuition, it is for the first time in this paper when the essential spectrum is localised without imposing any additional conditions (*e.g.*, about the decay of the derivatives of curvature at infinity, *cf* [14, 53, 13]). The progress has become possible due to a general characterisation of essential spectrum adopted from [11] (see our Lemma 10.1), which is for our purposes more suitable than the classical Weyl criterion. On the other hand, periodic strips are discussed as an illustration of asymptotically non-straight geometry which does change the essential spectrum.

The answer to the second question depends substantially on the choice of boundary conditions. First of all, notice that the question does not make sense for the Neumann strips because $E_1^N = 0$, *cf* Theorem 10.2. A characteristic property of the Dirichlet strips is that any bending of the reference curve Γ pushes the infimum of the spectrum below the spectral threshold $E_1^D > 0$ of the corresponding straight strip, *cf* Theorem 10.3.

This property was shown first in [31] for sufficiently thin strips and the proof for general cases was introduced in [39]. On the other hand, the case of combined Dirichlet-Neumann boundary condition was investigated in the recent letter [13]. The authors established the existence of spectrum below $E_1^{DN} > 0$ provided the total bending angle of Γ (*i.e.*, the integral of curvature, see (10.14)) has a suitable sign. In this paper, we generalise this result and add two new sufficient conditions, *cf* Theorem 10.4. We also derive an interesting result on the number of discrete eigenvalues of H^{DN}, *cf* Proposition 10.1

Finally, when Γ is curved only locally, we derive an upper bound on the spectral threshold, *i.e.* $\inf\sigma(H^\iota)$, for the Dirichlet strip and the one with combined Dirichlet-Neumann boundary condition, *cf* Theorem 10.5. In particular, we find important qualitative differences between these two respective results. Making the curvature small, the leading term in the estimate of the difference $\inf\sigma(H^D) - E_1^D$ is proportional to the fourth power of the total bending angle, while it is the second power what one obtains for the Dirichlet-Neumann case, *cf* Remark 10.9. Another interesting difference appears, when we are shrinking the width of the strip to zero, *cf* Remark 10.10. These estimates are new in the theory of curved quantum waveguides. We can only compare them with the eigenvalue asymptotics for mildly curved, respectively thin, Dirichlet strips established in [14]. Let us note that a similar estimate for straight, window-coupled waveguides was given in [34, 35], see also [3].

All our proofs of the statements concerning the existence and properties of the spectrum below E_1^ι are based on a variational strategy. The corner stone of them, *i.e.* the construction of a suitable trial function, follows the idea of [39], see also [14, 53].

The paper is organised as follows. Section 10.3 is devoted to some preliminary material in order to be able to state precisely the main results of the paper, *i.e.* Theorems 10.1–10.5, in the subsequent Section 10.4. The proofs and discussions of the Theorems are presented in Sections 10.5 and 10.6. We conclude the paper by Section 10.7, where some open problems and directions of a future research are mentioned.

10.3 Preliminaries

10.3.1 Configuration Space

Let Γ be a unit-speed infinite plane curve *i.e.* the (image of the) C^2-mapping $\Gamma : \mathbb{R} \to \mathbb{R}^2$ satisfying $|\dot\Gamma(s)| = 1$ for all $s \in \mathbb{R}$ (the arc-length parameter of the curve). The function $N := (-\dot\Gamma^2, \dot\Gamma^1)$ defines a unit normal vector field and the couple $(\dot\Gamma, N)$ gives a distinguished Frenet frame, *cf* [42, Chap. 1]. The curvature is defined through the Frenet-Serret formulae by $k := \det(\dot\Gamma, \ddot\Gamma)$. We note that k is a continuous function and the sign of $k(s)$ is defined uniquely up to the re-parameterisation $s \mapsto -s$ of the arc-length parameter. It is also worth to notice that the curve Γ is fully determined (except for its position and orientation in the plane) by the curvature function k only, *cf* [46, Sec. II. 20].

Let $d > 0$, $I := (0, d)$ and $\Omega_0 := \mathbb{R} \times I$ be a straight strip of width d. A curved strip of the same width based on Γ is defined via $\Omega := \mathcal{L}(\Omega_0)$, where

$$\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^2 : \{(s,u) \mapsto \Gamma(s) + u\, N(s)\}\,. \tag{10.1}$$

Then $s \mapsto \mathcal{L}(s,u)$ for u fixed traces out a parallel curve at a distance $|u|$ from Γ. Through all the paper, we shall always assume that

$\langle \mathsf{H} \rangle$ $\quad \Omega$ is not self-intersecting $\quad$ and $\quad k \in L^\infty(\mathbb{R})$ with $d\,\|k_+\|_\infty < 1$,

where $k_\pm := \max\{0, \pm k\}$. Then the mapping $\mathcal{L} : \Omega_0 \to \Omega$ is a C^1-diffeomorphism and its inverse determines a system of natural "coordinates" (s,u) in a neighbourhood of Γ. We remark that under our assumption $\langle \mathsf{H} \rangle$ the curve $\mathcal{L}(\mathbb{R} \times \{u\})$ is of class C^2 for any fixed $u \in \overline{I}$, in particular, this claim holds true for both the boundary curves.

Remark 10.1. In this paper, we adopt the standard component notation of the tensor analysis together with the repeated indices convention. The range of indices is $1, 2$ and they are associated with the above mentioned coordinates via $(1,2) \leftrightarrow (s,u)$. The partial derivatives are marked by a comma with the index.

By virtue of the Frenet-Serret formulae, the metric tensor of Ω in these coordinates, *i.e.* $G_{ij} := \mathcal{L}_{,i} \cdot \mathcal{L}_{,j}$ where "$\cdot$" denotes the scalar product in $\mathbb{R}^2$, has the following diagonal form

$$\big(G_{ij}(s,u)\big) = \begin{pmatrix} (1 - uk(s))^2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{10.2}$$

Its determinant, $G := \det(G_{ij})$, defines through $d\Omega := G(s,u)^{\frac{1}{2}} ds du$ the area element of the strip. By virtue of the second part of the assumption $\langle \mathsf{H} \rangle$, it is clear that the metric (10.2) is uniformly elliptic. In particular, we have the following useful estimates:

$$\forall (s,u) \in \Omega_0 \;:\quad C_- \le 1 - uk(s) \le C_+ \qquad \text{with} \qquad C_\pm := 1 \pm d\,\|k_\mp\|_\infty. \tag{10.3}$$

10.3.2 Hamiltonian

The infinite strip-like region Ω models the configuration space of a non-relativistic particle in a curved quantum waveguide. Putting $\hbar^2/(2m) = 1$, the particle Hamiltonian could be identified with the Laplace operator, $-\Delta$ on $L^2(\Omega)$. The dynamics is then well defined by means of the boundary conditions imposed on $\partial\Omega$. Using the mapping (10.1), we identify the Hilbert space $L^2(\Omega)$ with $\mathcal{H} := L^2(\Omega_0, d\Omega)$ and consider three different situations.

Dirichlet case. The easiest case is given by the Dirichlet boundary conditions. The corresponding Hamiltonian H^D is defined as a unique self-adjoint operator on $\mathcal{H}$ associated with the quadratic form

$$Q^D[\psi] := \left(\psi_{,i}, G^{ij}\psi_{,j}\right), \tag{10.4}$$

$$\mathrm{Dom}\, Q^D := W_0^{1,2}(\Omega_0, d\Omega). \tag{10.5}$$

Here and in what follows, (G^{ij}) stands for the inverse of (G_{ij}) and $(\cdot,\cdot)$ denotes the scalar product in $\mathcal{H}$; the induced norm will be denoted by $\|\cdot\|$.

Neumann case. The Hamiltonian H^N corresponding to the Neumann boundary conditions imposed on $\partial\Omega$ is defined as a unique self-adjoint operator on $\mathcal{H}$ associated with the quadratic form Q^N which acts like Q^D in (10.4), however, on a different domain:

$$\mathrm{Dom}\, Q^N := W^{1,2}(\Omega_0, d\Omega). \tag{10.6}$$

Dirichlet-Neumann case. The last situation considered here is given by the Dirichlet boundary condition imposed on the reference curve $\mathcal{L}(\mathbb{R}\times\{0\})$ and the Neumann one imposed on the opposite boundary $\mathcal{L}(\mathbb{R}\times\{d\})$. More precisely, the Hamiltonian H^{DN} in such a case is defined as a unique self-adjoint operator on $\mathcal{H}$ associated with the quadratic form Q^{DN} which acts like Q^D in (10.4), however, with

$$\mathrm{Dom}\, Q^{DN} := \left\{\psi \in W^{1,2}(\Omega_0, d\Omega) \mid \psi(s,0) = 0 \quad \text{for a.e. } s \in \mathbb{R}\right\}. \tag{10.7}$$

Here $\psi(\cdot,0)$ denotes the trace of the function ψ on the boundary part $\mathcal{L}(\mathbb{R}\times\{0\})$.

Remark 10.2. Since the metric (G_{ij}) is uniformly elliptic due to (10.3), it is not necessary to take into account the measure $d\Omega$ in (10.5), (10.6) and (10.7).

Sometimes it will be convenient to consider two or all of the three different situations simultaneously and employ the common superscript ι in order to unify the notation. Hereafter, whenever we use this abridged notation without specifying the range of ι explicitly, we shall assume $\iota \in \{D, N, DN\}$.

Remark 10.3 (Operators associated with Q^ι)**.** We have

$$H^\iota = -G^{-\frac{1}{2}}\partial_i G^{\frac{1}{2}} G^{ij}\partial_j\,, \tag{10.8}$$

which is a general expression for the Laplace operator in a manifold equipped with a metric (G_{ij}). The equality in (10.8) must be understood in the form sense if the curvature k is not differentiable (which is the case we are particularly concerned to deal with in this paper). Nevertheless, assuming that the reference curve Γ is, say, C^3-smooth, then the metric is differentiable and, putting (10.2) into (10.8), we can write

$$H^\iota = -\frac{1}{(1-uk(s))^2}\,\partial_s^2 - \frac{u\dot{k}(s)}{(1-uk(s))^3}\,\partial_s - \partial_u^2 + \frac{k(s)}{1-uk(s)}\,\partial_u$$

in the strong sense on $\mathrm{Dom}\, H^\iota$. Moreover, we can give the explicit form of the operator domain $\mathrm{Dom}\, H^\iota$, see [47]:

$$\begin{aligned}
\mathrm{Dom}\, H^D &= \left\{\psi \in W^{2,2}(\Omega_0) \mid \psi(s,0) = \psi(s,d) = 0 \quad \text{for a.e. } s \in \mathbb{R}\right\},\\
\mathrm{Dom}\, H^N &= \left\{\psi \in W^{2,2}(\Omega_0) \mid \psi_{,2}(s,0) = \psi_{,2}(s,d) = 0 \quad \text{for a.e. } s \in \mathbb{R}\right\},\\
\mathrm{Dom}\, H^{DN} &= \left\{\psi \in W^{2,2}(\Omega_0) \mid \psi(s,0) = \psi_{,2}(s,d) = 0 \quad \text{for a.e. } s \in \mathbb{R}\right\}.
\end{aligned}$$

10.3.3 Straight Strips

If the strip is straight in the sense that $k \equiv 0$, *i.e.* the curvature of the reference curve is equal to zero everywhere on $\mathbb{R}$, then the Hamiltonian coincides with the decoupled operator

$$H_0^\iota := \overline{-\Delta^{\mathbb{R}} \otimes Id + Id \otimes (-\Delta_\iota^I)} \qquad \text{on} \quad L^2(\mathbb{R}) \otimes L^2(I), \tag{10.9}$$

where Id denotes the identity operator on appropriate spaces. The operators on the transverse section, $-\Delta_\iota^I$, are the usual Laplacians on $L^2(I)$ with the Dirichlet boundary conditions if $\iota = D$, the Neumann conditions if $\iota = N$, or the Dirichlet condition at 0 and the Neumann one at d if $\iota = DN$. The eigenvalues of $-\Delta_\iota^I$ are given by

$$E_n^D := (\pi/d)^2 n^2, \quad E_n^N := (\pi/d)^2 (n-1)^2, \quad E_n^{DN} := (\pi/d)^2 (n - \tfrac{1}{2})^2, \tag{10.10}$$

where $n \in \mathbb{N} \setminus \{0\}$. The corresponding family of normalised eigenfunctions $\{\chi_n^\iota\}_{n=1}^\infty$ can be chosen in the following way:

$$\chi_n^\iota(u) := \sqrt{\tfrac{2}{d}} \sin \sqrt{E_n^\iota}\, u \quad \text{for} \quad \iota \in \{D, DN\}\,; \tag{10.11}$$

$$\chi_n^N(u) := \begin{cases} \sqrt{\frac{1}{d}} & \text{if} \quad n = 1, \\ \sqrt{\frac{2}{d}} \cos \sqrt{E_n^N}\, u & \text{if} \quad n \geq 2. \end{cases} \tag{10.12}$$

In view of (10.9), the straight strip has an absolutely continuous spectrum starting from the first eigenvalue of the transverse Laplacian, *i.e.*,

$$\sigma(H_0^\iota) = \sigma_{\mathrm{ess}}(H_0^\iota) = [E_1^\iota, \infty). \tag{10.13}$$

We shall use this trivial case of quantum strips as a comparative system whose spectrum is known explicitly.

10.4 Main Results

As we have seen, the essential spectrum of a straight waveguide, *i.e.* $k \equiv 0$, is the interval $[E_1^\iota, \infty)$. In Section 10.5, we prove that the same spectral result holds for any curved waveguide which is straight *asymptotically* in the sense that the curvature k vanishes at infinity, *i.e.*,

$\langle \mathsf{d} \rangle \qquad k(s) \xrightarrow[|s| \to \infty]{} 0\ .$

Theorem 10.1 (Essential spectrum)**.** *Suppose* $\langle \mathsf{H} \rangle$*. If the strip obeys* $\langle \mathsf{d} \rangle$*, then*

$$\sigma_{\mathrm{ess}}(H^\iota) = [E_1^\iota, \infty) \qquad \textit{for} \quad \iota \in \{D, N, DN\}.$$

To the best of our knowledge, the spectrum of the Neumann Laplacian H^N, has been previously investigated just for strips which were straight and contained an obstacle, [17, 10]. Hence, our Theorem 10.1 represents a quite new result concerning the spectral theory of curved Neumann strips.

The Dirichlet-Neumann case, *i.e.* $\iota = DN$, was previously considered just in the recent letter [13]. It is mentioned there that $\inf \sigma_{\mathrm{ess}}(H^{DN}) = E_1^{DN}$ provided k has a compact support. Here we have proved that the whole interval $[E_1^{DN}, \infty)$ is in the essential spectrum under much weaker condition $\langle \mathsf{d} \rangle$.

Although the case of Dirichlet strips, *i.e.* $\iota = D$, has already been considered in many works, our Theorem 10.1 represents a new result in this situation as well, since it is for the first time when the whole essential spectrum has been localised under a condition which does not contain derivatives of k. Some decay assumptions about the derivatives of the curvature were even required in order to localise the threshold $\inf \sigma_{\mathrm{ess}}(H^D)$ itself in the previous works, *cf* [14, 53]. (An exception is the paper [45] where, however, only a lower bound on the threshold is given.) Let us mention that the result of Theorem 10.1 was achieved in the thesis [44] under an additional condition about vanishing of the first derivative of k.

Since H^N is non-negative, it follows immediately from Theorem 10.1 that there is no discrete spectrum in asymptotically straight Neumann strips.

Theorem 10.2 (Neumann case)**.** *Suppose* $\langle \mathsf{H} \rangle$*. Then*

$$\inf \sigma(H^N) = E_1^N \equiv 0.$$

Consequently, if the strip is asymptotically straight, i.e. $\langle \mathsf{d} \rangle$*, then*

$$\sigma(H^N) = \sigma_{\mathrm{ess}}(H^N) = [0, \infty),$$

i.e., $\sigma_{\mathrm{disc}}(H^N) = \emptyset$.

Here the fact that the spectral threshold of H^N starts exactly at 0 for any strip can be easily proved by means of a suitable trial function (see Proposition 10.3).

An interesting result in the theory of quantum waveguides is that the curved geometry may produce a non-trivial spectrum below the energy E_1^ι for $\iota \in \{D, DN\}$. The phenomenon is examined in this paper. Notice that any result of the type $\inf\sigma(H^\iota) < E_1^\iota$ together with the decay condition $\langle \mathsf{d} \rangle$ yield that the spectrum below E_1^ι consists of isolated eigenvalues of finite multiplicity only, *i.e.* $\sigma_{\mathrm{disc}}(H^\iota) \neq \emptyset$. However, we do not restrict ourselves to the particular case of asymptotically straight strips, *i.e.*, the geometrically induced spectrum below E_1^ι may have a non-zero Lebesgue measure, too.

Sufficient conditions for the Hamiltonians H^ι with $\iota \in \{D, DN\}$ to have a non-empty spectrum below E_1^ι are known. In particular, any non-trivial curvature of the reference curve pushes the spectrum of H^D down the corresponding spectral threshold of the straight strip.

Theorem 10.3 (Dirichlet case)**.** *Suppose* $\langle \mathsf{H} \rangle$.

If $k \not\equiv 0$, *then* $\inf\sigma(H^D) < E_1^D$.

Consequently, if the strip is not straight but it is straight asymptotically, i.e. $\langle \mathsf{d} \rangle$, *then* H^D *has at least one eigenvalue of finite multiplicity below its essential spectrum* $[E_1^D, \infty)$, i.e., $\sigma_{\mathrm{disc}}(H^D) \neq \emptyset$.

This property was shown first in [31] for sufficiently thin strips with a rapidly decaying curvature and since various improvements have been achieved (see the references mentioned in Introduction, mainly [39]). Since the present paper is intended also as a survey paper, we find useful to make a proof of Theorem 10.3 in Section 10.6.1, simultaneously with the proof of the new result contained in condition (a) of Theorem 10.4 below.

As for the operator H^{DN}, its spectrum was studied for the first time in the recent letter [13]. It shows that the position of the infimum of spectrum essentially depends on the sign of the total bending angle

$$\alpha := \int_{\mathbb{R}} k(s)\, ds, \tag{10.14}$$

which is well defined if we assume that the curvature is integrable. In detail, the authors of [13] proved that: i) the spectrum of H^{DN} in a non-trivially curved strip starts below E_1^{DN} provided $\alpha \leq 0$ and the curvature k is non-positive out of some bounded interval. On the other hand, ii) if $k(s) \geq 0$ for all $s \in \mathbb{R}$, then the spectrum below the energy E_1^{DN} is empty. Our improvement is two-fold. Firstly, we generalise the first claim in the sense that we skip the condition on k. Secondly, we find a sufficient condition which guarantees the existence of spectrum below E_1^{DN} even for some strips with $\alpha > 0$. In addition to these substantial generalisations, we will derive the same result also for periodic waveguides. Let us summarise the spectral properties of H^{DN} into the following theorem.

Theorem 10.4 (Dirichlet-Neumann case)**.** *Suppose* $\langle \mathsf{H} \rangle$.

(i) *If* $k \not\equiv 0$, *then any of the three conditions*

   (a) $k \in L^1(\mathbb{R})$ *and* $\alpha \equiv \int_{\mathbb{R}} k(s)\, ds \leq 0$

   (b) k *is periodic*

   (c) $k_- \not\equiv 0$ *and* d *is small enough*

   is sufficient to guarantee that $\inf\sigma(H^{DN}) < E_1^{DN}$.

(ii) *If* $k_- \equiv 0$, *then* $\inf\sigma(H^{DN}) \geq E_1^{DN}$.

Consequently, if the strip is not straight but it is straight asymptotically, i.e. $\langle \mathsf{d} \rangle$, *then any of the conditions* (a) *or* (c) *is sufficient to guarantee that* H^{DN} *has at least one eigenvalue of finite multiplicity below its essential spectrum* $[E_1^{DN}, \infty)$, i.e., $\sigma_{\mathrm{disc}}(H^{DN}) \neq \emptyset$. *On the other hand, if the strip is asymptotically straight and* $k_- \equiv 0$, *then* $\sigma(H^{DN}) = \sigma_{\mathrm{ess}}(H^{DN}) = [E_1^{DN}, \infty)$, i.e., $\sigma_{\mathrm{disc}}(H^{DN}) = \emptyset$.

Remark 10.4. The signs of $k(s)$ and the corresponding total bending angle α change after the change of arc-length parameter given by $s \mapsto -s$. It has to be stressed here that such a re-parameterisation of the reference curve Γ leads to another strip due to (10.1) and, consequently, there is no ambiguity in stating the spectral results on H^{DN} in terms of the sign of α and k, see Figure 10.2.

The sufficient conditions (a)–(c) of the first part of the theorem are proved in Section 10.6.1, while the result of (ii) is already known from [13] (for completeness, we sketch the main ideas of the proof). A comparison of the condition (a) with the assumptions in [13] is done in Remark 10.6.

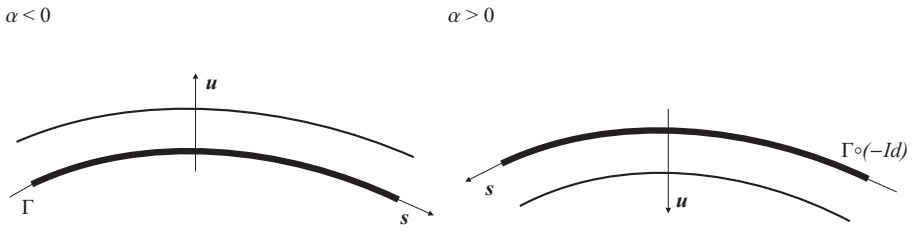

Figure 10.2: Inversion of orientation of the reference curve (given by the re-parameterisation $s \mapsto -s$). Thick lines denote the Dirichlet boundary condition, thin lines the Neumann one.

Consider now a situation when the discrete spectrum of H^ι, $\iota \in \{D, DN\}$, below the energy E_1^ι is not empty.

Although this paper in not intended to investigate the *number* of eigenvalues of H^ι, let us point out the following remarkable property of H^{DN} which we establish at the end of Section 10.6.1.

Proposition 10.1 (Number of bound states in the DN case)**.** *Suppose* $\langle \mathsf{H} \rangle$ *and* $\langle \mathsf{d} \rangle$*. If* $k_- \not\equiv 0$ *then*

$$\forall n \in \mathbb{N} \quad \exists d_n > 0 \ : \quad d < d_n \implies N(H^{DN}) \geq n \,,$$

where $N(H^{DN})$ *denotes the number of discrete eigenvalues of* H^{DN}*, counting multiplicity.*

The number of bound states in thin strips is another property, which demonstrates a significant influence of the choice of boundary conditions on the spectrum. To see it, we recall that an upper SKN-type (*cf* [54, 41, 50]) bound on the number of bound states in thin Dirichlet strips was derived in [14, Sec. 2.3] and it showed that $N(H^D)$ is bounded from above by a finite constant which *does not* depend on the strip width d. On the other hand, Proposition 10.1 shows that $N(H^{DN})$ can reach arbitrarily large value by shrinking the strip width to zero.

The last objective of this paper is to estimate the distance between the spectral threshold, *i.e.* the lowest eigenvalue of H^ι, and E_1^ι. We derive the following upper bounds, which are again qualitatively different for the Dirichlet and mixed Dirichlet-Neumann situations, respectively.

Theorem 10.5 (Estimates of the spectral threshold)**.** *Suppose* $\langle \mathsf{H} \rangle$ *and assume that* k *has a compact support in an interval of width* $2s_0$*.*

(i) *If* $\alpha \leq 0$*, then* $\inf \sigma(H^{DN}) \leq E_1^{DN} - C^{DN}(s_0, d, \alpha)^2 \, \alpha^2$*, where*

$$C^{DN}(s_0, d, \alpha) := \sqrt{E_1^{DN}} \, \frac{\sqrt{3}/\pi}{1 + \sqrt{1 - \frac{3}{2}\frac{\alpha s_0}{d} + \frac{3}{4}\alpha^2 \left(\frac{1}{2} + \frac{2}{\pi^2}\right)}} \,.$$

(ii) $\inf \sigma(H^D) \leq E_1^D - C^D(s_0, d, \alpha)^2 \, \alpha^4$*, where*

$$C^D(s_0, d, \alpha) := \frac{2^4}{3^3} \, \frac{\sqrt{3}/\pi^2}{d\left(\frac{s_0}{d} - \frac{\alpha}{4} + \frac{2}{3\,\pi}\right)} \, \frac{1}{1 + \sqrt{1 + \left(\frac{4\,\alpha}{3\,\pi}\right)^2 \frac{4s_0 - \alpha\, d}{4s_0 - \alpha\, d + \frac{8d}{3\,\pi}}}} \,.$$

These estimates are new in the theory of quantum waveguides and we derive them in Section 10.6. One can immediately see that for small total bending angles, the leading term in the estimate (i) is proportional to the second power of α, while it is the fourth power of α in the estimate (ii). Another essential difference in our estimates appears in the limit case of thin strips. We discuss these interesting disparities in Remarks 10.9 and 10.10. We also compare there the result (ii) with the exact eigenvalue asymptotics obtained in [14] by perturbation methods applied to mildly curved or thin strips, respectively.

10.5 Essential Spectrum

This section is devoted to the proof of Theorem 10.1. It is achieved in two steps. Firstly, in Lemma 10.2, we employ a Neumann bracketing argument in order to show that the threshold of the essential spectrum does not descend below the energy E_1^ι. Secondly, in Lemma 10.3, we prove that all energies above E_1^ι belong to the spectrum by means of the following general characterisation of essential spectrum, which we have adopted from [11].

Lemma 10.1. *Let H be a non-negative self-adjoint operator in a complex Hilbert space $\mathcal{H}$ and Q be the associated quadratic form. Then $\eta \in \sigma_{\rm ess}(H)$ if and only if*

$$\exists\{\psi_n\}_{n=1}^{\infty} \subset \mathrm{Dom}\, Q : \begin{cases} \text{(i)} & \forall n \in \mathbb{N}\setminus\{0\} : \ \|\psi_n\| = 1, \\ \text{(ii)} & \psi_n \xrightarrow[n\to\infty]{w} 0 \quad \textit{in } \mathcal{H}, \\ \text{(iii)} & (H-\eta)\psi_n \xrightarrow[n\to\infty]{} 0 \quad \textit{in } (\mathrm{Dom}\, Q)^* . \end{cases}$$

Here $(\mathrm{Dom}\, Q)^*$ denotes the dual of the space $\mathrm{Dom}\, Q$. We note that $H+1 : \mathrm{Dom}\, Q \to (\mathrm{Dom}\, Q)^*$ is an isomorphism and

$$\|\psi\|_{-1} := \|\psi\|_{(\mathrm{Dom}\, Q)^*} = \sup_{\phi\in\mathrm{Dom}\, Q\setminus\{0\}} \frac{|(\phi,\psi)|}{\|\phi\|_1} \tag{10.15}$$

with

$$\|\phi\|_1 := \sqrt{Q[\phi] + \|\phi\|^2}\,.$$

Lemma 10.1 is proved in a quite similar fashion as the Weyl criterion, [63, Thms. 7.22–7.24]. The advantage of the present characterisation is that it requires to find a sequence from the form domain of H only, and not from $\mathrm{Dom}\, H$ as it is required by the Weyl criterion. Moreover, in order to check the limit from (iii), it is still sufficient to consider the operator H in the form sense, *i.e.*, we will not need to assume that (G_{ij}) is differentiable in our case.

We start by an estimate on the threshold of the essential spectrum.

Lemma 10.2. *If $\langle \mathsf{d}\rangle$ holds true, then* $\inf\sigma_{\rm ess}(H^\iota) \geq E_1^\iota$.

Proof. Since the curvature vanishes at infinity, for any fixed $\epsilon > 0$, there exists s_ϵ such that

$$\forall (s,u) \in \Omega_{\rm ext} \ : \ (1-d\epsilon) \leq 1 - u\,k(s) \leq (1+d\epsilon), \tag{10.16}$$

where $\Omega_{\rm ext} := \Omega_0 \setminus \overline{\Omega}_{\rm int}$ with $\Omega_{\rm int} := (-s_\epsilon, s_\epsilon)\times I$. Denote by H_N^ι the operator H^ι with a supplementary Neumann boundary condition imposed on the two segments $\{\pm s_\epsilon\}\times I$, that is, the operator associated with the form $Q_N^\iota := Q_N^{\iota,\rm int} \oplus Q_N^{\iota,\rm ext}$, where

$$\begin{aligned} Q_N^{\iota,\omega}[\psi] &:= \big(\psi_{,i}, G^{ij}\psi_{,j}\big)_{L^2(\Omega_\omega, d\Omega)}\,, \\ \mathrm{Dom}\, Q_N^{D,\omega} &:= \big\{\psi \in W^{1,2}(\Omega_\omega, d\Omega)\,|\,\psi(s,0) = \psi(s,d) = 0 \text{ for a.e. } s \in \mathbb{R}\cap\overline{\Omega}_\omega\big\}\,, \\ \mathrm{Dom}\, Q_N^{N,\omega} &:= W^{1,2}(\Omega_\omega, d\Omega)\,, \\ \mathrm{Dom}\, Q_N^{DN,\omega} &:= \big\{\psi \in W^{1,2}(\Omega_\omega, d\Omega)\,|\,\psi(s,0) = 0 \text{ for a.e. } s \in \mathbb{R}\cap\overline{\Omega}_\omega\big\} \end{aligned}$$

for $\omega \in \{\rm int, ext\}$. Since $H^\iota \geq H_N^\iota$ and the spectrum of the operator associated with $Q_N^{\iota,\rm int}$ is purely discrete, *cf* [10, Chap. 7], the minimax principle gives the estimate

$$\inf\sigma_{\rm ess}(H^\iota) \geq \inf\sigma_{\rm ess}(H_N^{\iota,\rm ext}) \geq \inf\sigma(H_N^{\iota,\rm ext}),$$

where $H_N^{\iota,\rm ext}$ denotes the operator associated with $Q_N^{\iota,\rm ext}$. Neglecting the non-negative "longitudinal" part of the Laplacian in (10.8) (*i.e.*, the term where one sums over $i = j = 1$) and using the estimates (10.16), we arrive easily at the following lower bound

$$H_N^{\iota,\rm ext} \geq \frac{1-d\epsilon}{1+d\epsilon}\, E_1^\iota \qquad \text{in} \quad L^2(\Omega_{\rm ext}, d\Omega),$$

which holds in the form sense (see also proof of Theorem 4.1 in [16]). The claim then follows by the fact that ϵ can be chosen arbitrarily small. □

Remark 10.5 (Neumann case)**.** Since $E_1^N = 0$ and H^N is a non-negative operator, the claim of Lemma 10.2 holds trivially true for the Neumann boundary conditions, *i.e.*, $\iota = N$, even without the assumption $\langle \mathsf{d}\rangle$.

Example 10.1 (Periodic waveguides)**.** The periodic strip (*i.e.* assumption $\langle \mathsf{d}\rangle$ is not obeyed) is the simplest example for which

$$\inf\sigma_{\rm ess}(H^\iota) < E_1^\iota\,, \qquad \iota \in \{D, DN\}.$$

Let $k \not\equiv 0$ be a periodic function of a period $L > 0$, *i.e.*, $\forall s \in \mathbb{R} : \ k(s+L) = k(s)$, and such that the hypothesis $\langle \mathsf{H}\rangle$ holds true for some $d > 0$. Then the operator H^ι is invariant w.r.t. the transformation $s \mapsto s + jL$

for every $j \in \mathbb{Z}$, which implies that there is no discrete eigenvalue in its spectrum, *i.e.* $\sigma(H^\iota) = \sigma_{\mathrm{ess}}(H^\iota)$. However, Theorem 10.3 and Theorem 10.4 state that $\inf \sigma(H^\iota) < E_1^\iota$.

According to a common belief, second order elliptic differential operators with sufficiently regular periodic coefficients should not have degenerate bands in their spectra, or, in other words, their spectra should be purely absolutely continuous (see [55] and references therein). An elegant rigorous proof of this fact for Dirichlet and Neumann periodic waveguides was given by E. Shargorodsky and A. Sobolev in [55].

The precedent Lemma 10.2 together with the following one establish Theorem 10.1.

Lemma 10.3. *If* $\langle \mathsf{d} \rangle$ *holds true, then* $\sigma_{\mathrm{ess}}(H^\iota) \supseteq [E_1^\iota, \infty)$.

Proof. Let $n \in \mathbb{N} \setminus \{0\}$. We shall construct a sequence $\{\psi_n^\iota\}$ satisfying (i)–(iii) of Lemma 10.1 with $\eta^\iota := \lambda^2 + E_1^\iota$ for all $\lambda \in \mathbb{R}$. We start with the following family of functions

$$\hat{\psi}_n^\iota(s,u) := \varphi_n(s)\, \chi_1^\iota(u)\, e^{i\lambda s}\,,$$

where χ_1^ι is the lowest transverse-mode function (10.11) if $\iota \in \{D, DN\}$, or (10.12) if $\iota = N$, respectively, and $\varphi_n(s) := \varphi(n^{-1}s - n)$ with φ being a C^∞-smooth function with a compact support in $(-1,1)$. Note that $\operatorname{supp}\varphi_n \subset (n^2 - n, n^2 + n)$ and, consequently, the sequence $\{\varphi_n\}$ is "localised at $+\infty$" for large n. It is clear that $\hat{\psi}_n^\iota$ belongs to the form domain of H^ι. Since it is not normalised in $\mathcal{H}$, we introduce $\psi_n^\iota := \hat{\psi}_n^\iota / \|\hat{\psi}_n^\iota\|$. Hereafter we shall use the equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_{L^2(\Omega_0)}$, which follows by (10.3). In particular, one has

$$C_- \|\varphi_n\|^2_{L^2(\mathbb{R})} \le \left\|\hat{\psi}_n^\iota\right\|^2 \le C_+ \|\varphi_n\|^2_{L^2(\mathbb{R})} \tag{10.17}$$

due to the normalisation of χ_1^ι.

First of all, we note that one does not need to check the weak convergence. Indeed, if we show the condition (iii) for our sequence $\{\psi_n^\iota\}$ for all $\lambda \in \mathbb{R}$, we get that $[E_1^\iota, \infty)$ belongs to the spectrum of H^ι. Then it follows immediately that it must be the essential part of the spectrum because intervals have no isolated points.

Hence, it remains to check that $\|(H^\iota - \eta^\iota)\psi_n\|_{-1} \to 0$ as $n \to \infty$. Employing the diagonal form (10.2) of the metric tensor, we can split the Hamiltonian (10.8) into a sum of two parts, $H^\iota = H_1^\iota + H_2^\iota$, where H_i^ι, $i \in \{1,2\}$, corresponds to the term with G^{ii} in (10.8). This decomposition leads to the trivial bound

$$\left\|\left(H^\iota - \eta^\iota\right)\psi_n^\iota\right\|_{-1} \le \left\|\left(H_1^\iota - \lambda^2\right)\psi_n^\iota\right\|_{-1} + \left\|\left(H_2^\iota - E_1^\iota\right)\psi_n^\iota\right\|_{-1}. \tag{10.18}$$

We will show that the norms at the r.h.s. of this inequality tends to zero as $n \to \infty$ separately. Denote $\|f\|_{\infty,n} := \sup\{|f(s,u)| \,|\, (s,u) \in \operatorname{supp}\varphi_n \times I\}$.

An explicit calculation using (10.3) and the fact that χ_1^ι is an eigenfunction of $-\Delta_\iota^I$ corresponding to the energy E_1^ι yields

$$\left|\left(\phi, \left(H_2^\iota - E_1^\iota\right)\hat{\psi}_n^\iota\right)\right| = \left|\left(\phi, k\,\hat{\psi}_{n,2}^\iota\right)_{L^2(\Omega_0)}\right| \le C_-^{-1}\sqrt{E_1^\iota}\, \|k\|_{\infty,n}\, \|\phi\|\, \|\hat{\psi}_n^\iota\| \tag{10.19}$$

for all $\phi \in \operatorname{Dom} Q^\iota$. Consequently, the second term at the r.h.s. of (10.18) goes to zero as $n \to \infty$ by the assumption $\langle \mathsf{d} \rangle$.

A little more toilsome but still direct calculation yields

$$\begin{aligned}\left(\phi, \left(H_1^\iota - \lambda^2\right)\hat{\psi}_n^\iota\right) \;&=\; \lambda^2 \left(\phi, \left(1 - G^{\frac{1}{2}}\right)\hat{\psi}_n^\iota\right)_{L^2(\Omega_0)} \\ &+\; \left(\phi_{,1}, \left(G^{\frac{1}{2}}G^{11} - 1\right)\left(\dot{\varphi}_n + i\lambda\varphi_n\right)\chi_1^\iota e^{i\lambda s}\right)_{L^2(\Omega_0)} \\ &-\; \left(\phi, \left(\ddot{\varphi}_n + 2i\lambda\dot{\varphi}_n\right)\chi_1^\iota e^{i\lambda s}\right)_{L^2(\Omega_0)}\end{aligned}$$

for all $\phi \in \operatorname{Dom} Q^\iota$. Estimating all the terms at the r.h.s. of this equality in the same way as in (10.19), it is enough to show that the following sequences

$$\left\|1 - G^{\frac{1}{2}}\right\|_{\infty,n}, \quad \left\|G^{\frac{1}{2}}G^{11} - 1\right\|_{\infty,n}, \quad \frac{\|\dot{\varphi}_n\|_{L^2(\mathbb{R})}}{\|\varphi_n\|_{L^2(\mathbb{R})}}, \quad \frac{\|\ddot{\varphi}_n\|_{L^2(\mathbb{R})}}{\|\varphi_n\|_{L^2(\mathbb{R})}},$$

has the zero limit as $n \to \infty$. However, this is evident for the first and second ones by virtue of (10.2) and $\langle \mathsf{d} \rangle$, while for the rest it follows by the definition of the sequence $\{\varphi_n\}$. □

If the strip is asymptotically straight, *i.e.* $\langle \mathsf{d} \rangle$, then $\sigma(H^N) = [0, \infty)$ by Theorem 10.1, (10.10) and non-negativity of H^N; see also Theorem 10.2. We conclude this section by proving the following result about the spectral threshold of the operators H^D and H^{DN}.

Proposition 10.2. *Suppose* $\langle \mathsf{H} \rangle$*. If the strip obeys* $\langle \mathsf{d} \rangle$*, then*

$$\inf \sigma(H^\iota) > 0 \qquad \textit{for} \quad \iota \in \{D, DN\}.$$

Proof. We have $H^\iota \geq 0$ and $E_1^\iota > 0$. By virtue of Theorem 10.1, it is enough to prove that $0 \notin \sigma_{\mathrm{p}}(H^\iota)$. Assume that there exists $\psi \in \mathrm{Dom}\, H^\iota$ such that $H^\iota \psi = 0$. Then $\psi \in \mathrm{Dom}\, Q^\iota$ and $0 = (\psi, H^\iota \psi) = Q^\iota[\psi] \equiv \int_{\Omega_0} \overline{\psi}_{,i} G^{ij} \psi_{,j}\, d\Omega$ with (G_{ij}) being a strictly positive definite matrix, hence $\psi = 0$. □

10.6 Curvature-Induced Spectrum

Now we will be interested in the proofs concerning the existence and properties of the spectrum of H^ι below the energy E_1^ι. Since H^N is a non-negative operator and $E_1^N = 0$, only the situations $\iota \in \{D, DN\}$ are relevant here, however, we do not exclude the Neumann case from the preliminary considerations here in order to establish a minor result contained in Proposition 10.3 below.

All the proofs of the following subsections are based on the variational strategy of finding a trial function ψ^ι from the form domain of H^ι such that

$$Q_1^\iota[\psi^\iota] := Q^\iota[\psi^\iota] - E_1^\iota \, \|\psi^\iota\|^2 < 0. \tag{10.20}$$

We construct such a trial function by modifying the generalised eigenfunction (10.11) of energy E_1^ι for the straight strip. This idea goes back to J. Goldstone and R. L. Jaffe, [39]; see also [14, 16, 13, 45].

As a preliminary, let us express the form (10.20) in the situation when the variables are separated in the following way:

$$\psi^\iota(s,u) := \varphi(s)\, \chi_1^\iota(u), \tag{10.21}$$

where χ_1^ι is the first transverse mode (10.11) or (10.12) and φ is a suitable function from $W^{1,2}(\mathbb{R})$. In view of (10.3), it is clear that ψ^ι belongs to $\mathrm{Dom}\, Q^\iota$, given by (10.5), (10.6) or (10.7), respectively. An explicit calculation yields

$$Q_1^\iota[\psi^\iota] = \left(\dot\varphi, \langle G^{-\frac{1}{2}} \rangle_\iota \, \dot\varphi\right)_{L^2(\mathbb{R})} + \tfrac{1}{2} \left[\chi_1^\iota(d)^2 - \chi_1^\iota(0)^2\right] (\varphi, k\,\varphi)_{L^2(\mathbb{R})}\,, \tag{10.22}$$

where $\langle \cdot \rangle_\iota$ denotes the expectation w.r.t. χ_1^ι, *i.e.* $\langle f \rangle_\iota := \int_I f(\cdot,u) \chi_1^\iota(u)^2 du$ with $f \in L^\infty(\Omega_0)$. It is clear from (10.11) and (10.12) that the second term at the r.h.s. of (10.22) is absent for $\iota \in \{D, N\}$, while $\chi_1^{DN}(d) = \sqrt{2/d}$ and $\chi_1^{DN}(0) = 0$.

10.6.1 The Existence

***Proof of* Theorem 10.3 and Theorem 10.4, *condition* (a).** We set

$$\psi_n^\iota(s,u) := \varphi(s;n)\, \chi_1^\iota(u), \tag{10.23}$$

where $\varphi : \mathbb{R} \times (0,\infty) \to [0,1]$ is supposed to satisfy:

(i) $\forall n \in (0,\infty) : \ \varphi(\cdot;n) \in W^{1,2}(\mathbb{R})$,

(ii) $\varphi(s;n) \xrightarrow[n\to\infty]{} 1 \quad$ for a.e. $s \in \mathbb{R}$,

(iii) $\|\varphi_{,1}(\cdot;n)\|_{L^2(\mathbb{R})} \xrightarrow[n\to\infty]{} 0$,

that is, φ is a suitable mollifier of 1 (for an example of such a function, see (10.32) below). Substituting this trial function to (10.22), we get

$$Q_1^D[\psi_n] \xrightarrow[n\to\infty]{} 0\,, \qquad Q_1^N[\psi_n] \xrightarrow[n\to\infty]{} 0\,, \qquad Q_1^{DN}[\psi_n] \xrightarrow[n\to\infty]{} \frac{\alpha}{d}\,, \tag{10.24}$$

where α is the total bending angle (10.14). The limits hold true by virtue of the required properties of φ, the fact that $\langle G^{-\frac{1}{2}} \rangle_\iota$ are bounded functions and, in the case $\iota = DN$, also by the dominated convergence theorem. That is why we need to assume in addition that k is integrable for $\iota = DN$. Consequently, if α is strictly negative, then there exists a finite $n_0 > 0$ such that $Q_1^{DN}[\psi_{n_0}] < 0$ and the proof for $\iota = DN$ is finished in this case.

To obtain the result for $\iota = DN$ in the limit case $\alpha = 0$, and for any Dirichlet strip, we modify the function ψ_n^ι, in a curved part of the waveguide. We define

$$\psi_{n,\varepsilon}^\iota(s,u) := \psi_n^\iota(s,u) + \varepsilon\, \phi(s)\, v^\iota(u)\, \chi_1^\iota(u), \qquad \iota \in \{D, DN\} \tag{10.25}$$

where $\varepsilon \in \mathbb{R}$, $\phi \in W^{1,2}(\mathbb{R})$ is a real, non-negative function with compact support contained in a bounded interval in $\mathbb{R}$ where k is not zero and does not change sign (such an interval surely exists because $k \not\equiv 0$ and is continuous), and $\upsilon^D(u) := -2u/d$ and $\upsilon^{DN}(u) := 1$. The family $\{\psi^\iota_{n,\varepsilon}\}$ is a subset of $\mathrm{Dom}\, Q^\iota$ and we can write

$$Q^\iota_1[\psi^\iota_{n,\varepsilon}] = Q^\iota_1[\psi^\iota_n] + 2\varepsilon\, Q^\iota_1(\phi\upsilon^\iota\chi^\iota_1, \psi^\iota_n) + \varepsilon^2\, Q^\iota_1[\phi\upsilon^\iota\chi^\iota_1]. \tag{10.26}$$

The last term at the r.h.s. of (10.26) does not depend on n, while the first one tends to zero as $n \to \infty$ by (10.24). An explicit calculation of the central term gives (*cf* (10.22) for $\iota = DN$)

$$Q^\iota_1(\phi\upsilon^\iota\chi^\iota_1, \psi_n) = \left(\dot\phi, \langle \upsilon^\iota G^{-\frac12}\rangle_\iota\, , \dot\varphi_n\right)_{L^2(\mathbb{R})} + \tfrac{1}{d}\,(\phi, k\,\varphi_n)_{L^2(\mathbb{R})}\,,$$

where we have denoted $\varphi_n := \varphi(\cdot; n)$ and $\dot\varphi_n := \varphi_{,1}(\cdot; n)$. Using then the properties of the function φ together with the dominated convergence theorem (notice that $\phi\, k \in L^1(\mathbb{R})$), we have

$$Q^\iota_1[\psi^\iota_{n,\varepsilon}] \xrightarrow[n\to\infty]{} \tfrac{2}{d}\,\varepsilon\,(\phi, k)_{L^2(\mathbb{R})} + \varepsilon^2\, Q^\iota_1[\phi\upsilon^\iota\chi^\iota_1]. \tag{10.27}$$

Since the integral $(\phi, k)_{L^2(\mathbb{R})} \neq 0$ by the construction of ϕ, we can take ε sufficiently small and of an appropriate sign so that the sum of the last two terms at the r.h.s. of (10.27) is negative, and then choose n sufficiently large so that $Q^\iota_1[\psi^\iota_{n,\varepsilon}] < 0$. □

The intermediate results (10.24) of the precedent proof give the following upper bounds on the spectral threshold of H^ι:

Proposition 10.3. *Suppose* $\langle\mathsf{H}\rangle$. *One has*

(i) $\inf\sigma(H^\iota) \le E^\iota_1$ *for* $\iota \in \{D, N\}$;

(ii) $\inf\sigma\left(H^{DN} - \frac{k(s)}{d(1-uk(s))}\right) \le E^{DN}_1$ *provided* $k \in L^1(\mathbb{R})$.

Actually, in view of Theorem 10.3, a stronger result than (i) holds for any Dirichlet strip. The assertion (i) for the Neumann case, together with the fact that H^N is non-negative, establishes the first claim of Theorem 10.2.

Remark 10.6 (Condition (a) of Theorem 10.4 *vs* the assumptions in [13])**.** The non-positivity of the total bending angle, *i.e.* $\alpha \le 0$, is a nice sufficient condition which guarantees the existence of geometrically induced spectrum for H^{DN}. This was established already in [13] under the additional hypothesis that "k is non-positive everywhere outside of some bounded interval". Since the latter is not assumed in this paper, we extend significantly the class of admissible geometries. Nevertheless, in order to justify the use of the dominated convergence theorem, we need to assume that "k is integrable" instead; *cf* the condition (a) of Theorem 10.4. Hence, a natural question is to ask whether the assumptions in [13] may after all present an alternative criterion which is not contained in our condition (a). The answer is negative due to the following (purely geometrical) result, which can be easily shown using the so-called "Umlaufsatz", [42, Thm. 2.2.1].

Lemma 10.4. *Let* Γ *be an infinite plane* C^2*-smooth curve of bounded curvature* k*. If there exists a compact* $\Gamma_c \subset \Gamma$ *such that* $\left|\int_{\Gamma_a} k\right| > 2\pi$ *for any compact* Γ_a *obeying* $\Gamma_c \subseteq \Gamma_a \subset \Gamma$*, then any tubular neighbourhood of* Γ *overlaps.*

That is, any reference curve satisfying the assumptions of [13] but having a non-integrable curvature leads to a violation of the basic hypothesis $\langle\mathsf{H}\rangle$ (which is assumed in [13] as well).

***Proof of* Theorem 10.4, *condition* (b).** Let $L > 0$ be the period of k, *i.e.*, $\forall s \in \mathbb{R}:\ k(s+L) = k(s)$. We take the trial function of the form

$$\psi^{DN}_{n,\varepsilon}(s,u) := \varphi_n(s)\ (1 + \varepsilon\,\phi(s))\ \chi^{DN}_1(u)\,,$$

cf (10.25), where the functions φ_n and ϕ are defined as follows. Let $\varphi_1 \in C^\infty_0((-L, 2L))$ be a real function which is equal to 1 on the period cell $(0, L)$. We set, for any $n \in \mathbb{N}\setminus\{0\}$,

$$\varphi_n(s) := \begin{cases} \varphi_1(s) & \text{if} \quad s \in (-\infty, L), \\ 1 & \text{if} \quad s \in [L, nL], \\ \varphi_1\,(s - (n-1)L) & \text{if} \quad s \in (nL, (n+1)L]\,, \\ 0 & \text{if} \quad s \in ((n+1)L, +\infty)\,. \end{cases}$$

Let $\phi \in C^\infty(\mathbb{R})$ be non-negative, L-periodic, and such that $\operatorname{supp}\phi \restriction (0,L)$ is contained in an interval where k is not zero and does not change sign. Then $(\phi,k)_{L^2((0,L))} \neq 0$. Finally, let $\varepsilon \in \mathbb{R}$ be chosen in such a way that (*cf* (10.27))

$$\begin{aligned} A &:= \big(\psi_{1,\varepsilon}^{DN}, \big(H^{DN} - E_1^{DN}\big)\psi_{1,\varepsilon}^{DN}\big)_{L^2((0,L)\times I, d\Omega)} \\ &= \tfrac{2}{d}\,\varepsilon\,(\phi,k)_{L^2((0,L))} + \varepsilon^2\,\big(\phi\chi_1^{DN}, \big(H^{DN} - E_1^{DN}\big)\phi\chi_1^{DN}\big)_{L^2((0,L)\times I, d\Omega)} \end{aligned} \tag{10.28}$$

is negative. By virtue of the definition of φ_1 and the fact that $\int_0^L k(s)ds = 0$ (*cf* Lemma 10.4), it is clear that

$$Q_1^{DN}[\psi_{1,\varepsilon}^{DN}] = A + B,$$

where B is defined as the integral at the first line of (10.28), however, with the range of integration being the set $((-L,0)\cup(L,2L))\times I$. Using the periodicity of the coefficients of H^{DN} together with the definition of φ_n, we continue by induction and arrive at the identity

$$\forall n \in \mathbb{N}\setminus\{0\}: \quad Q_1^{DN}[\psi_{n,\varepsilon}^{DN}] = nA + B,$$

which becomes negative for n sufficiently large. □

Remark 10.7 (Integrability of k)**.** If $k \not\equiv 0$ is periodic, then the curvature is not integrable. However, one has for every $n \in \mathbb{N}$, $\int_{-nL}^{nL} k(s)\,ds = 0$ due to the periodicity (*cf* Lemma 10.4). This indicates that the requirement $k \in L^1(\mathbb{R})$ in the condition (a) of Theorem 10.4 may be rather a technical hypothesis.

***Proof of* Theorem 10.4, *condition* (c).** We take the trial function ψ^{DN} of the form (10.21). Since k is continuous and $k_- \not\equiv 0$, there exists an interval $J \subset \mathbb{R}$, such that $k(s) < 0$ for all $s \in J$. Choosing $\varphi \in W^{1,2}(\mathbb{R})$ such that $\operatorname{supp}\varphi \subseteq \overline{J}$ and substituting it to (10.22), obvious estimates yield

$$Q_1^{DN}[\psi^{DN}] \;\leq\; \|\dot\varphi\|^2_{L^2(J)} \;+\; \frac{1}{d}\int_J |\varphi(s)|^2\,k(s)\,ds. \tag{10.29}$$

The second term at the r.h.s. of the last inequality is obviously negative, while the first one does not depend on d. Hence for all d sufficiently small their sum is negative. □

Remark 10.8 (An estimate of the critical width)**.** The claim of Theorem 10.4 with the condition (c) can be rewritten as follows. Suppose $\langle \mathsf{H} \rangle$. There exists a positive d_0 such that for all $d < d_0$ the condition $k_- \not\equiv 0$ implies that $\inf\sigma(H^{DN}) < E_1^{DN}$. Let us estimate the critical width d_0 from below here. First of all, notice that the assumption $\langle \mathsf{H} \rangle$ estimates d from above by $\|k_+\|_\infty^{-1}$, hence d_0 can be infinite in the case $k_+ \equiv 0$ (for instance, in view of the condition (a) of Theorem 10.4, this happens if in addition $k \in L^1(\mathbb{R})$).

We define for $s_1, s_2 \in \mathbb{R}$, $s_1 < s_2$,

$$\alpha(s_1,s_2) := \int_{s_1}^{s_2} k(s)\,ds\,, \qquad k_m(s_1,s_2) := \sup_{s\in\mathbb{R}\setminus(s_1,s_2)} k_+(s)$$

and

$$d_1 := \sup\left\{ \frac{1}{k_m(s_1,s_2)}\,\frac{\alpha(s_1,s_2)^2}{4\,+\,\alpha(s_1,s_2)^2} \;\middle|\; s_1,s_2\in\mathbb{R}, s_1<s_2,\, \alpha(s_1,s_2)<0 \right\}.$$

Notice that $d_1 > 0$ because it is supposed that $k_- \not\equiv 0$, and $k \in L^\infty(\mathbb{R})$ by $\langle \mathsf{H} \rangle$ By the definition of the supremum, for any positive $d < d_1$, there exist $s_{1,d}, s_{2,d} \in \mathbb{R}$ such that $s_{1,d} < s_{2,d}$, $\alpha_d := \alpha(s_{1,d}, s_{2,d}) < 0$ and

$$d < \frac{1}{k_d}\,\frac{\alpha_d^2}{4\,+\,\alpha_d^2}, \tag{10.30}$$

where $k_d := k_m(s_{1,d}, s_{2,d})$. If $k_d = 0$, then necessarily $k \in L^1(\mathbb{R})$ by virtue of Lemma 10.4, thus the condition (a) of Theorem 10.4 is satisfied and the only restriction on d is given by the natural condition $\langle \mathsf{H} \rangle$. Let us assume therefore $k_d > 0$ so that the r.h.s. of (10.30) is well defined. We set

$$\varphi_d(s) := \begin{cases} \exp\Big(\sqrt{k_d\,(1-d\,k_d)/d}\,(s-s_{1,d})\Big) & \text{if} \quad s\in(-\infty,s_{1,d})\,, \\ 1 & \text{if} \quad s\in[s_{1,d},s_{2,d}]\,, \\ \exp\Big(-\sqrt{k_d\,(1-d\,k_d)/d}\,(s-s_{2,d})\Big) & \text{if} \quad s\in(s_{2,d},+\infty)\,. \end{cases}$$

Again, $\psi_d(s,u) := \varphi_d(s)\,\chi_1^{DN}(u)$ belongs to the domain of Q^{DN} and we have, *cf* (10.22),

$$\begin{aligned} Q_1^{DN}[\psi_d] &\le \frac{1}{1-d\,k_d}\,\|\dot\varphi_d\|^2_{L^2(\mathbb{R}\setminus(s_{1,d},s_{2,d}))} + \frac{k_d}{d}\,\|\varphi_d\|^2_{L^2(\mathbb{R}\setminus(s_{1,d},s_{2,d}))} + \frac{\alpha_d}{d} \\ &= \frac{2}{d}\sqrt{\frac{d\,k_d}{1-d\,k_d}} + \frac{\alpha_d}{d}. \end{aligned}$$

The inequality uses the definition of k_d and the equality is a result of an explicit calculation of the norms. However, using (10.30) and the fact that $\alpha_d < 0$, we arrive at $Q_1^{DN}[\psi_d] < 0$. Hence, together with the assumption $\langle \mathsf{H} \rangle$, we get $d_0 \ge \min\left\{\|k_+\|_\infty^{-1}, d_1\right\}$ (with the convention $d_0 := +\infty$ if $k_+ \equiv 0$).

***Proof of* Theorem 10.4, *part* (ii).** Let us only sketch here the main ideas of the proof, for more details see [12, Prop. 3]. Let $k_- \equiv 0$. We show that then the functional $Q_1^{DN}[\Phi]$ is non-negative for every $\Phi \in \operatorname{Dom} Q^{DN}$. To do so, we decompose the function Φ to a transverse orthonormal basis, *i.e.*

$$\Phi(s,u) = \sum_{n=1}^{\infty} \phi_n(s)\,\chi_n(s;u),$$

where, for almost every value of the parameter $s \in \mathbb{R}$,

$$\phi_n(s) := \left(\Phi(s,\cdot)\,,\,\chi_n(s;\cdot)\right)_{L^2(I,(1-uk(s))du)}$$

and $\chi_n(s;\cdot)$ are normalised eigenfunctions of the self-adjoint operator $h(s)$ acting in $L^2\,(I,(1-uk(s))du)$ associated with the quadratic form

$$\begin{aligned} q(s)[\psi] &:= \Big(\psi_{,2}(s;\cdot),\psi_{,2}(s;\cdot)\Big)_{L^2(I,(1-uk(s))du)}\,, \\ \operatorname{Dom} q(s) &:= \left\{\psi(s;\cdot) \in W^{1,2}(I) \mid \psi(s;0) = 0\right\}\,. \end{aligned}$$

Let us denote by $\lambda_n(s)$ the eigenvalues of this operator corresponding to eigenfunctions $\chi_n(s;\cdot)$. Neglecting the non-negative longitudinal term (*i.e.* the integral (10.4) for $i,j=1$), we obtain

$$Q_1^{DN}[\Phi] \ge \sum_{n=1}^{\infty} \int_{\mathbb{R}} |\phi_n(s)|^2 \left(\lambda_n(s) - E_1^{DN}\right) ds\,.$$

Hence it is enough to show that for almost every $s \in \mathbb{R}$, the lowest eigenvalue $\lambda_1(s)$ is greater than or equal to E_1^{DN}. This can be shown by a spectral analysis of the ordinary differential operator h associated with q, *i.e.*, for a.e. $s \in \mathbb{R}$,

$$\begin{aligned} h(s) &= -\partial_u^2 + \frac{k(s)}{1-uk(s)}\,\partial_u\,, \\ \operatorname{Dom} h(s) &= \left\{\psi(s;\cdot) \in W^{2,2}(I) \mid \psi(s;0) = \psi_{,2}(s;d) = 0\right\}. \end{aligned}$$

More specifically, using an expansion of the explicit solutions of the corresponding eigenvalue problem and the minimax principle together with a unitary transformation, the authors of [13] prove that for a.e. $s \in \mathbb{R}$, $h(s)$ has no spectrum in $[0, E_1^{DN})$ provided $k_- \equiv 0$. □

***Proof of* Proposition 10.1.** The claim is trivial for $n = 0$. Let us fix an integer $n \in \mathbb{N} \setminus \{0\}$. We shall find a critical width d_n such that for all $d < d_n$, there are at least n discrete eigenvalues in the spectrum of H^{DN}, counting multiplicity. As in the proof of Theorem 10.4, condition (c), let $J \subset \mathbb{R}$ be a bounded interval such that $k(s) < 0$ for all $s \in J$. We set $s_0 := \inf J$ and $s_j := s_0 + j\,|J|/n$ for every $j \in \{1,\dots,n\}$. Let φ_0 be a non-zero function from $W^{1,2}(\mathbb{R})$ such that $\operatorname{supp}\varphi_0 \subset (s_0,s_1)$. We define for every $j \in \{1,\dots,n\}$ and $s \in \mathbb{R}$,

$$\begin{aligned} N_j^{-2} &:= \int_{s_{j-1}}^{s_j} |\varphi_0(s_0+s-s_{j-1})|^2\ \langle G^{\frac12}\rangle_{DN}(s)\,ds\,, \\ \varphi_j(s) &:= N_j\,\varphi_0(s_0+s-s_{j-1})\,. \end{aligned}$$

Putting $\psi_j^{DN}(s,u) := \varphi_j(s)\chi_1^{DN}(u)$ for every $j \in \{1,\dots,n\}$, *cf* (10.21), we obtain an orthonormal basis of a subset of $\operatorname{Dom} Q^{DN}$. Moreover, $Q^{DN}(\psi_j^{DN},\psi_\ell^{DN}) = 0$ whenever $j \neq \ell$ because φ_j and φ_ℓ have disjoint supports. Therefore, it follows by [10, Lemma 4.5.4] and Theorem 10.1 that a sufficient condition for H^{DN} to

have at least n discrete eigenvalue is $Q^{DN}[\psi_j^{DN}] < E_1^{DN}$, *i.e.* $Q_1^{DN}[\psi_j^{DN}] < 0$, for every $j \in \{1, \dots n\}$. However, according to (10.29),

$$Q_1^{DN}\left[\psi_j^{DN}\right] \;\leq\; N_j^2 \left\|\dot\varphi_0\right\|^2_{L^2(\mathbb{R})} + \frac{N_j^2}{d} \int_{s_{j-1}}^{s_j} \left|\varphi_0(s_0 + s - s_{j-1})\right|^2 \, k(s)\, ds\,.$$

The r.h.s. of the last inequality is obviously negative for all $j \in \{1, \dots, n\}$ provided that $d < d_n$ with

$$d_n := \min_{j\in\{1,\dots,n\}} \frac{1}{\left\|\dot\varphi_0\right\|^2_{L^2(\mathbb{R})}} \int_{s_0}^{s_1} |\varphi_0(s)|^2 \; |k(s - s_0 + s_{j-1})| \; ds\,.$$

□

10.6.2 The estimates on the spectral threshold

Throughout this subsection, we consider only $\iota \in \{D, DN\}$. Obviously,

$$\inf\sigma(H^\iota) - E_1^\iota = \inf_{\psi\in\mathrm{Dom}\, Q^\iota} \frac{Q_1^\iota[\psi]}{\|\psi\|^2} \leq \inf_{\psi\in T^\iota} \frac{Q_1^\iota[\psi]}{\|\psi\|^2}\,, \tag{10.31}$$

where T^ι is an arbitrary subset of $\mathrm{Dom}\, Q^\iota$. Our strategy will be to choose a suitable T^ι and then explicitly find the infimum of the quotient at the r.h.s. of (10.31).

In Theorem 10.5, the curvature is supposed to have a compact support contained in an interval of width $2s_0$; without loss of generality we may assume that the reference curve is parameterised in such a way that $\mathrm{supp}\, k \subseteq [-s_0, s_0]$.

***Proof of* Theorem 10.5, *part* (i).** Let $\psi_{n,c}(s,u) := \varphi_c(s;n)\, \chi_1^{DN}(u)$ be the trial function from the beginning of the proof of the condition (a) of Theorem 10.4 in Section 10.6.1 with the mollifier $\varphi_c(\cdot;n)$ given explicitly by

$$\varphi_c(s;n) := \begin{cases} 1 & \text{if} \quad |s| \in [0, n), \\ (c\,n - |s|)/\left((c-1)\,n\right) & \text{if} \quad |s| \in [n, cn), \\ 0 & \text{if} \quad |s| \in [cn, \infty), \end{cases} \qquad c > 1. \tag{10.32}$$

We set $T^{DN} := \{\psi_{n,c} \,|\, n \geq s_0\,,\, c > 1\}$. An easy calculation yields

$$Q_1^{DN}[\psi_{n,c}] = \frac{2}{(c-1)\,n} + \frac{\alpha}{d}\,, \qquad \|\psi_{n,c}\|^2 = \frac{2}{3}\,(c+2)\,n - \alpha\,\langle u\rangle\,,$$

where

$$\langle u\rangle := \int_I u\, \chi_1^{DN}(u)^2\, du = d\,\left(\frac{1}{2} + \frac{2}{\pi^2}\right).$$

Hence, denoting by $f(n,c)$ the quotient at the r.h.s. of (10.31), we have

$$f(n,c) = \frac{\frac{2}{c-1} + \frac{\alpha}{d} n}{\frac{2}{3}(c+2)\,n^2 - \alpha\,\langle u\rangle\, n}\,. \tag{10.33}$$

Now we shall seek the infimum of the continuous function f in the region $[s_0, \infty)\times(1,\infty)$; the result establishes the bound from Theorem 10.5.

Let us solve the equation $f_{,2}(n,c) = 0$. Calculating the derivative

$$f_{,2}(n,c) = \frac{2}{3}\, \frac{-\frac{\alpha\, n}{d}\,(c-1)^2 - 2(2c+1) + \frac{3\alpha\langle u\rangle}{n}}{(c-1)^2\,\left(\frac{2}{3}(c+2)\,n - \alpha\,\langle u\rangle\right)^2}\,,$$

we see that its numerator is a quadratic polynom in c which has two roots

$$c_\pm(n) := -\frac{2d}{\alpha\, n} + 1 \mp \frac{d}{\alpha\, n}\,\sqrt{-6\frac{\alpha\, n}{d} + 4 + 3\frac{\alpha^2\,\langle u\rangle}{d}}\,. \tag{10.34}$$

Since $\alpha < 0$, it is evident that $c_+ > 1$, while $c_- < 1$ is not allowed. A straightforward calculation leads to the following (negative) value

$$f\left(n, c_+(n)\right) = \frac{-3\,\alpha^2/d^2}{4\,\left(1 + \sqrt{1 - \frac{3}{2}\alpha\, n/d + \frac{3}{4}\alpha^2\langle u\rangle/d}\right)^2}\,, \tag{10.35}$$

which is nothing else than the global minimum of the function $c \mapsto f(n,c)$ for any fixed $n \in [s_0,\infty)$. Obviously, $g(n) := f(n, c_+(n))$ is a strictly increasing function of the variable n in the interval $[s_0,\infty)$, *i.e.* $\forall n \in [s_0,\infty)$: $\dot g(n) > 0$. However,

$$0 < \dot g(n) = f_{,1}(n,c_+) + f_{,2}(n,c_+)\,\dot c_+(n) = f_{,1}(n,c_+)\,,$$

and that is why there is no candidate for a local minimum of the function f in the interior of its domain, *i.e.* for every point $(n,c) \in (s_0,\infty)\times(1,\infty)$, $f_{,1}(n,c) \neq 0$ or $f_{,2}(n,c) \neq 0$.

Thus the problem reduces to the study of the behaviour of f on the boundary set $\{s_0\}\times(1,\infty)$ and its limits as $n\to\infty$, $c\to 1$ and $c\to\infty$, respectively. Using the estimate $f(n,c) \geq \alpha/(2dn)$, we obtain

$$\liminf_{n\to\infty} f(n,c) \geq 0$$

uniformly in c. Hence, there exists a (finite) $n_0 > s_0$ such that for every $n > n_0$ holds true $f(n,c) \geq f(s_0, c_+(s_0))$ (recall that $f(s_0,c_+(s_0)) < 0$, *cf* (10.35)) uniformly in c. Therefore since we seek the infimum of f we can consider only $n \in [s_0,n_0]$ in the rest of the proof. However, for those values of n we have

$$f(n,c) \geq \frac{6}{(c-1)\,(2(c+2)n_0^2 - 3\alpha\langle u\rangle n_0)} + \frac{3\alpha/d}{2(c+2)n - 3\alpha\langle u\rangle} \tag{10.36}$$

and since

$$\lim_{c\to 1} \frac{6}{(c-1)\,(2(c+2)n_0^2 - 3\alpha\langle u\rangle n_0)} = \infty\,, \qquad \left|\lim_{c\to 1} \frac{3\alpha/d}{2(c+2)n - 3\alpha\langle u\rangle}\right| < \frac{1}{d\langle u\rangle}\,,$$

we obtain

$$\lim_{c\to 1} f(n,c) = \infty$$

uniformly in n. Finally,

$$\lim_{c\to\infty} \frac{6}{(c-1)\,(2(c+2)n_0^2 - 3\alpha\langle u\rangle n_0)} = 0\,,$$
$$\lim_{c\to\infty} \frac{3\alpha/d}{2(c+2)n - 3\alpha\langle u\rangle} \geq \frac{\alpha}{dn}\,\lim_{c\to\infty}\frac{3}{2(c+2)} = 0$$

because $n \mapsto \alpha/(dn)$ is bounded on $[s_0,n_0]$; hence, in view of (10.36),

$$\liminf_{c\to\infty} f(n,c) \geq 0$$

uniformly in n. Since the infimum of f should be negative, we infer from the above results that

$$\inf_{(n,c)\in[s_0,\infty)\times(1,\infty)} f(n,c) = \inf_{c\in(1,\infty)} f(s_0,c) = f\left(s_0, c_+(s_0)\right),$$

where $f(s_0,c_+(s_0)) < 0$ given by (10.35) provides an upper bound on the r.h.s. of (10.31) for the case $\iota = DN$. □

***Proof of* Theorem 10.5, *part* (ii).** In the Dirichlet case, we use the mollifier (10.32) with the fixed $n = s_0$ for the construction of the functions from T^D. We set for any c_1, $c_2 > 1$ and $\varepsilon \in \mathbb{R}$,

$$\psi_{c_1,c_2,\varepsilon}(s,u) := \varphi_{c_1}(s;s_0)\,\chi_1^D(u) + \varepsilon\,\varphi_{c_2}(s;s_0)\,\chi_2^D(u) \tag{10.37}$$

and $T^D := \{\psi_{c_1,c_2,\varepsilon}\,|\,c_1,\,c_2>1\,,\,\varepsilon\in\mathbb{R}\}$. Easy explicit calculations give

$$Q_1^D[\psi_{c_1,c_2,\varepsilon}] = \frac{\pi^2}{d}\,\left(h(c_1) + \frac{16}{3\pi^2}\alpha\varepsilon + \varepsilon^2(2g(c_2)+h(c_2))\right),$$

$$\|\psi_{c_1,c_2,\varepsilon}\|^2 = \frac{2d}{3}\left(g(c_1) + \frac{16}{3\pi^2}\alpha\varepsilon + \varepsilon^2 g(c_2)\right),$$

where

$$h(c) := \frac{2}{\pi^2}\,\frac{d}{s_0}\,\frac{1}{c-1}\,, \qquad g(c) := \frac{s_0}{d}(c+2) - \frac{3}{4}\alpha\,.$$

Thus, the quotient at the r.h.s. of (10.31) can be written as

$$\tilde{f}(c_1,c_2,\varepsilon) := \frac{3\pi^2}{2d^2}\,\frac{h(c_1)+\frac{16}{3\pi^2}\alpha\varepsilon+\varepsilon^2(2g(c_2)+h(c_2))}{g(c_1)+\frac{16}{3\pi^2}\alpha\varepsilon+\varepsilon^2 g(c_2)}\,. \tag{10.38}$$

Clearly, $\tilde{f}$ is a continuous function of the three variables defined in the region $(1,\infty)^2\times\mathbb{R}$ (the denominator is positive since it is the squared norm of a nonzero function) and one could look for its infimum. However, from the technical point of view, it seems to be a rather complicated task and that is why we make first the following simplification.

We start by verifying that the infimum of $\tilde{f}$ is negative, *i.e.*, $\psi_{c_1,c_2,\varepsilon}$ is an admissible trial function to estimate $\inf(H^D)-E_1^D<0$, *cf* Theorem 10.3. Obviously, $h(c)>0$ for any $c\in(1,\infty)$. Using the definition of α, the assumption $\langle\mathsf{H}\rangle$ and obvious estimates, we check that the same holds true for g:

$$g(c) > 3\left(\frac{s_0}{d}-\frac{\alpha}{4}\right) > 3\left(\frac{s_0}{d}-\frac{1}{2}s_0\|k_+\|_\infty\right) > \frac{3}{2}\,\frac{s_0}{d}\,. \tag{10.39}$$

Hence, the only term in the numerator of $\tilde{f}$ which can attain negative values is the term linear in ε. However, for any given $c_2>0$, there exists $\varepsilon\in\mathbb{R}$ of such a sign that $\alpha\,\varepsilon<0$ and with a sufficiently small absolute value so that the negative term linear in ε dominates over the quadratic one. Then we can find c_1 large enough to make the numerator of the r.h.s. of (10.38) negative. Recalling that the denominator is positive, we can restrict ourselves to those values of the triple (c_1,c_2,ε), for which $\tilde{f}(c_1,c_2,\varepsilon)<0$; let us denote $\mathcal{N}:=\{(c_1,c_2,\varepsilon)\in(1,\infty)^2\times\mathbb{R}\,|\,\tilde{f}(c_1,c_2,\varepsilon)<0\}$. Setting for any $(c_1,c_2,\varepsilon)\in\mathcal{N}$,

$$f(c_1,c_2,\varepsilon) := \frac{3\pi^2}{2d^2}\,\frac{h(c_1)+\frac{16}{3\pi^2}\alpha\varepsilon+\varepsilon^2(2g(c_2)+h(c_2))}{g(c_1)}\,, \tag{10.40}$$

we arrive easily at the inequality $\tilde{f}(c_1,c_2,\varepsilon)\le f(c_1,c_2,\varepsilon)$, because the (positive) denominator in (10.38) is bounded from above by $g(c_1)$ due to the above considerations. Consequently,

$$\inf\sigma(H^D)-E_1^D \le \inf_{(c_1,c_2,\varepsilon)\in\mathcal{N}} f(c_1,c_2,\varepsilon). \tag{10.41}$$

Calculating the partial derivatives of f, it is straightforward to see that the system of equations $f_{,i}=0$, $i=1,2,3$, can be cast into the following form:

$$\begin{aligned}
\frac{s_0}{d}\,A(c_2,\varepsilon)\,(c_1-1)^2+\frac{4}{\pi^2}\,(c_1-1)+\frac{6}{\pi^2}\,\frac{d}{s_0}\,\left(\frac{s_0}{d}-\frac{\alpha}{4}\right)&=0\,,\\
(c_2-1)^2-\left(\frac{d}{\pi s_0}\right)^2&=0\,,\\
\varepsilon+\frac{8\alpha}{3\pi^2}\,\frac{1}{h(c_2)+2g(c_2)}&=0\,,
\end{aligned}$$

respectively, where, for any $(c_1,c_2,\varepsilon)\in\mathcal{N}$,

$$A(c_2,\varepsilon) := \frac{16}{3\pi^2}\alpha\varepsilon+\varepsilon^2(2g(c_2)+h(c_2))<0.$$

¿From the second equation we can immediately express c_2; of course, we choose that root c_{2+} which is greater than 1. Substituting c_{2+} to the third equation of our system, we obtain the root ε_0 (notice that really $\alpha\,\varepsilon_0<0$). Finally, putting c_{2+} and ε_0 to the first equation, we choose that root c_{1+} which is greater than 1. A tedious but straightforward calculation yields

$$f(c_{1+},c_{2+},\varepsilon_0) = -\frac{3\pi^4}{4d^2}\,\frac{A(c_{2+},\varepsilon_0)^2}{\left(1+\sqrt{1-\frac{3}{2}A(c_{2+},\varepsilon_0)\pi^2\left(\frac{s_0}{d}-\frac{\alpha}{4}\right)}\right)^2}$$

with

$$A(c_{2+},\varepsilon_0) = -\frac{32\alpha^2}{9\pi^4}\,\frac{1}{\frac{2}{\pi}+3(\frac{s_0}{d}-\frac{\alpha}{4})}\,.$$

(Recall that $\frac{s_0}{d}-\frac{\alpha}{4}>0$, so the square root in the first formula is well defined in $\mathbb{R}$.) Hence really $(c_{1+},c_{2+},\varepsilon_0)\in\mathcal{N}$. Moreover, one can check that the matrix of second derivatives of f is in the point $(c_{1+},c_{2+},\varepsilon_0)$ diagonal with all positive elements, that is, the function f reaches its local minimum in that point.

To see that it is the global minimum too, we study the behaviour of the limits of f as $c_i \to 1, \infty$, $i \in \{1,2\}$ and $\varepsilon \to \pm\infty$. We restrict ourselves to that cases, where the limit is reached by negative values of f; the rest of the "boundary" of the set $\mathcal{N}$ consists of those triples $(\tilde{c}_1, \tilde{c}_2, \tilde{\varepsilon})$, for which $f(\tilde{c}_1, \tilde{c}_2, \tilde{\varepsilon}) = 0$, that is, $f(\tilde{c}_1, \tilde{c}_2, \tilde{\varepsilon}) > f(c_{1_+}, c_{2_+}, \varepsilon_0)$. Since (10.39) gives

$$g(c) > \frac{3\,|\alpha|}{4}\,, \tag{10.42}$$

we obtain

$$f(c_1, c_2, \varepsilon) > \frac{3\,\pi^2}{2\,d^2}\,\frac{\frac{16}{3\,\pi^2}\alpha\,\varepsilon + \frac{3\,|\alpha|}{2}\varepsilon^2}{g(c_1)}$$

and the condition $f(c_1, c_2, \varepsilon) < 0$ yields

$$|\varepsilon| < \frac{32}{9\,\pi^2}\,. \tag{10.43}$$

Hence we do not study the limits as $\varepsilon \to \pm\infty$ and we may assume in the following that ε is bounded. Using (10.42) in the denominator of (10.40), neglecting $h(c_1)$ and minimising the remaining polynom in ε in the numerator of (10.40), we arrive at the lower bound

$$f(c_1, c_2, \varepsilon) > -\frac{128}{9\,\pi^2\,d^2}\,\frac{|\alpha|}{h(c_2) + 2\,g(c_2)}$$

for any $c_2 \in (1, \infty)$. Thus

$$\liminf_{c_2 \to \infty} f(c_1, c_2, \varepsilon) \geq 0\,, \qquad \liminf_{c_2 \to 1} f(c_1, c_2, \varepsilon) \geq 0$$

uniformly in c_1 and ε. Finally, using (10.43) we can see that

$$f(c_1, c_2, \varepsilon) > \frac{3\,\pi^2}{2\,d^2}\,\frac{h(c_1)}{g(c_1)} - \frac{256\,|\alpha|}{9\,\pi^2\,d^2}\,\frac{1}{g(c_1)}$$

and therefore

$$\liminf_{c_1 \to \infty} f(c_1, c_2, \varepsilon) \geq 0\,, \qquad \lim_{c_1 \to 1} f(c_1, c_2, \varepsilon) = \infty$$

uniformly in c_2 and ε. Summing up the considerations, we conclude that $f(c_{1_+}, c_{2_+}, \varepsilon_0)$ is the global minimum and the claim (ii) of Theorem 10.5 then follows from (10.41). □

Remark 10.9 (Mildly curved strips)**.** Let us compare our estimate (ii) of Theorem 10.5 with the exact ground-state eigenvalue asymptotics derived in [14, Thm. 4.1] for *mildly* curved Dirichlet strips by the Birman-Schwinger perturbation technique. We consider families of generating curves Γ_β characterised by the curvature $k_\beta(s) := \beta\,k(s)$, where k is a fixed curvature function and $\beta > 0$ is a small parameter. Since $\alpha_\beta := \int_{\mathbb{R}} k_\beta(s)ds = \beta\,\alpha$, we see that β controls the total bending of the strip, too. The result of [14] can be written as

$$\inf(H^D) = E_1^D \,-\, C(d,k)^2\,\beta^4 + \mathcal{O}(\beta^5)\,,$$

where $C(d,k)$ is a positive constant depending only on the fixed width d and (integrals of) k, while our estimate (ii) yields

$$\inf(H^D) \leq E_1^D \,-\, C^D(s_0,d,0)^2\,\alpha^4\,\beta^4 + \mathcal{O}(\beta^5)\,.$$

Hence we observe the same dependence of the leading terms on the perturbation parameter β. Let us quantitatively compare the actual gap-width asymptotic given by $C(d,k)^2$ with our estimate $C^D(s_0,d,0)^2\alpha^4$. Since $C(d,k)$ has rather a complicated structure, we restrict ourselves to small values of the width d when

$$C(d,k) = \frac{1}{8}\,\|k\|^2_{L^2(\mathbb{R})} + \mathcal{O}(d^2)\,. \tag{10.44}$$

We have $C^D(s_0,d,0) = 8/(9\sqrt{3}\pi^2 s_0) + \mathcal{O}(d)$. Since $\alpha^2 \leq 2s_0\|k\|^2_{L^2(\mathbb{R})}$ by the Schwarz inequality, we see that

$$\frac{C^D(s_0,0,0)\,\alpha^2}{C(0,k)} \leq \frac{128}{9\sqrt{3}\,\pi^2} \approx 0.83\,.$$

As for the mixed Dirichlet-Neumann case, our estimate (i) of Theorem 10.5 leads to

$$\inf(H^{DN}) \leq E_1^{DN} - \frac{3\,\alpha^2}{8\,d^2}\,\beta^2 + \mathcal{O}(\beta^3)$$

and we observe that the leading term is proportional to the second power of β now. In particular, it is much greater than the leading term in the identical mildly curved strip with the pure Dirichlet boundary condition. Unfortunately, no exact asymptotics are known for $\inf\sigma(H^{DN})$, so we cannot perform any comparison in this case.

Remark 10.10 (Thin strips)**.** Another natural perturbation parameter is the strip width d. Calculating the expansions w.r.t. d of the constants $C^\iota(s_0, d, \alpha)$ from our Theorem 10.5, we arrive at

$$E_1^{DN} - \inf(H^{DN}) \geq -\frac{\alpha}{2\, s_0\, d} + \mathcal{O}(d^{-\frac{1}{2}})\,,$$
$$E_1^{D} - \inf(H^{D}) \geq \frac{2^8\, \alpha^4}{3^5\, \pi^4\, s_0^2 \left(1 + \sqrt{1 + \left(\frac{4\alpha}{3\pi}\right)^2}\right)^2} + \mathcal{O}(d).$$

Again, we observe qualitatively different behaviour of our estimates w.r.t. the perturbation parameter.

In particular, the leading term in our lower estimate of the gap between the essential spectrum threshold and the lowest Dirichlet eigenvalue is independent of the strip width. This is in accordance with the perturbation expansion of the ground-state eigenvalue derived in [14, Thm. 5.1]:

$$E_1^{D} - \inf(H^{D}) = -\lambda(k) + \mathcal{O}(d)\,.$$

Here $\lambda(k)$ denotes the first (negative) eigenvalue of the one-dimensional Schrödinger operator $l := -\Delta - \frac{1}{4}\,k^2$ on $L^2(\mathbb{R})$ with $\mathrm{Dom}\, l := W^{2,2}(\mathbb{R})$, which is naturally associated with the problem and reflects the geometry of Γ only. (We remark that, under our assumptions, the operator l has always a negative eigenvalue, *cf* [59, Thm. XIII.11].)

The leading term in the Dirichlet-Neumann estimate tends to $+\infty$ as $d \to 0$ (notice, however, that this fact does not conflict with anything because $E_1^{DN} = \mathcal{O}(d^{-2})$). That is, we again observe the effect of stronger binding of the particle in the case when a Dirichlet boundary curve of the strip is replaced by the Neumann one. A similar asymptotic estimate can be also deduced directly from the crude bound (ii) of Proposition 10.3. Since no perturbation expansion w.r.t. d for the lowest eigenvalue in the Dirichlet-Neumann case is known yet, we cannot compare our estimate with exact asymptotics.

10.7 Conclusions

Motivated by the theory of curved quantum waveguides, we were interested in spectral properties of the Laplace operator in a strip built over an infinite planar curve, see Figure 10.1, subject to three different types of boundary conditions (Dirichlet, Neumann or a combination of these ones, respectively). We localised the essential spectrum as a set under a very natural and weak condition about vanishing of curvature at infinity only, *cf* Theorem 10.1. We stress that no condition about the decay of derivatives of the curvature was required throughout this paper (the derivatives may not even exist because the reference curve is supposed to be C^2-smooth only). Then we were interested in the geometrically induced spectrum, *i.e.*, the spectrum below the spectral threshold of the corresponding straight strip; we made a survey of known results and established new ones, *cf* Theorems 10.2–10.4. Here the most important progress was achieved in the case of combined Dirichlet-Neumann boundary conditions where we generalised the only one known result of [13] and established two new sufficient conditions which guaranteed the existence of geometrically induced spectrum, *cf* Theorem 10.4. We recall that the geometrically induced spectrum consists of discrete eigenvalues only, whenever the above asymptotic behaviour of curvature holds true. Finally, we established two upper bounds on the infimum of the spectral threshold in a situation when the geometrically induced spectrum is present, *cf* Theorem 10.5. These estimates are new in the theory of curved quantum waveguides and their remarkable behaviour in the limit of mild curvature or small width of the strip was discussed, *cf* Remarks 10.9 and 10.10. Summing up briefly the main contribution of the paper, we gave answers to the two questions formulated in Section 10.2.

Let us now mention some directions in which the above mentioned results could be strengthened or extended.

In Theorem 10.1, we succeeded to localise the essential spectrum as a set, however, an open problem is to examine its nature. Here a particularly interesting question is whether the curved geometry may produce a singular continuous spectrum.

Theorem 10.3 concerning the existence of geometrically induced spectrum in Dirichlet strips is optimal in the sense that no better result can be achieved without violating the basic hypothesis $\langle \mathsf{H} \rangle$. One is of course tempted to ask which more general regions (than the curved asymptotically straight strips) still possess a non-trivial discrete spectrum. For instance, it is easy to see that the existence result does not change if the boundary of the strip is deformed locally and in such a way that the resulting deformed region lies in the exterior of the strip, *cf* [53], however, more complicated deformations of the boundary represent a difficult problem even in the straight case, [5, 4]. In this context, it is worth to recall that the existence of discrete spectrum in V-shaped waveguides was demonstrated in [32, 2, 7] (the computed bound-state energy has been verified experimentally in a flat electromagnetic waveguide in [6]).

The Neumann case is trivial from the point of view of the existence of discrete spectrum in asymptotically straight strips, *cf* Theorem 10.2. As for the Dirichlet-Neumann strip, while our Theorem 10.4 covers various wide classes of geometries for which the geometrically induced spectrum exists, it does not represent an ultimate result. For instance, it remains to be clarified whether one can include also some thick strips with a positive total bending angle. Another open question concerning the strips with combined boundary condition is the study of the behaviour of eigenvalues in mildly curved, respectively thin, strips, *cf* Remarks 10.9 and 10.10.

The upper bounds on the spectral threshold we presented in Theorem 10.5 can be surely improved. First of all, one should include the situations when the total bending angle is equal to zero and/or the strip is curved globally.

As we have already mentioned in Introduction, the Dirichlet Laplacian in the curved strip represents a reasonable model for a quantum Hamiltonian of a particle restricted to move in a strip-like nanostructure. Assuming that the boundary is sufficiently regular, to impose the Dirichlet boundary conditions means to require the vanishing of wavefunctions, however, as pointed out in [38], this may be in general too restrictive and one should rather require the vanishing of the probability current only. The latter leads in our case to a general boundary condition of the type

$$a_0\psi(\cdot,0) + b_0\psi_{,2}(\cdot,0) = 0\,, \qquad a_d\psi(\cdot,d) + b_d\psi_{,2}(\cdot,d) = 0\,,$$

where $\psi \in \mathcal{H}$ denotes the wavefunction and $(a_0, a_d), (b_0, b_d) \in \mathbb{R}^2 \setminus \{(0,0)\}$. However, at least from the mathematical point of view, it would be interesting to examine the influence of the choice of particular boundary conditions on the spectral properties of the Hamiltonian. Finally, it would be also possible to let the coefficients a_0, a_d, b_0, b_d depend on the longitudinal variable s.

Other obvious extensions are to consider the Laplacian in tubular neighbourhoods of non-compact submanifolds of general Riemannian manifolds. Here the spectral problem has been studied only for Dirichlet tubes, [39, 14], and layers, [16, 8], in $\mathbb{R}^3$, or strips in two-dimensional manifolds, [45]; more general boundary conditions and/or higher-dimensional generalisations are completely missing.

Acknowledgements

One of the authors (D.K.) would like to thank Viorel Iftimie for pointing out to him the general characterisation of essential spectrum (Lemma 10.1) which gives a better result than our original proof based on the Weyl criterion; discussions on this subject with Ingrid Beltita are also acknowledged. We are very grateful to Jaroslav Dittrich, Pierre Duclos, Pavel Exner, and Pedro Freitas for useful remarks and suggestions. The work has been partially supported by GA AS CR grant IAA 1048101. The first author was partially funded by FCT and FSE, Portugal.

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Chapter 11

A narrow curved strip with combined Dirichlet and Neumann boundary conditions

Published in: ESAIM: Control, Optimisation and Calculus of Variations **15** (2009), 555–568

Spectrum of the Laplacian in a narrow curved strip with combined Dirichlet and Neumann boundary conditions

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Abstract. We consider the Laplacian in a domain squeezed between two parallel curves in the plane, subject to Dirichlet boundary conditions on one of the curves and Neumann boundary conditions on the other. We derive two-term asymptotics for eigenvalues in the limit when the distance between the curves tends to zero. The asymptotics are uniform and local in the sense that the coefficients depend only on the extremal points where the ratio of the curvature radii of the Neumann boundary to the Dirichlet one is the biggest. We also show that the asymptotics can be obtained from a form of norm-resolvent convergence which takes into account the width-dependence of the domain of definition of the operators involved.

11.1 Introduction

Given an open interval $I \subseteq \mathbb{R}$ (bounded or unbounded), let $\gamma \in C^2(\overline{I}; \mathbb{R}^2)$ be a unit-speed planar curve. The derivative $\dot\gamma \equiv (\dot\gamma^1, \dot\gamma^2)$ and $n := (-\dot\gamma^2, \dot\gamma^1)$ define unit tangent and normal vector fields along γ, respectively. The curvature is defined through the Frenet-Serret formulae by $\kappa := \det(\dot\gamma, \ddot\gamma)$; it is a bounded and uniformly continuous function on I.

For any positive ε, we introduce a mapping $\mathcal{L}_\varepsilon$ from $\overline{I} \times [0,1]$ to $\mathbb{R}^2$ by

$$\mathcal{L}_\varepsilon(s,t) := \gamma(s) + \varepsilon\, t\, n(s)\,. \tag{11.1}$$

Assuming that $\mathcal{L}_\varepsilon$ is injective and that ε is so small that the supremum norm of κ is less than ε^{-1}, $\mathcal{L}_\varepsilon$ induces a diffeomorphism and the image

$$\Omega_\varepsilon := \mathcal{L}_\varepsilon\big(I \times (0,1)\big) \tag{11.2}$$

has a geometrical meaning of an open non-self-intersecting strip, contained between the parallel curves $\gamma(I)$ and $\gamma_\varepsilon(I) := \mathcal{L}_\varepsilon(I \times \{1\})$, and, if ∂I is not empty, the straight lines $\mathcal{L}_\varepsilon\big(\{\inf I\} \times (0,1)\big)$ and $\mathcal{L}_\varepsilon\big(\{\sup I\} \times (0,1)\big)$. The geometry is set in such a way that $\kappa > 0$ implies that the parallel curve γ_ε is "locally shorter" than γ, and vice versa, *cf* Figure 11.1.

Let $-\Delta_{DN}^{\Omega_\varepsilon}$ be the Laplacian in $L^2(\Omega_\varepsilon)$ with Dirichlet and Neumann boundary conditions on γ and γ_ε, respectively. If ∂I is not empty, we impose Dirichlet boundary conditions on the remaining parts of $\partial\Omega_\varepsilon$.

For any self-adjoint operator H which is bounded from below, we denote by $\{\lambda_j(H)\}_{j=1}^\infty$ the non-decreasing sequence of numbers corresponding to the spectral problem of H according to the Rayleigh-Ritz variational formula [10, Sec. 4.5]. Each $\lambda_j(H)$ represents either a (discrete) eigenvalue (repeated according to multiplicity) below the essential spectrum or the threshold of the essential spectrum of H. All the eigenvalues below the essential spectrum may be characterized by this variational/minimax principle.

Under the above assumptions, our main result reads as follows:

Theorem 11.1. *For all $j \geq 1$,*

$$\lambda_j(-\Delta_{DN}^{\Omega_\varepsilon}) \;=\; \left(\frac{\pi}{2\varepsilon}\right)^2 + \frac{\inf\kappa}{\varepsilon} + o(\varepsilon^{-1}) \qquad \textit{as} \qquad \varepsilon \to 0\,. \tag{11.3}$$

Theorem 11.1 has important consequences for the spectral properties of the operator $-\Delta_{DN}^{\Omega_\varepsilon}$, especially in the physically interesting situation $I = \mathbb{R}$. In this case, assuming that the curvature κ vanishes at infinity, the leading term $\pi^2/(2\varepsilon)^2$ of (11.3) coincides with the threshold of the essential spectrum of $-\Delta_{DN}^{\Omega_\varepsilon}$. The next term in the expansion then tells us that

(a) the discrete spectrum exists whenever κ assumes a negative value and ε is sufficiently small;

(b) the number of the eigenvalues increases to infinity as $\varepsilon \to 0$.

This provides an insight into the mechanism which is behind the qualitative results obtained by Dittrich and Kříž in their 2002 letter [5]. Using $-\Delta_{DN}^{\Omega_\varepsilon}$ as a model for the Hamiltonian of a quantum waveguide, they show that the discrete eigenvalues exist if, and only if, the reference curve γ of sign-definite κ is curved "in the right direction", namely if the Neumann boundary condition is imposed on the "locally longer" boundary (*i.e.* $\kappa < 0$

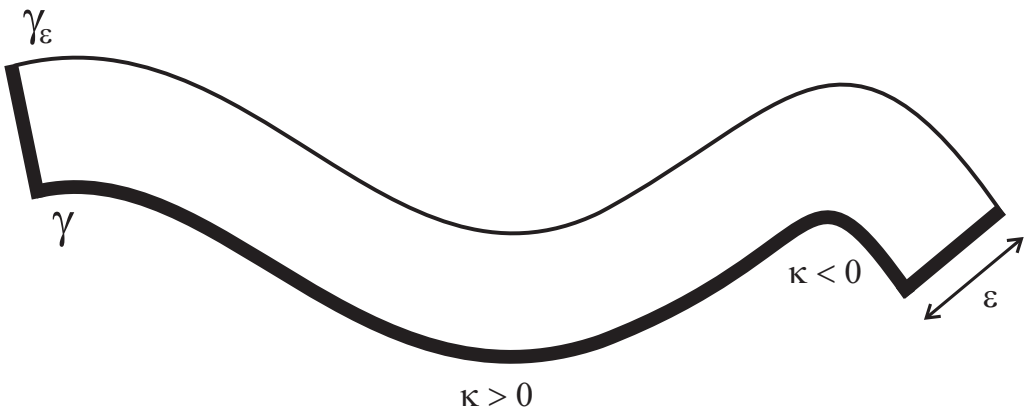

Figure 11.1: The geometry of the strip Ω_ε in a case of bounded I. The thick and thin lines correspond to Dirichlet and Neumann boundary conditions, respectively.

in our setting), and that (b) holds. The results were further generalized in [21, 12], numerically tested in [24], and established in a different physical model in [17].

The purely Dirichlet or Neumann strips differ from the present situation in many respects (see [21] for a comparison). The case of the Neumann Laplacian $-\Delta_N^{\Omega_\varepsilon}$ is trivial in the sense that

$$\lambda_1(-\Delta_N^{\Omega_\varepsilon}) = \lambda_1(-\Delta_N^I) = 0\,,$$

independently of the geometry and smallness of ε, where $-\Delta_N^I$ denotes the Neumann Laplacian in $L^2(I)$. For $j \geq 2$, one has

$$\lambda_j(-\Delta_N^{\Omega_\varepsilon}) = \lambda_j(-\Delta_N^I) + o(1) \qquad \text{as} \qquad \varepsilon \to 0\,,$$

independently of the geometry. More generally, it is well known that the spectrum of the Neumann Laplacian on an ε-tubular neighbourhood of a Riemannian manifold converges when $\varepsilon \to 0$ to the spectrum of the Laplace-Beltrami operator on the manifold [26].

As for the Dirichlet Laplacian $-\Delta_D^{\Omega_\varepsilon}$, it is well known [8, 15, 6, 21] that the existence of discrete spectrum in unbounded strips is robust, *i.e.* independent of the sign of κ. This is also reflected in the asymptotic formula, $j \geq 1$,

$$\lambda_j(-\Delta_D^{\Omega_\varepsilon}) = \Big(\frac{\pi}{\varepsilon}\Big)^2 + \lambda_j\big(-\Delta_D^I - \frac{\kappa^2}{4}\big) + o(1) \qquad \text{as} \qquad \varepsilon \to 0\,, \tag{11.4}$$

known for many years [18, 6], where $-\Delta_D^I$ denotes the Dirichlet Laplacian in $L^2(I)$. That is, contrary to Theorem 11.1, in the purely Dirichlet case the second term in the asymptotic expansion is independent of ε, always negative unless γ is a straight line, and its value is determined by the *global* geometry of γ.

The local character of (11.3) rather resembles the problem of a straight narrow strip of *variable* width studied recently by Friedlander and Solomyak [13, 14], and also by Borisov and Freitas [1] – see also [9] for related work. In view of their asymptotics, the spectrum of the Dirichlet Laplacian is basically determined by the points where the strip is the widest. In our model the cross-section is uniform but the curvature *and* boundary conditions are not homogeneous.

The purely Dirichlet case with uniform cross-section differs from the present situation also in the direct method of the proof of (11.4). Using the parametrization (11.1), the spectral problem for the Laplacian $-\Delta_D^{\Omega_\varepsilon}$ in the "curved" and ε-dependent Hilbert space $L^2(\Omega_\varepsilon)$ is transferred to the spectral problem for a more complicated operator H_ε^D in $L^2\big(I \times (0,1)\big)$. Inspecting the dependence of the coefficients of H_ε^D on ε, it turns out that the operator is in the limit $\varepsilon \to 0$ decoupled into a sum of the "transverse" Laplacian multiplied by ε^{-2} and of the ε-independent Schrödinger operator on γ. At this stage, the minimax principle is sufficient to establish (11.4). Furthermore, since the "straightened" Hilbert space is independent of ε, it is also possible to show that (11.4) is obtained as a consequence of some sort of norm-resolvent convergence [6, 10]. An alternative approach is based on the Γ-convergence method [2]. See also [16] for a recent survey of the thin-limit problem in a wider context.

The above procedure does not work in the present situation because the transformed operator H_ε^{DN} does not decouple as $\varepsilon \to 0$, at least at the stage of the elementary usage of the minimax principle. Moreover, the operator domain of H_ε^{DN} becomes dependent on ε; contrary to the Dirichlet boundary condition, the Neumann one is transferred to an ε-dependent and variable Robin-type boundary condition (*cf* Remark 11.1 below). In this paper we propose an alternative approach, which enables us to treat the case of combined boundary conditions. Our method of proof is based on refined applications of the minimax principle.

In the following Section 11.2, we prove Theorem 11.1 as a consequence of upper and lower bounds to $\lambda_j(-\Delta_{DN}^{\Omega_\varepsilon})$. More specifically, these estimates imply

Theorem 11.2. *For all $j \geq 1$,*

$$\lambda_j(-\Delta_{DN}^{\Omega_\varepsilon}) = \Big(\frac{\pi}{2\varepsilon}\Big)^2 + \lambda_j\big(-\Delta_D^I + \frac{\kappa}{\varepsilon}\big) + \mathcal{O}(1) \qquad as \qquad \varepsilon \to 0\,. \tag{11.5}$$

Then Theorem 11.1 follows at once as a weaker version of Theorem 11.2, by using known results about the strong-coupling/semiclassical asymptotics of eigenvalues of the one-dimensional Schrödinger operator. Indeed, for all $j \geq 1$, one has

$$\lambda_j\big(-\Delta_D^I + \frac{\kappa}{\varepsilon}\big) = \frac{\inf\kappa}{\varepsilon} + o(\varepsilon^{-1}) \qquad \text{as} \qquad \varepsilon \to 0\,. \tag{11.6}$$

This result seems to be well known; we refer to [23, App. A] for a proof in any dimension.

Another goal of the present paper is to show that the eigenvalue convergence of Theorem 11.1 can be obtained as a consequence of the norm-resolvent "convergence" of $-\Delta_{DN}^{\Omega_\varepsilon}$ to $-\Delta_D^I + \kappa/\varepsilon$ as $\varepsilon \to 0$. We use the quotation marks because the latter operator is in fact ε-dependent and the norm-resolvent convergence should be rather interpreted as the convergence of the difference of corresponding resolvent operators in norm. However, the operators act in different Hilbert spaces and the norm-resolvent convergence still requires a meaningful reinterpretation. Because of the technical complexity, we postpone the statement of this convergence result until Section 11.3.

The paper is concluded by Section 11.4 in which we discuss possible extensions of our main results.

11.2 Spectral convergence

In this section we give a simple proof of Theorem 11.2 by establishing upper and lower bounds to $\lambda_j(-\Delta_{DN}^{\Omega_\varepsilon})$. We begin with necessary geometric preliminaries.

11.2.1 Curvilinear coordinates

As usual, the Laplacian $-\Delta_{DN}^{\Omega_\varepsilon}$ is introduced as the self-adjoint operator in $L^2(\Omega_\varepsilon)$ associated with the quadratic form $Q_{DN}^{\Omega_\varepsilon}$ defined by

$$Q_{DN}^{\Omega_\varepsilon}[\Psi] := \int_{\Omega_\varepsilon} |\nabla\Psi(x)|^2\,dx\,,$$
$$\Psi \in D(Q_{DN}^{\Omega_\varepsilon}) := \big\{\Psi \in W^{1,2}(\Omega_\varepsilon) \mid \quad \Psi = 0 \quad \text{on} \quad \partial\Omega_\varepsilon \setminus \gamma_\varepsilon(I)\big\}\,.$$

Here Ψ on $\partial\Omega_\varepsilon$ is understood in the sense of traces. It is natural to express the Laplacian in the "coordinates" (s,t) determined by the inverse of $\mathcal{L}_\varepsilon$.

As stated in Introduction, under the hypotheses that $\mathcal{L}_\varepsilon$ is injective and

$$\varepsilon\,\sup|\kappa| < 1\,, \tag{11.7}$$

the mapping (11.1) induces a global diffeomorphism between $I \times (0,1)$ and Ω_ε. This is readily seen by the inverse function theorem and the expression for the Jacobian $\det(\partial_1\mathcal{L}_\varepsilon, \partial_2\mathcal{L}_\varepsilon) = \varepsilon h_\varepsilon$ of $\mathcal{L}_\varepsilon$, where

$$h_\varepsilon(s,t) := 1 - \kappa(s)\,\varepsilon\,t\,. \tag{11.8}$$

In fact, (11.7) yields the uniform estimates

$$0 < 1 - \varepsilon\sup\kappa \leq h_\varepsilon \leq 1 - \varepsilon\inf\kappa < \infty\,, \tag{11.9}$$

where the lower bound ensures that the Jacobian never vanishes in $\overline{I} \times [0,1]$.

The passage to the natural coordinates (together with a simple scaling) is then performed via the unitary transformation

$$U_\varepsilon : L^2(\Omega_\varepsilon) \to \mathcal{H}_\varepsilon := L^2\big(I \times (0,1), h_\varepsilon(s,t)\,ds\,dt\big) : \big\{\Psi \mapsto \sqrt{\varepsilon}\;\Psi \circ \mathcal{L}_\varepsilon\big\}\,.$$

This leads to a unitarily equivalent operator $H_\varepsilon := U_\varepsilon(-\Delta_{DN}^{\Omega_\varepsilon})U_\varepsilon^{-1}$ in $\mathcal{H}_\varepsilon$, which is associated with the quadratic form Q_ε defined by

$$Q_\varepsilon[\psi] := \int_{I\times(0,1)} \frac{|\partial_1\psi(s,t)|^2}{h_\varepsilon(s,t)}\,ds\,dt + \int_{I\times(0,1)} \frac{|\partial_2\psi(s,t)|^2}{\varepsilon^2}\,h_\varepsilon(s,t)\,ds\,dt\,,$$
$$\psi \in D(Q_\varepsilon) := \big\{\psi \in W^{1,2}\big(I\times(0,1)\big) \mid \psi = 0 \quad \text{on} \quad \partial\big(I\times(0,1)\big) \setminus \big(I \times \{1\}\big)\big\}\,.$$

As a consequence of (11.9), $\mathcal{H}_\varepsilon$ and $L^2\big(I\times(0,1)\big)$ can be identified as vector spaces due to the equivalence of norms, denoted respectively by $\|\cdot\|_\varepsilon$ and $\|\cdot\|$ in the following. More precisely, we have

$$1-\varepsilon\sup\kappa\le\frac{\|\psi\|_\varepsilon^2}{\|\psi\|^2}\le 1-\varepsilon\inf\kappa\,. \tag{11.10}$$

That is, the fraction of norms behaves as $1+\mathcal{O}(\varepsilon)$ as $\varepsilon\to 0$.

11.2.2 Upper bound

Let ψ be a test function from the domain $D(Q_\varepsilon)$ of the form

$$\psi(s,t):=\varphi(s)\chi_1(t)\,,\qquad\text{where}\qquad \chi_1(t):=\sqrt{2}\sin(\pi t/2) \tag{11.11}$$

and $\varphi\in W_0^{1,2}(I)$ is arbitrary. Note that χ_1 is a normalized eigenfunction corresponding to the lowest eigenvalue of $-\Delta_{DN}^{(0,1)}$, *i.e.* the Laplacian in $L^2((0,1))$, subject to the Dirichlet and Neumann boundary condition at 0 and 1, respectively. A straightforward calculation yields

$$Q_\varepsilon[\psi]-\Big(\frac{\pi}{2\varepsilon}\Big)^2\|\psi\|_\varepsilon^2=\int_I\bigg(a_\varepsilon(s)\,|\varphi'(s)|^2+\frac{\kappa(s)}{\varepsilon}\,|\varphi(s)|^2\bigg)\,ds\,,$$

where

$$a_\varepsilon(s):=\int_0^1\frac{|\chi_1(t)|^2}{h_\varepsilon(s,t)}\,dt\,.$$

Note that $\sup a_\varepsilon=1+\mathcal{O}(\varepsilon)$ due to (11.9) and the normalization of χ_1. Using in addition the boundedness of κ and $\|\varphi\|_{L^2(I)}=\|\psi\|$ together with (11.10), we can therefore write

$$\frac{Q_\varepsilon[\psi]}{\|\psi\|_\varepsilon^2}-\Big(\frac{\pi}{2\varepsilon}\Big)^2-\mathcal{O}(1)\;\le\;\big[1+\mathcal{O}(\varepsilon)\big]\,\frac{\int_I\Big(|\varphi'(s)|^2+\frac{\kappa(s)}{\varepsilon}\,|\varphi(s)|^2\Big)\,ds}{\int_I|\varphi(s)|^2\,ds}\,.$$

From this inequality, the minimax principle gives the upper bound

$$\begin{aligned}\lambda_j(H_\varepsilon)-\Big(\frac{\pi}{2\varepsilon}\Big)^2\;&\le\;\big[1+\mathcal{O}(\varepsilon)\big]\,\lambda_j\big(-\Delta_D^I+\frac{\kappa}{\varepsilon}\big)+\mathcal{O}(1)\\ &=\;\lambda_j\big(-\Delta_D^I+\frac{\kappa}{\varepsilon}\big)+\mathcal{O}(1)\qquad\text{as}\qquad\varepsilon\to 0\end{aligned} \tag{11.12}$$

for all $j\ge 1$. Here the equality follows by (11.6).

11.2.3 Lower bound

For all $\psi\in D(Q_\varepsilon)$, we have

$$Q_\varepsilon[\psi]\ge\int_{I\times(0,1)}\frac{|\partial_1\psi(s,t)|^2}{h_\varepsilon(s,t)}\,ds\,dt+\int_{I\times(0,1)}\frac{\nu\big(\varepsilon\kappa(s)\big)}{\varepsilon^2}\,|\psi(s,t)|^2\,h_\varepsilon(s,t)\,ds\,dt\,,$$

where $\nu(\epsilon)\equiv\lambda_1(T_\epsilon)$ denotes the lowest eigenvalue of the operator T_ϵ in the Hilbert space $L^2\big((0,1),(1-\epsilon t)dt\big)$ defined by

$$\begin{aligned}(T_\epsilon\chi)(t)&:=-\chi''(t)+\frac{\epsilon}{1-\epsilon t}\,\chi'(t)\,,\\ \chi\in D(T_\epsilon)&:=\big\{\chi\in W^{2,2}\big((0,1)\big)\;\big|\quad\chi(0)=\chi'(1)=0\big\}\,.\end{aligned}$$

Note that $\nu(0)=(\pi/2)^2$ and that the corresponding eigenfunction for $\epsilon=0$ can be identified with χ_1. The analytic perturbation theory yields

$$\nu(\epsilon)=\Big(\frac{\pi}{2}\Big)^2+\epsilon+\mathcal{O}\big(\epsilon^2\big)\qquad\text{as}\qquad\epsilon\to 0\,. \tag{11.13}$$

Using this expansion and the boundedness of κ, we can estimate

$$Q_\varepsilon[\psi]-\Big(\frac{\pi}{2\varepsilon}\Big)^2\|\psi\|_\varepsilon^2\ge\int_{I\times(0,1)}\bigg(\frac{|\partial_1\psi(s,t)|^2}{1-\varepsilon\inf\kappa}+\frac{\kappa}{\varepsilon}\,|\psi(s,t)|^2-C\,|\psi(s,t)|^2\bigg)\,ds\,dt\,,$$

where C is a positive constant depending uniquely on $\|\kappa\|_{L^\infty(I)}$. Using in addition (11.10), we therefore get

$$\frac{Q_\varepsilon[\psi]}{\|\psi\|^2_\varepsilon} - \Big(\frac{\pi}{2\varepsilon}\Big)^2 - \mathcal{O}(1) \;\geq\; \big[1+\mathcal{O}(\varepsilon)\big]\,\frac{\int_{I\times(0,1)}\Big(|\partial_1\psi(s,t)|^2 + \frac{\kappa(s)}{\varepsilon}\,|\psi(s,t)|^2\Big)\,ds}{\int_{I\times(0,1)}|\psi(s,t)|^2\,ds}\,.$$

Consequently, the minimax principle gives

$$\begin{aligned}\lambda_j(H_\varepsilon) - \Big(\frac{\pi}{2\varepsilon}\Big)^2 \;&\geq\; \big[1+\mathcal{O}(\varepsilon)\big]\,\lambda_j\big(-\Delta^I_D + \frac{\kappa}{\varepsilon}\big) + \mathcal{O}(1) \\ &=\; \lambda_j\big(-\Delta^I_D + \frac{\kappa}{\varepsilon}\big) + \mathcal{O}(1) \qquad \text{as} \qquad \varepsilon\to 0 \end{aligned} \tag{11.14}$$

for all $j\geq 1$. Again, here the equality follows by (11.6).

In view of the unitary equivalence of H_ε with $-\Delta^{\Omega_\varepsilon}_{DN}$, the estimates (11.12) and (11.14) prove Theorem 11.2.

11.3 Norm-resolvent convergence

In this section we study the mechanism which is behind the eigenvalue convergence of Theorem 11.1 in more details. First we explain what we mean by the norm-resolvent convergence of the family of operators $\{-\Delta^{\Omega_\varepsilon}_{DN}\}_{\varepsilon>0}$.

11.3.1 The reference Hilbert space and the result

In Section 11.2.1, we identified the Laplacian $-\Delta^{\Omega_\varepsilon}_{DN}$ with a Laplace-Beltrami-type operator H_ε in $\mathcal{H}_\varepsilon$. Now it is more convenient to pass to another unitarily equivalent operator $\hat H_\varepsilon$ which acts in the "fixed" (*i.e.* ε-independent) Hilbert space

$$\mathcal{H}_0 := L^2\big(I\times(0,1)\big)\,.$$

This is enabled by means of the unitary mapping

$$\hat U_\varepsilon : \mathcal{H}_\varepsilon \to \mathcal{H}_0 : \big\{\psi\mapsto\sqrt{h_\varepsilon}\;\psi\big\}\,,$$

provided that the curvature κ is differentiable in a weak sense; henceforth we assume that

$$\kappa'\in L^\infty(I)\,. \tag{11.15}$$

We set $\hat H_\varepsilon := \hat U_\varepsilon H_\varepsilon \hat U_\varepsilon^{-1}$. As a comparison operator to $\hat H_\varepsilon$ for small ε, we consider the decoupled operator

$$\hat H_0 := \Big(-\Delta^I_D + \frac{\kappa}{\varepsilon}\Big)\otimes 1 + 1\otimes\Big(-\frac{1}{\varepsilon^2}\,\Delta^{(0,1)}_{DN}\Big) \qquad \text{in} \qquad L^2(I)\otimes L^2\big((0,1)\big)\,.$$

Here the subscript 0 is just a notational convention, of course, since $\hat H_0$ still depends on ε. Using natural isomorphisms, we may reconsider $\hat H_0$ as an operator in $\mathcal{H}_0$.

We clearly have

$$\hat H_0 \geq \Big(\frac{\pi}{2\varepsilon}\Big)^2 + \frac{\inf\kappa}{\varepsilon}\,. \tag{11.16}$$

At the same time,

$$\hat H_\varepsilon \geq \frac{\nu(\varepsilon\kappa)}{\varepsilon^2} \geq \frac{\nu(\varepsilon\inf\kappa)}{\varepsilon^2} = \Big(\frac{\pi}{2\varepsilon}\Big)^2 + \frac{\inf\kappa}{\varepsilon} + \mathcal{O}(1)\,, \tag{11.17}$$

where the first inequality was established (for the unitarily equivalent operator H_ε) in the beginning of Section 11.2.3, the second inequality holds due to the monotonicity of $\varepsilon\mapsto\nu(\varepsilon)$ proved in [24, Thm. 2] and the equality follows from (11.13). (Alternatively, we could use Theorem 11.1 to get (11.17), however, one motivation of the present section is to show that the former can be obtained as a consequence of Theorem 11.3 below.) Fix any number

$$k > -\inf\kappa\,. \tag{11.18}$$

It follows that $\hat H_\varepsilon - \pi^2/(2\varepsilon)^2 + k/\varepsilon$ and $\hat H_0 - \pi^2/(2\varepsilon)^2 + k/\varepsilon$ are positive operators for all sufficiently small ε.

Now we are in a position to state the main result of this section.

Theorem 11.3. *In addition to the injectivity of $\mathcal{L}_\varepsilon$ and the boundedness of κ, let us assume* (11.15)*. Then there exist positive constants ε_0 and C_0, depending uniquely on k and the supremum norms of κ and κ', such that for all $\varepsilon\in(0,\varepsilon_0)$:*

$$\bigg\|\Big[\hat H_\varepsilon - \Big(\frac{\pi}{2\varepsilon}\Big)^2 + \frac{k}{\varepsilon}\Big]^{-1} - \Big[\hat H_0 - \Big(\frac{\pi}{2\varepsilon}\Big)^2 + \frac{k}{\varepsilon}\Big]^{-1}\bigg\| \;\leq\; C_0\,\varepsilon^{3/2}\,.$$

The theorem is proved in several steps divided into the following subsections. In particular, it follows as a direct consequence of Lemmata 11.1 and 11.2 below. In the final subsection we show how it implies the convergence of eigenvalues of Theorem 11.1.

11.3.2 The transformed Laplacian

Let us now find an explicit expression for the quadratic form $\hat{Q}_\varepsilon$ associated with the operator $\hat{H}_\varepsilon$. By definition, it is given by

$$\hat{Q}_\varepsilon[\psi] := Q_\varepsilon[\hat{U}_\varepsilon^{-1}\psi]\,, \qquad \psi \in D(\hat{Q}_\varepsilon) := \hat{U}_\varepsilon D(Q_\varepsilon)\,.$$

One easily verifies that

$$D(\hat{Q}_\varepsilon) = D(Q_\varepsilon) =: \mathcal{Q}\,, \tag{11.19}$$

which is actually independent of ε. Furthermore, for any $\psi \in \mathcal{Q}$, we have

$$\hat{Q}_\varepsilon[\psi] = \hat{Q}_\varepsilon^1[\psi] + \hat{Q}_\varepsilon^2[\psi]\,,$$

where

$$\begin{aligned}
\hat{Q}_\varepsilon^1[\psi] &:= \int \frac{\left|\partial_1(h_\varepsilon^{-1/2}\psi)\right|^2}{h_\varepsilon} && = \int \left\{ \frac{|\partial_1\psi|^2}{h_\varepsilon^2} + V_\varepsilon^1\,|\psi|^2 + V_\varepsilon^2\,\Re\big(\overline{\psi}\partial_1\psi\big) \right\}, \\
\hat{Q}_\varepsilon^2[\psi] &:= \int \frac{\left|\partial_2(h_\varepsilon^{-1/2}\psi)\right|^2}{\varepsilon^2}\,h_\varepsilon && = \int \left\{ \frac{|\partial_2\psi|^2}{\varepsilon^2} + V_\varepsilon^3\,|\psi|^2 + V_\varepsilon^4\,\Re\big(\overline{\psi}\partial_2\psi\big) \right\},
\end{aligned}$$

with

$$\begin{aligned}
V_\varepsilon^1(s,t) &:= \frac{1}{4}\frac{\kappa'(s)^2\varepsilon^2 t^2}{h_\varepsilon(s,t)^4}\,, && V_\varepsilon^2(s,t) := \frac{\kappa'(s)\varepsilon t}{h_\varepsilon(s,t)^3}\,, \\
V_\varepsilon^3(s,t) &:= \frac{1}{4}\frac{\kappa(s)^2}{h_\varepsilon(s,t)^2}\,, && V_\varepsilon^4(s,t) := \frac{\kappa(s)}{\varepsilon h_\varepsilon(s,t)}\,.
\end{aligned}$$

Here and in the sequel the integral sign $\int$ refers to an integration over $I \times (0,1)$. Integrating by parts in the expression for $\hat{Q}_\varepsilon^2[\psi]$, we finally arrive at

$$\hat{Q}_\varepsilon[\psi] = \int \left\{ \frac{|\partial_1\psi|^2}{h_\varepsilon^2} + \frac{|\partial_2\psi|^2}{\varepsilon^2} + (V_\varepsilon^1 - V_\varepsilon^3)|\psi|^2 + V_\varepsilon^2\,\Re\big(\overline{\psi}\partial_1\psi\big) \right\} + \int_\partial v_\varepsilon\,|\psi|^2\,,$$

where

$$v_\varepsilon(s,t) := \frac{1}{2}\frac{\kappa(s)}{\varepsilon\big(1-\varepsilon\kappa(s)\big)}\,.$$

Here and in the sequel the integral sign $\int_\partial$ refers to an integration over the boundary $I \times \{1\}$.

Remark 11.1. $\hat{H}_\varepsilon$ is exactly the operator H_ε^{DN} mentioned briefly in Introduction. Let us remark in this context that, contrary to the form domains (11.19), the operator domains of H_ε and $\hat{H}_\varepsilon$ do differ (unless the curvature κ vanishes identically). Indeed, under additional regularity conditions about γ, it can be shown that while functions from $D(H_\varepsilon)$ satisfy Neumann boundary conditions on $I \times \{1\}$, the functions ψ from $D(\hat{H}_\varepsilon)$ satisfy non-homogeneous Robin-type boundary conditions $\partial_2\psi + \varepsilon^2 v_\varepsilon\psi = 0$ on $I \times \{1\}$. This is the reason why the decoupling of $\hat{H}_\varepsilon$ for small ε is not obvious in this situation. At the same time, we see that the operator domain of $\hat{H}_\varepsilon$ heavily depends on the geometry of γ. For our purposes, however, it will be enough to work with the associated quadratic form $\hat{Q}_\varepsilon$ whose domain is independent of ε and κ.

11.3.3 Renormalized operators and resolvent bounds

It will be more convenient to work with the shifted operators

$$L_\varepsilon := \hat{H}_\varepsilon - \left(\frac{\pi}{2\varepsilon}\right)^2 + \frac{k}{\varepsilon} \qquad \text{and} \qquad L_0 := \hat{H}_0 - \left(\frac{\pi}{2\varepsilon}\right)^2 + \frac{k}{\varepsilon}\,.$$

Let l_ε and l_0 denote the associated quadratic forms. It is important that they have the same domain $\mathcal{Q}$. More precisely, $\hat{H}_0$ was initially defined as a direct sum, however, using natural isomorphisms, it is clear that we can identify the form domain of L_0 with $\mathcal{Q}$ and

$$l_0[\psi] = \int \left\{ |\partial_1\psi|^2 + \frac{1}{\varepsilon^2}\left[|\partial_2\psi|^2 - \left(\frac{\pi}{2}\right)^2 |\psi|^2\right] + \frac{k+\kappa}{\varepsilon}\,|\psi|^2 \right\}$$

for all $\psi \in \mathcal{Q}$.

It will be also useful to have an intermediate operator L, obtained from L_ε after neglecting its non-singular dependence on ε but keeping the boundary term. For simplicity, henceforth we assume that ε is less than one and that it is in fact so small that (11.7) holds with a number less than one on the right hand side. Consequently,

$$|h_\varepsilon - 1| \le C\varepsilon\,,\ |V_\varepsilon^1| \le C\varepsilon^2\,,\ |V_\varepsilon^2| \le C\varepsilon\,,\ |V_\varepsilon^3| \le C\,,\ |V_\varepsilon^4| \le C\varepsilon^{-1}\,,\ |v_\varepsilon| \le C\varepsilon^{-1}\,. \tag{11.20}$$

Here and in the sequel, we use the convention that C and c are positive constants which possibly depend on k and the supremum norms of κ and κ', and which may vary from line to line. In view of these estimates, it is reasonable to introduce L as the operator associated with the quadratic form l defined by $D(l) := \mathcal{Q}$ and

$$l[\psi] := \int \left\{ |\partial_1\psi|^2 + \frac{1}{\varepsilon^2}\Big[|\partial_2\psi|^2 - \Big(\frac{\pi}{2}\Big)^2 |\psi|^2\Big] + \frac{k}{\varepsilon}\,|\psi|^2 \right\} + \int_\partial v_\varepsilon\,|\psi|^2$$

for all $\psi \in \mathcal{Q}$. Indeed, it follows from (11.20) that

$$\begin{aligned} \big|l_\varepsilon[\psi] - l[\psi]\big| &\le \int \Big\{ |h_\varepsilon^{-2} - 1||\partial_1\psi|^2 + |V_\varepsilon^1 - V_\varepsilon^3||\psi|^2 + |V_\varepsilon^2|\,|\psi||\partial_1\psi| \Big\} \\ &\le C\big(\varepsilon\|\partial_1\psi\|^2 + \|\psi\|^2\big) \end{aligned} \tag{11.21}$$

for all $\psi \in \mathcal{Q}$.

Let us now argue that, for every $\psi \in \mathcal{Q}$ and for all sufficiently small ε (which precisely means that ε has to be less than an explicit constant depending on k and the supremum norms of κ and κ'), we have

$$\min\{l_\varepsilon[\psi], l_0[\psi], l[\psi]\} \ge c\,\big(\|\partial_1\psi\|^2 + \varepsilon^{-1}\|\psi\|^2\big)\,. \tag{11.22}$$

Here the bound for l_0 follows at once by improving the crude bound (11.16) and recalling (11.18). The bound for l follows from that for l_ε and from (11.21). As for the bound for l_ε, we first remark that the estimates (11.17) actually hold for the part of $\hat{H}_\varepsilon$ associated with $\hat{Q}_\varepsilon^2$. Second, using (11.20) and some elementary estimates, we have $\hat{Q}_\varepsilon^1[\psi] \ge (c - C\varepsilon)\|\partial_1\psi\|^2 - C\|\psi\|^2$. Hence, for ε small enough, we indeed conclude with the bound for l_ε.

The estimates (11.22) imply that, for all sufficiently small ε,

$$\|L_\varepsilon^{-1}\| \le C\varepsilon\,, \qquad \|L_0^{-1}\| \le C\varepsilon\,, \qquad \|L^{-1}\| \le C\varepsilon\,. \tag{11.23}$$

11.3.4 An intermediate convergence result

As the first step in the proof of Theorem 11.3, we show that it is actually enough to establish the norm-resolvent convergence for a simpler operator L instead of L_ε.

Lemma 11.1. *Under the assumptions of Theorem 11.3, there exist positive constants ε_0 and C_0, depending uniquely on k and the supremum norms of κ and κ', such that for all $\varepsilon \in (0, \varepsilon_0)$:*

$$\big\|L_\varepsilon^{-1} - L^{-1}\big\| \ \le\ C_0\,\varepsilon^2\,.$$

Proof. We are inspired by [13, Sec. 3]. Adapting the estimate (11.21) for the sesquilinear form generated by $l_\varepsilon - l$ and using (11.22), we get

$$\begin{aligned} \big|l_\varepsilon(\phi,\psi) - l(\phi,\psi)\big| &\le C\sqrt{\varepsilon\|\partial_1\phi\|^2 + \|\phi\|^2}\,\sqrt{\varepsilon\|\partial_1\psi\|^2 + \|\psi\|^2} \\ &\le (C/c)\,\varepsilon\sqrt{l[\phi]\,l_\varepsilon[\psi]} \end{aligned}$$

for every $\phi, \psi \in \mathcal{Q}$. Choosing $\phi := L^{-1}f$ and $\psi := L_\varepsilon^{-1}g$, where $f, g \in \mathcal{H}_0$ are arbitrary, we arrive at

$$\big|(f, L^{-1}g) - (f, L_\varepsilon^{-1}g)\big| \le (C/c)\,\varepsilon\sqrt{(f, L^{-1}f)(g, L_\varepsilon^{-1}g)} \le (C^2/c)\,\varepsilon^2\,\|f\|\,\|g\|\,.$$

Here $(\cdot,\cdot)$ denotes the inner product in $\mathcal{H}_0$ and the second inequality follows from (11.23). This completes the proof with $C_0 := C^2/c$. □

11.3.5 An orthogonal decomposition of the Hilbert space

Contrary to Lemma 11.1, the convergence of $\|L^{-1} - L_0^{-1}\|$ is less obvious. We follow the idea of [13] and decompose the Hilbert space $\mathcal{H}_0$ into an orthogonal sum

$$\mathcal{H}_0 = \mathfrak{H}_1 \oplus \mathfrak{H}_1^\perp\,,$$

where the subspace $\mathfrak{H}_1$ consists of functions ψ_1 such that

$$\psi_1(s,t) = \varphi_1(s)\chi_1(t)\,. \tag{11.24}$$

Recall that χ_1 has been introduced in (11.11). Since χ_1 is normalized, we clearly have $\|\psi_1\| = \|\varphi_1\|_{L^2(I)}$. Given any $\psi \in \mathcal{H}_0$, we have the decomposition

$$\psi = \psi_1 + \psi_\perp \qquad \text{with} \qquad \psi_1 \in \mathfrak{H}_1,\ \psi_\perp \in \mathfrak{H}_1^\perp\,, \tag{11.25}$$

where ψ_1 has the form (11.24) with $\varphi_1(s) := \int_0^1 \psi(s,t)\chi_1(t)dt$. Note that $\psi_1 \in \mathcal{Q}$ if $\psi \in \mathcal{Q}$. The inclusion $\psi_\perp \in \mathfrak{H}_1^\perp$ means that

$$\int_0^1 \psi_\perp(s,t)\,\chi_1(t)\,dt = 0 \qquad \text{for a.e.} \quad s \in I\,. \tag{11.26}$$

If in addition $\psi_\perp \in \mathcal{Q}$, then one can differentiate the last identity to get

$$\int_0^1 \partial_1\psi_\perp(s,t)\,\chi_1(t)\,dt = 0 \qquad \text{for a.e.} \quad s \in I\,. \tag{11.27}$$

11.3.6 A complementary convergence result

Now we are in a position to prove the following result, which together with Lemma 11.1 establishes Theorem 11.3.

Lemma 11.2. *Under the assumptions of Theorem 11.3, there exist positive constants ε_0 and C_0, depending uniquely on k and the supremum norms of κ and κ', such that for all $\varepsilon \in (0,\varepsilon_0)$:*

$$\big\|L^{-1} - L_0^{-1}\big\| \ \le\ C_0\,\varepsilon^{3/2}\,.$$

Proof. Again, we use some of the ideas of [13, Sec. 3]. As a consequence of (11.26) and (11.27), we get that $l_0(\psi_1,\psi_\perp) = 0$; therefore

$$l_0[\psi] = l_0[\psi_1] + l_0[\psi_\perp] \tag{11.28}$$

for every $\psi \in \mathcal{Q}$. At the same time, for sufficiently small ε,

$$\begin{aligned} l_0[\psi_1] &\ge c\left(\|\varphi_1'\|^2_{L^2(I)} + \varepsilon^{-1}\,\|\varphi_1\|^2_{L^2(I)}\right)\,,\\ l_0[\psi_\perp] &\ge c\Big(\|\partial_1\psi_\perp\|^2 + \varepsilon^{-2}\,\|\partial_2\psi_\perp\|^2 + \varepsilon^{-2}\,\|\partial\psi_\perp\|^2\Big)\,, \end{aligned} \tag{11.29}$$

where the second inequality is based on

$$\int |\partial_2\psi_\perp|^2 \ge \pi^2 \int |\psi_\perp|^2\,. \tag{11.30}$$

Let us now compare l_0 with l. For every $\psi \in \mathcal{Q}$, we define

$$m[\psi] := l[\psi] - l_0[\psi] = \int_\partial v_\varepsilon\,|\psi|^2 - \int \frac{\kappa}{\varepsilon}\,|\psi|^2\,.$$

Using the estimates (11.20) and (11.29), we get, for any $\psi \in \mathcal{Q}$ decomposed as in (11.25),

$$\begin{aligned} m[\psi_1] &= \int_I \frac{\kappa(s)^2}{1-\varepsilon\kappa(s)}\,|\varphi_1(s)|^2 ds \le C\,\|\varphi_1\|^2_{L^2(\mathbb{R})} \le (C/c)\,\varepsilon\,l_0[\psi_1]\,,\\ |m[\psi_\perp]| &\le C\varepsilon^{-1}\Big(\|\psi_\perp\|\|\partial_2\psi_\perp\| + \|\psi_\perp\|^2\Big) \le 2(C/c)\,\varepsilon\,l_0[\psi_\perp]\,,\\ |m(\psi_1,\psi_\perp)| &= \left|\int_\partial v_\varepsilon\,\overline{\psi_1}\psi_\perp\right| \le C\varepsilon^{-1}\|\psi_1\|\sqrt{\|\psi_\perp\|\|\partial_2\psi_\perp\|}\\ &\le (C/c)\,\varepsilon^{1/2}\sqrt{l_0[\psi_1]\,l_0[\psi_\perp]}\,. \end{aligned}$$

Except for $m[\psi_1]$, here the boundary integral was estimated via

$$\int_\partial |\psi|^2 = \int \partial_2|\psi|^2 = \int 2\,\Re\big(\overline{\psi}\partial_2\psi\big) \le 2\,\|\psi\|\,\|\partial_2\psi\|\,. \tag{11.31}$$

Taking (11.28) into account, we conclude with the estimate (which can be again adapted for the corresponding sesquilinear form)

$$|m[\psi]| \leq C\,\varepsilon^{1/2}\; l_0[\psi]$$

valid for every $\psi \in \mathcal{Q}$ and all sufficiently small ε. In particular, this implies the crude estimates

$$c\, l_0[\psi] \leq l[\psi] \leq C\, l_0[\psi]\,.$$

Summing up, we have the crucial bound

$$\big|l(\phi,\psi) - l_0(\phi,\psi)\big| \leq C\,\varepsilon^{1/2}\,\sqrt{l_0[\phi]\,l[\psi]}\,,$$

valid for arbitrary $\phi, \psi \in \mathcal{Q}$. The rest of the proof then follows the lines of the proof of Lemma 11.1. □

11.3.7 Convergence of eigenvalues

As an application of Theorem 11.3, we shall show now how it implies the eigenvalue asymptotics of Theorem 11.1. Recall that the numbers $\lambda_j(H)$ represent either eigenvalues below the essential spectrum or the threshold of the essential spectrum of H. In particular, they provide a complete information about the spectrum of H if it is an operator with compact resolvent. In our situation, this will be the case if I is bounded, but let us stress that we allow infinite or semi-infinite intervals, too.

We begin with analysing the spectrum of the comparison operator.

Lemma 11.3. *Let κ be bounded. One has*

$$\lambda_1(\hat{H}_0) = \lambda_1\big(-\Delta_D^I + \frac{\kappa}{\varepsilon}\big) + \Big(\frac{\pi}{2\varepsilon}\Big)^2\,.$$

Moreover, for any integer $N \geq 2$, there exists a positive constant ε_0 depending on N, κ and I such that for all $\varepsilon < \varepsilon_0$:

$$\forall j \in \{1,\dots,N\}, \qquad \lambda_j(\hat{H}_0) = \lambda_j\big(-\Delta_D^I + \frac{\kappa}{\varepsilon}\big) + \Big(\frac{\pi}{2\varepsilon}\Big)^2\,.$$

Proof. Since $\hat{H}_0$ is decoupled, we know that (*cf* [25, Corol. of Thm. VIII.33])

$$\Big\{\lambda_j(\hat{H}_0)\Big\}_{j=1}^{\infty} = \Big\{\lambda_j\big(-\Delta_D^I + \frac{\kappa}{\varepsilon}\big)\Big\}_{j=1}^{\infty} + \left\{\left(\frac{j\pi}{2\varepsilon}\right)^2\right\}_{j=1}^{\infty}\,,$$

and it only remains to arrange the sum of the numbers on the right hand side into a non-decreasing sequence. The assertion for $N = 1$ is therefore trivial. Let $j \geq 2$ and assume by induction that $\lambda_{j-1}(\hat{H}_0) - \pi^2/(2\varepsilon)^2 = \lambda_{j-1}(-\Delta_D^I + \kappa/\varepsilon)$. Then

$$\lambda_j(\hat{H}_0) - \pi^2/(2\varepsilon)^2 = \min\big\{\lambda_{j-1}(-\Delta_D^I + \kappa/\varepsilon), 3\pi^2/(2\varepsilon)^2\big\}$$

and the assertion of Lemma follows at once due to the asymptotics (11.6). □

Now, fix $j \geq 1$ and assume that ε is so small that the conclusions of Theorem 11.3 and Lemma 11.3 hold. By virtue of Theorem 11.3, we have

$$\left|\left[\lambda_j(\hat{H}_\varepsilon) - \Big(\frac{\pi}{2\varepsilon}\Big)^2 + \frac{k}{\varepsilon}\right]^{-1} - \left[\lambda_j(\hat{H}_0) - \Big(\frac{\pi}{2\varepsilon}\Big)^2 + \frac{k}{\varepsilon}\right]^{-1}\right| \;\leq\; C_0\,\varepsilon^{3/2}\,,$$

since the left hand side is estimated by the norm of the resolvent difference. Using now Lemma 11.3, the above estimate is equivalent to

$$\left|\frac{1}{\varepsilon\big[\lambda_j(\hat{H}_\varepsilon) - \pi^2/(2\varepsilon)^2\big] + k} - \frac{1}{\varepsilon\,\lambda_j(-\Delta_D^I + \kappa/\varepsilon) + k}\right| \;\leq\; C_0\,\varepsilon^{1/2}\,. \tag{11.32}$$

Consequently, recalling (11.6), we conclude with

$$\lim_{\varepsilon\to 0} \varepsilon\left[\lambda_j(\hat{H}_\varepsilon) - \Big(\frac{\pi}{2\varepsilon}\Big)^2\right] = \lim_{\varepsilon\to 0} \varepsilon\,\lambda_j\big(-\Delta_D^I + \frac{\kappa}{\varepsilon}\big) = \inf\kappa\,.$$

This is indeed equivalent to Theorem 11.1 because $\hat{H}_\varepsilon$ and $-\Delta_{DN}^{\Omega_\varepsilon}$ are unitarily equivalent (therefore isospectral).

Remark 11.2. Because of the lack of one half in the power of ε in Theorem 11.3, it turns out that (11.32) yields a slightly worse result than Theorem 11.2. Let us also emphasize that Theorem 11.1 and 11.2 have been proved in Section 11.2 without the need to assume the extra condition (11.15). On the other hand, Theorem 11.3 contains an operator-type convergence result of independent interest.

11.4 Possible extensions

11.4.1 Different boundary conditions

The result of Theorem 11.2 readily extends to the case of other boundary conditions imposed on the sides $\mathcal{L}_\varepsilon\big(\{\inf I\}\times(0,1)\big)$ and $\mathcal{L}_\varepsilon\big(\{\sup I\}\times(0,1)\big)$, provided that the boundary conditions for the one-dimensional Schrödinger operator are changed accordingly.

It is more interesting to impose different boundary conditions on the approaching parallel curves as $\varepsilon\to 0$. As an example, let us keep the Dirichlet boundary conditions but replace the Neumann boundary condition by the Robin condition of the type

$$\frac{\partial\Psi}{\partial n} + (\alpha\circ\ell_\varepsilon^{-1})\,\Psi = 0 \qquad \text{on} \qquad \gamma_\varepsilon(I)\,.$$

Here $\ell_\varepsilon := \mathcal{L}_\varepsilon(\cdot,1)$ and $\alpha : I\to\mathbb{R}$ is assumed to be bounded and uniformly continuous. Let us denote the corresponding Laplacian by $-\Delta^{\Omega_\varepsilon}_{DR_\alpha}$. Then the method of the present paper gives

Theorem 11.4. *For all $j\geq 1$,*

$$\begin{aligned}\lambda_j(-\Delta^{\Omega_\varepsilon}_{DR_\alpha}) &= \Big(\frac{\pi}{2\varepsilon}\Big)^2 + \lambda_j\big(-\Delta^I_D + \frac{\kappa+2\alpha}{\varepsilon}\big) + \mathcal{O}(1)\\ &= \Big(\frac{\pi}{2\varepsilon}\Big)^2 + \frac{\inf(\kappa+2\alpha)}{\varepsilon} + o(\varepsilon^{-1}) \qquad \textit{as} \qquad \varepsilon\to 0\,.\end{aligned}$$

Let us mention that strips with this combination of Dirichlet and Robin boundary conditions were studied in [12].

11.4.2 Higher-dimensional generalization

Let Ω_ε be a three-dimensional layer instead of the planar strip. That is, we keep the definition (11.2) with (11.1), but now $\gamma : I\subseteq\mathbb{R}^2\to\mathbb{R}^3$ is a parametrization of a two-dimensional surface embedded in $\mathbb{R}^3$ and $n := (\partial_1\gamma)\times(\partial_2\gamma)$, where the cross denotes the vector product in $\mathbb{R}^3$. Let M be the corresponding mean curvature. Proceeding in the same way as in Section 11.2 (we omit the details but refer to [7] for a necessary geometric background), we get

Theorem 11.5. *For all $j\geq 1$,*

$$\begin{aligned}\lambda_j(-\Delta^{\Omega_\varepsilon}_{DN}) &= \Big(\frac{\pi}{2\varepsilon}\Big)^2 + \lambda_j\big(-\Delta^\gamma_D + \frac{2M}{\varepsilon}\big) + \mathcal{O}(1)\\ &= \Big(\frac{\pi}{2\varepsilon}\Big)^2 + \frac{2\inf M}{\varepsilon} + o(\varepsilon^{-1}) \qquad \textit{as} \qquad \varepsilon\to 0\,,\end{aligned}$$

where $-\Delta^\gamma_D$ denotes the Laplace-Beltrami operator in $L^2\big(\gamma(I)\big)$, subject to Dirichlet boundary conditions.

Notice that the leading geometric term in the asymptotic expansions depends on the extrinsic curvature only. This suggests that the spectral properties of the Dirichlet-Neumann layers will differ significantly from the purely Dirichlet case studied in [7, 3, 22, 23], where the Gauss curvature of γ plays a crucial role.

11.4.3 Curved ambient space

The results of the present paper extend to the case of strips embedded in an abstract two-dimensional Riemannian manifold $\mathcal{A}$ instead of the Euclidean plane. Indeed, it follows from [19] (see also [20] and [26, Sec. 5]) that the quadratic form Q_ε has the same structure; the only difference is that in this more general situation h_ε is obtained as the solution of the Jacobi equation

$$\partial_2^2 h_\varepsilon + \varepsilon^2\,K\,h_\varepsilon = 0 \qquad \text{with} \qquad \begin{cases} h_\varepsilon(\cdot,0) = 1\,,\\ \partial_2 h_\varepsilon(\cdot,0) = -\varepsilon\,\kappa\,,\end{cases}$$

where K is the Gauss curvature of $\mathcal{A}$. Here κ is the curvature of $\gamma : I\to\mathcal{A}$ (it is in fact the geodesic curvature of γ if the ambient space $\mathcal{A}$ is embedded in $\mathbb{R}^3$). Consequently, up to higher-order terms in ε, the function h_ε coincides with the expression (11.8) for the flat case $K=0$. Namely,

$$h_\varepsilon(s,t) = 1-\kappa(s)\,\varepsilon\,t + \mathcal{O}(\varepsilon^2) \qquad \text{as} \qquad \varepsilon\to 0\,.$$

Following the lines of the proof in Section 11.2, it is then possible to check that Theorems 11.1 and 11.2 remain valid without changes. The curvature of the ambient space comes into the asymptotics via higher-order terms only.

Acknowledgment

The work has been supported by the Czech Academy of Sciences and its Grant Agency within the projects IRP AV0Z10480505 and A100480501, and by the project LC06002 of the Ministry of Education, Youth and Sports of the Czech Republic.

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Errata

The estimates of Section 11.2.3 imply the desired bound (11.14) only for $j = 1$. To get the lower bound also for $j \geq 2$, one has to employ in addition the Hilbert space decomposition of Section 11.3.5. Integrating by parts, we first establish the identity

$$\begin{aligned} Q_\varepsilon[\psi] - \left(\frac{\pi}{2\varepsilon}\right)^2 \|\psi\|_\varepsilon^2 \;&=\; \int \left\{ \frac{|\partial_1\psi|^2}{h_\varepsilon} + \frac{\kappa}{\varepsilon}\,|\psi_1|^2 - \frac{2\kappa}{\varepsilon}\Re(\overline{\psi}_1\partial_2\psi_\perp) + \frac{1}{\varepsilon^2}|\partial_2\psi_\perp|^2 h_\varepsilon - \left(\frac{\pi}{2\varepsilon}\right)^2 |\psi_\perp|^2 h_\varepsilon \right\} \\ &\quad + \int_\partial \frac{2\kappa}{\varepsilon}\Re(\overline{\psi}_1\psi_\perp)\,. \end{aligned}$$

Here the mixed terms can be estimated as follows

$$\begin{aligned} \int \left| \frac{2\kappa}{\varepsilon}\Re(\overline{\psi}_1\partial_2\psi_\perp) \right| &\leq \frac{2\,\|\kappa\|_\infty}{\varepsilon}\,\|\psi_1\|\,\|\partial_2\psi_\perp\| \leq \frac{\|\kappa\|_\infty^2}{\delta}\,\|\psi_1\|^2 + \frac{\delta}{\varepsilon^2}\,\|\partial_2\psi_\perp\|^2\,, \\ \left| \int_\partial \frac{2\kappa}{\varepsilon}\Re(\overline{\psi}_1\psi_\perp) \right| &\leq \frac{2\,\|\kappa\|_\infty}{\varepsilon}\,\|\psi_1\|_\partial\,\|\psi_\perp\|_\partial \leq \frac{\|\kappa\|_\infty^2}{\delta}\,\|\psi_1\|_\partial^2 + \frac{\delta}{\varepsilon^2}\,\|\psi_\perp\|_\partial^2 \\ &\leq \frac{\|\kappa\|_\infty^2}{\delta}\left(\|\psi_1\|^2 + \|\partial_2\psi_1\|^2\right) + \frac{\delta}{\varepsilon^2}\left(\|\psi_\perp\|^2 + \|\partial_2\psi_\perp\|^2\right)\,, \end{aligned}$$

for any $\delta \in (0,1)$, where $\|\cdot\|_\partial$ denotes the norm in $L^2(I\times\{1\})$ which is estimated via (11.31). At the same time, we write

$$\int \frac{\kappa}{\varepsilon}\,|\psi_1|^2 = \int \frac{\kappa}{\varepsilon}\,|\psi|^2 - \int \frac{\kappa}{\varepsilon}\,|\psi_\perp|^2 \geq \int \frac{\kappa}{\varepsilon}\,|\psi|^2 - \frac{\|\kappa\|_\infty}{\varepsilon}\|\psi_\perp\|^2\,.$$

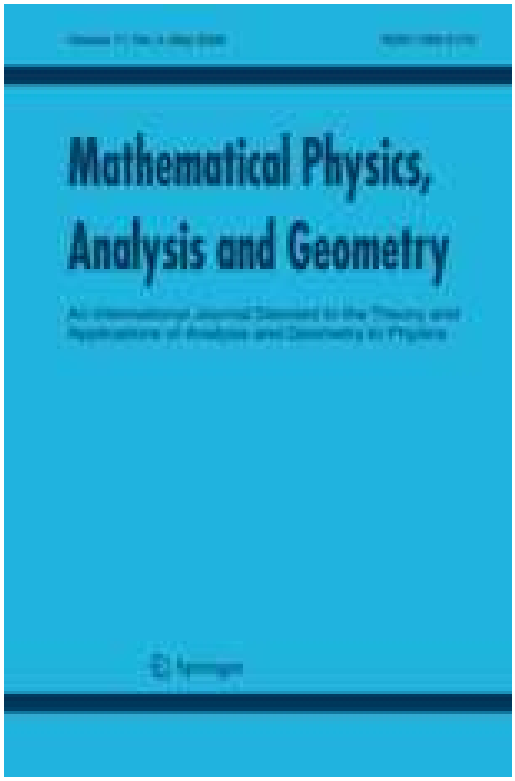
Choosing ε and δ sufficiently small, the above estimates together with (11.9) and (11.30) enable us to conclude that

$$\frac{Q_\varepsilon[\psi]}{\|\psi\|_\varepsilon^2} - \left(\frac{\pi}{2\varepsilon}\right)^2 \;\geq\; \frac{1}{(1+\|\kappa\|_\infty\varepsilon)^2}\,\frac{\int \left\{ |\partial_1\psi|^2 + \frac{\kappa}{\varepsilon}\,|\psi|^2 + \frac{c}{\varepsilon^2}|\psi_\perp|^2 \right\}}{\int |\psi|^2} - C$$

with some constants C and c depending on $\|\kappa\|_\infty$. This estimate enables us to establish (11.14) for every $j \geq 1$ via the minimax principle.

Chapter 12

Waveguides with combined Dirichlet and Robin boundary conditions



Published in: Mathematical Physics, Analysis and Geometry **9** (2006), 335–352

Joint work with: Pedro Freitas

Waveguides with combined Dirichlet and Robin boundary conditions

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Abstract. We consider the Laplacian in a curved two-dimensional strip of constant width squeezed between two curves, subject to Dirichlet boundary conditions on one of the curves and variable Robin boundary conditions on the other. We prove that, for certain types of Robin boundary conditions, the spectral threshold of the Laplacian is estimated from below by the lowest eigenvalue of the Laplacian in a Dirichlet-Robin annulus determined by the geometry of the strip. Moreover, we show that an appropriate combination of the geometric setting and boundary conditions leads to a Hardy-type inequality in infinite strips. As an application, we derive certain stability of the spectrum for the Laplacian in Dirichlet-Neumann strips along a class of curves of sign-changing curvature, improving in this way an initial result of Dittrich and Kříž [9].

12.1 Introduction

The Laplacian in an unbounded tubular region Ω has been extensively studied as a reasonable model for the Hamiltonian of electronic transport in long and thin semiconductor structures called *quantum waveguides*. We refer to [10, 29] for the physical background and references. In this model, it is more natural to consider Dirichlet boundary conditions on $\partial\Omega$ corresponding to a large chemical potential barrier (*cf* [17, 20, 10]).

However, Neumann boundary conditions or a combination of Dirichlet and Neumann boundary conditions have been also investigated. We refer to [27, 28] for the former and to [9, 30, 25] for the latter. Moreover, these types of boundary conditions are relevant to other physical systems (*cf* [13, 8, 21]).

Although we are not aware of any work in the literature where more general boundary conditions have been considered in the case of quantum waveguides, it is possible to think also of Robin boundary conditions as modelling impenetrable walls of Ω in the sense that there is no probability current through the boundary. Furthermore, Robin boundary conditions may in principle be relevant for different types of interphase in a solid.

Moreover, the interplay between boundary conditions, geometry and spectral properties is an interesting mathematical problem in itself. To illustrate this, let us recall that it has been known for more than a decade that the curved geometry of an unbounded planar strip of uniform width may produce eigenvalues below the essential spectrum. We refer to the pioneering work [17] of Exner and Šeba and the sequence of papers [20, 32, 10, 25, 4] for the existence results under rather simple and general geometric conditions.

However, it has not been noticed until the recent letter [9] of Dittrich and Kříž that the existence of eigenvalues in fact depends heavily on the geometrical setting provided the uniform Dirichlet boundary conditions are replaced by a combination of Dirichlet and Neumann ones. In particular, the discrete spectrum may be eliminated provided the Dirichlet-Neumann strip is "curved appropriately", *i.e.*, the Neumann boundary condition is imposed on the "locally shorter" boundary curve.

Recently, it has also been shown that the discrete spectrum may be eliminated by adding a local magnetic field perpendicular to a planar Dirichlet strip [11, 3], by embedding the strip into a curved surface [24] or by twisting a three-dimensional Dirichlet tube of non-circular cross-section [12].

The aim of the present paper is to examine further the interplay between boundary conditions, geometry and spectral properties in the case of Ω being a planar strip with a combination of Dirichlet and (variable) Robin boundary conditions on $\partial\Omega$. Our main result is a lower bound to the spectral threshold of the Laplacian in a (bounded or unbounded) Dirichlet-Robin strip. This enables us to prove quite easily non-existence results about the discrete spectrum for certain waveguides, and generalize in this way the results of Dittrich and Kříž [9]. Moreover, we show that certain combinations of boundary conditions and geometry lead to Hardy-type inequalities for the Laplacian in unbounded strips. These inequalities are new in the theory of quantum waveguides with combined boundary conditions. As an application, we further extend the class of Dirichlet-Neumann strips with empty discrete spectrum.

12.2 Scope of the paper

In this section we precise the problem we deal with in the present paper and state our main results.

12.2.1 The model

Given an open interval $I \subseteq \mathbb{R}$ (bounded or unbounded), let $\Gamma \equiv (\Gamma^1, \Gamma^2) : I \to \mathbb{R}^2$ be a unit-speed C^2-smooth plane curve. We assume that Γ is an embedding. The function $N := (-\dot\Gamma^2, \dot\Gamma^1)$ defines a unit normal vector field along Γ and the couple $(\dot\Gamma, N)$ gives a distinguished Frenet frame (*cf* [42, Chap. 1]). The curvature of Γ is defined through the Serret-Frenet formulae by $\kappa := \det(\dot\Gamma, \ddot\Gamma)$; it is a continuous function of the arc-length parameter. We assume that κ is bounded. It is worth noticing that the curve Γ is fully determined (except for its position and orientation in the plane) by the curvature function κ alone (*cf* [26, Sec. II.20]).

Let a be a given positive number. We define the mapping

$$\mathcal{L} : I \times [-a, a] \to \mathbb{R}^2 : \big\{(s,t) \mapsto \Gamma(s) + N(s)\, t\big\} \tag{12.1}$$

and make the hypotheses that

$$\|\kappa\|_\infty\, a < 1 \qquad \text{and} \qquad \mathcal{L} \quad \text{is injective.} \tag{12.2}$$

Then the image

$$\Omega := \mathcal{L}\big(I \times (-a, a)\big) \tag{12.3}$$

has a geometrical meaning of an open non-self-intersecting strip, contained between the parallel curves

$$\Gamma_\pm := \mathcal{L}(I \times \{\pm a\})$$

at the distance a from Γ, and, if ∂I is not empty, the straight lines $L_- := \mathcal{L}\big(\{\inf I\} \times (-a, a)\big)$ and $L_+ := \mathcal{L}\big(\{\sup I\} \times (-a, a)\big)$. The geometry is set in such a way that $\kappa > 0$ implies that the parallel curve Γ_+ is locally shorter than Γ_-, and vice versa. We refer to [17, App. A] for a sufficient condition ensuring the validity of the second hypothesis in (12.2).

Given a bounded continuous function $\tilde\alpha : \Gamma_+ \to \mathbb{R}$, let $-\Delta_{\kappa,\alpha}$ denote the (non-negative) Laplacian on $L^2(\Omega)$, subject to uniform Dirichlet boundary conditions on the parallel curve Γ_-, uniform Neumann boundary conditions on $L_- \cup L_+$ (*i.e.* none if ∂I is empty) and the Robin boundary conditions of the form

$$\frac{\partial u}{\partial N} + \tilde\alpha\, u = 0 \qquad \text{on} \qquad \Gamma_+\,, \tag{12.4}$$

where $u \in D(-\Delta_{\kappa,\alpha})$. Hereafter we shall rather use $\alpha := \tilde\alpha\big(\mathcal{L}(\cdot, a)\big)$, a function on I. Notice that the choice $\alpha = 0$ corresponds to uniform Neumann boundary conditions on Γ_+ and $\alpha \to +\infty$ approaches uniform Dirichlet boundary conditions on Γ_+; for this reason, we shall sometimes use "$\alpha = +\infty$" to refer to the latter. The Laplacian $-\Delta_{\kappa,\alpha}$ is properly defined in Section 12.3 below by means of a quadratic-form approach.

12.2.2 A lower bound to the spectral threshold

If the curvature κ is a constant function, then the image Ω can be identified with a segment of an annulus or a straight strip. We prove that, in certain situations, this constant geometry minimizes the spectrum of $-\Delta_{\kappa,\alpha}$, within all admissible functions κ and α considered as parameters.

More precisely, let us denote by $\mathcal{D}(r)$ the open disc of radius $r > 0$ and let $\mathcal{A}(r_1, r_2) := \mathcal{D}(r_2) \setminus \overline{\mathcal{D}(r_1)}$ be an annulus of radii $r_2 > r_1 > 0$. Abusing the notation for κ and α slightly, we introduce a function $\lambda : (-a, a) \times \mathbb{R} \to \mathbb{R}$ by means of the following definition:

Definition 12.1. Given two real numbers α and κ, with κ in $(-1/a, 1/a)$, we denote by $\lambda(\kappa, \alpha)$ the spectral threshold of the Laplacian on

$$\mathcal{A}_\kappa := \begin{cases} \mathcal{A}\big(|\kappa|^{-1} - a, |\kappa|^{-1} + a\big) & \text{if} \quad \kappa \neq 0\,, \\ \mathbb{R} \times (-a, a) & \text{if} \quad \kappa = 0\,, \end{cases}$$

subject to uniform Dirichlet boundary condition on

$$\begin{cases} \partial\mathcal{D}(\kappa^{-1} + a) & \text{if} \quad \kappa > 0\,, \\ \mathbb{R} \times \{-a\} & \text{if} \quad \kappa = 0\,, \\ \partial\mathcal{D}(|\kappa|^{-1} - a) & \text{if} \quad \kappa < 0\,, \end{cases}$$

and uniform Robin boundary conditions of the type (12.4) (with α constant and N being the outward unit normal on $\partial\mathcal{A}_\kappa$) on the other connected part of the boundary.

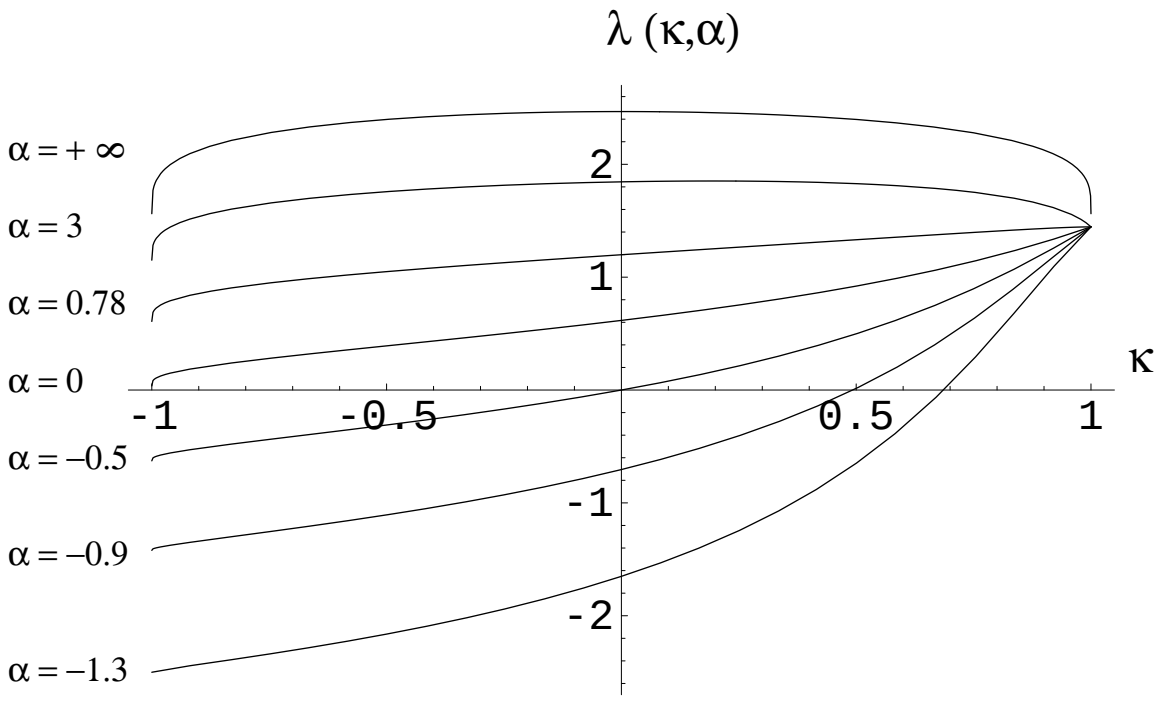

Figure 12.1: Dependence of the lower bound $\lambda(\kappa,\alpha)$ on the curvature κ for $a=1$ and different values of α. (All curves meet at $\kappa = 1/a$, the small gap for the curve with $\alpha=+\infty$ is due to a numerical inaccuracy.)

The most general result of the present paper reads as follows:

Theorem 12.1. *Given a positive number a and a bounded continuous function κ, let Ω be the strip defined by (12.3) with (12.1) and satisfying (12.2). Let α be a bounded continuous function. Then*

$$\inf\sigma(-\Delta_{\kappa,\alpha}) \geq \lambda(\inf\kappa, \inf\alpha) \quad \textit{provided } \kappa\leq 0 \textit{ or } \alpha\leq 0. \tag{12.5}$$

The lower bound $\lambda(\kappa,\alpha)$ as a function of curvature κ for certain values of α is depicted in Figure 12.1. We prove the following properties which are important for (12.5):

Theorem 12.2. *λ satisfies the following properties:*

(i) $\forall\kappa\in(-1/a,1/a), \quad \alpha\mapsto\lambda(\kappa,\alpha):\mathbb{R}\to\mathbb{R}$ *is continuous and increasing,*

(ii) $\forall\alpha\in\mathbb{R}, \quad \kappa\mapsto\lambda(\kappa,\alpha):(-1/a,1/a)\to\mathbb{R}$ *is continuous,*

(iii) $\forall\alpha\in\mathbb{R}, \quad \kappa\mapsto\lambda(\kappa,\alpha):(-1/a,0]\to\mathbb{R}$ *is increasing,*

(iv) $\forall\alpha\in(-\infty,0], \quad \kappa\mapsto\lambda(\kappa,\alpha):(-1/a,1/a)\to\mathbb{R}$ *is increasing,*

(v) $\forall\alpha\in\mathbb{R}, \quad \lim\limits_{\kappa\to-1/a}\lambda(\kappa,\alpha)=\nu(\alpha), \quad \lim\limits_{\kappa\to1/a}\lambda(\kappa,\alpha)=\nu(+\infty),$

where $\nu(\alpha)$, with $\alpha\in\mathbb{R}\cup\{+\infty\}$, denotes the first eigenvalue of the Laplacian in the disc $\mathcal{D}(2a)$, subject to uniform Robin boundary conditions of the type (12.4) if $\alpha\in\mathbb{R}$ (with α constant and N being the outward unit normal on $\partial\mathcal{D}(2a)$) or uniform Dirichlet boundary conditions if $\alpha=+\infty$.

Of course, $\nu(+\infty)=j_{0,1}^2/(2a)^2$, where $j_{0,1}$ denotes the first zero of the Bessel function J_0, and $\nu(0)=0$.

Theorem 12.1 is a natural continuation of efforts to estimate the spectral threshold in curved Dirichlet tubes [2, 14]. More specifically, in the recent article [14], Exner and the present authors establihed a lower bound of the type (12.5) for the case $\alpha=+\infty$, *i.e.* for pure Dirichlet strips (the results in that paper are more general in the sense that the tubes considered there were multi-dimensional and of arbitrary cross-section). Namely,

$$\inf\sigma(-\Delta_{\kappa,+\infty}) \geq \lambda\big(\|\kappa\|_\infty,+\infty\big)\,,$$

where $\lambda(\kappa,+\infty)$ is the spectral threshold of the Dirichlet Laplacian in $\mathcal{A}_\kappa$. It is also established in [14] that $\kappa\mapsto\lambda(\kappa,+\infty)$ is an even function, decreasing on $[0,1/a)$, reaching its maximum $\pi^2/(2a)^2$ for $\kappa=0$ (a straight strip) and approaching its infimum $\nu(+\infty)$ as $\kappa\to1/a$ (a disc). The style and the main idea (*i.e.* the intermediate lower bound (12.14) below) of the present paper are similar to that of [14]. However, we have to use different techniques to establish the properties of λ (Theorem 12.2), and consequently (12.5).

12.2.3 A Hardy inequality in infinite strips

Theorem 12.1 is optimal in the sense that the lower bound (12.5) is achieved by a strip (along a curve of constant curvature). On the other hand, since the minimizer is bounded if the curvature is non-trivial, a better lower bound is expected to hold for unbounded strips. Indeed, in certain unbounded situations, we prove that the lower bound of Theorem 12.1 can be improved by a Hardy-type inequality.

Let us therefore consider the infinite case $I = \mathbb{R}$ in this subsection. Let α_0 be a given real number. If κ is equal to zero identically (*i.e.* Ω is a straight strip) and α is equal to α_0 identically, it is easy to see that

$$\sigma(-\Delta_{0,\alpha_0}) = \sigma_{\rm ess}(-\Delta_{0,\alpha_0}) = \big[\lambda(0,\alpha_0),\infty\big)\,. \tag{12.6}$$

Although the results below hold under more general conditions about vanishing of κ and the difference $\alpha-\alpha_0$ at infinity (*cf* Section 12.7 below), for simplicity, we restrict ourselves to strips which are deformed only locally in the sense that κ and $\alpha-\alpha_0$ have compact support. Under these hypotheses, it is easy to verify that the essential spectrum is preserved:

$$\sigma_{\rm ess}(-\Delta_{\kappa,\alpha}) = \big[\lambda(0,\alpha_0),\infty\big)\,. \tag{12.7}$$

A harder problem is to decide whether this interval exhausts the spectrum of $-\Delta_{\kappa,\alpha}$, or whether there exists discrete eigenvalues below $\lambda(0,\alpha_0)$.

On the one hand, Dittrich and Kříž [9] showed that the curvature which is negative in a suitable sense creates eigenvalues below the threshold $\lambda(0,0)$ in the uniform Dirichlet-Neumann case (*i.e.* in the case $\alpha = 0$ identically). For instance, using, in analogy to [9], a modification of the "generalized eigenfunction" of $-\Delta_{0,\alpha_0}$ corresponding to $\lambda(0,\alpha_0)$ as a test function, it is straightforward to extend a result of [9] to the case of uniform Robin boundary conditions:

Proposition 12.1. *Let $I = \mathbb{R}$. If $\alpha(s) = \alpha_0$ for all $s \in \mathbb{R}$ and $\int_{\mathbb{R}} \kappa(s)\,ds < 0$, then*

$$\inf\sigma(-\Delta_{\kappa,\alpha_0}) < \lambda(0,\alpha_0)\,.$$

In particular, Proposition 12.1 together with (12.7) implies that the discrete spectrum of $-\Delta_{\kappa,\alpha_0}$ exists if the strip is appropriately curved and asymptotically straight. Notice also that the discrete spectrum may be created by variable α even if Ω is straight (*cf* [15, 16] for this type of results in a similar model).

On the other hand, Dittrich and Kříž [9] showed that the spectrum of $-\Delta_{\kappa,0}$ coincides with the interval (12.7) with $\alpha_0 = 0$ provided the curvature κ is non-negative and of compact support. More precisely, they proved that

$$\inf\sigma(-\Delta_{\kappa,0}) \geq \lambda(0,0) \qquad \text{provided} \qquad \kappa \geq 0\,, \tag{12.8}$$

which implies the result in view of (12.7). Of course, not only the lower bound (12.8) is contained in our Theorem 12.1, but the latter also generalizes the former to variable Robin boundary conditions:

Corollary 12.1. *Let $I = \mathbb{R}$ and assume that κ and $\alpha-\alpha_0$ have compact support. Under the hypotheses of Theorem 12.1,*

$$\sigma(-\Delta_{\kappa,\alpha}) = \sigma_{\rm ess}(-\Delta_{\kappa,\alpha}) = \big[\lambda(0,\alpha_0),\infty\big) \qquad \textit{if} \qquad \kappa \geq 0\,, \quad \alpha_0 \leq \alpha \leq 0\,.$$

Apart from this generalization, Theorem 12.1 provides an alternative and, we believe, more elegant, proof of (12.8). Indeed, the proof of Dittrich and Kříž in [9] is very technical, based on a decomposition of $-\Delta_{\kappa,0}$ into an orthonormal basis and an analysis of solutions of Bessel type to an associated ordinary differential operator, while the proof of Theorem 12.1 does not require any explicit solutions whatsoever.

Furthermore, we obtain a stronger result, namely, that a Hardy-type inequality actually holds true in positively curved Dirichlet-Robin strips:

Theorem 12.3. *Let $I = \mathbb{R}$. Given a positive number a and a bounded continuous function κ, let Ω be the strip defined by (12.3) with (12.1) and satisfying (12.2). Let α be a bounded continuous function such that $\alpha_0 \leq \alpha \leq 0$. Assume that κ is non-negative and that either one of κ or $\alpha-\alpha_0$ is not identically equal to zero. Then, for any s_0 such that $\kappa(s_0) > 0$ or $\alpha(s_0) > \alpha_0$, we have*

$$-\Delta_{\kappa,\alpha} \;\geq\; \lambda(0,\alpha_0) + \frac{c}{(\rho\circ\mathcal{L}^{-1})^2} \tag{12.9}$$

*in the sense of quadratic forms (*cf *(12.30) below). Here c is a positive constant which depends on s_0, a, κ and α, $\rho(s,t) := \sqrt{1+(s-s_0)^2}$ and $\mathcal{L}$ is given by (12.1).*

It is possible to find an explicit lower bound for the constant c; we give an estimate in (12.29) below.

Theorem 12.3 implies that the presence of a positive curvature or of suitable Robin boundary conditions represents a repulsive interaction in the sense that there is no spectrum below $\lambda(0,\alpha_0)$ for all small potential-type perturbations having a sufficiently fast decay at infinity. This provides certain stability of the spectrum of the type established in Corollary 12.1.

Moreover, in the uniform Dirichlet-Neumann case, we use Theorem 12.3 to show that the spectrum is stable even if κ is allowed to be negative:

Corollary 12.2. *Given a positive number a and a bounded continuous function κ of compact support, let Ω be the strip defined by (12.3) with (12.1) and satisfying (12.2). Assume that*

$$|\kappa_-| \le \varepsilon \qquad \text{with} \qquad \varepsilon \ge 0\,,$$

while κ_+ is independent of ε and not identically equal to zero. Then there exists a positive number ε_0 such that for all $\varepsilon \le \varepsilon_0$ we have

$$\sigma(-\Delta_{\kappa,0}) = \sigma_{\mathrm{ess}}(-\Delta_{\kappa,0}) = \big[\lambda(0,0),\infty\big)\,.$$

Here ε_0 depends on a, κ_+ and I.

Corollary 12.2 follows as a consequence of (12.7) and the Hardy inequality (12.32) below. This generalizes a result of [9] to strips with sign-changing curvature.

12.2.4 Contents

The present paper is organized as follows.

In the following section we introduce the Laplacian $-\Delta_{\kappa,\alpha}$ in the curved strip Ω by means of its associated quadratic form and express it in curvilinear coordinates defined by (12.1). We obtain in this way an operator of the Laplace-Beltrami form in the straight strip $I\times(-a,a)$.

In Section 12.4 we show that the structure of the Laplace-Beltrami operator leads quite easily to a "variable" lower bound (12.14), expressed in terms of the function λ of Definition 12.1. We call this lower bound "intermediate" since this and Theorem 12.2 imply Theorem 12.1 at once.

In Section 12.5 we prove Theorem 12.2 using a combination of a number of techniques, such as the minimax principle, the maximum principle, perturbation theory, *etc.*

Section 12.6 is devoted to infinite strips, namely, to the proofs of Theorem 12.3 and its Corollary 12.2. The former is based on an improved intermediate lower bound, Theorem 12.2 and the classical one-dimensional Hardy inequality.

In the closing section we discuss possible extensions and refer to some open problems.

12.3 The Laplacian

The Laplacian $-\Delta_{\kappa,\alpha}$ is properly defined as follows. We introduce on the Hilbert space $L^2(\Omega)$ the quadratic form $Q_{\kappa,\alpha}$ defined by

$$\begin{aligned} Q_{\kappa,\alpha}[u] &:= \int_\Omega |\nabla u(x)|^2\,dx + \int_{\Gamma_+} \tilde\alpha(\sigma)\,|u(\sigma)|^2\,d\sigma\,, \\ u\in D(Q_{\kappa,\alpha}) &:= \big\{u\in W^{1,2}(\Omega)\,\big|\; u(\sigma)=0 \quad \text{for a.e. } \sigma\in\Gamma_-\big\}\,, \end{aligned} \tag{12.10}$$

where $u(\sigma)$ with $\sigma\in\Gamma_+\cup\Gamma_-$ is understood as the trace of the function u on that part of the boundary $\partial\Omega$ (*cf* Remark 12.1 below). The associated sesquilinear form is symmetric, densely defined, closed and bounded from below (the latter is not obvious unless $\tilde\alpha\ge 0$, but it follows from the results (12.14) and (12.17) below). Consequently, $Q_{\kappa,\alpha}$ gives rise (*cf* [40, Sec. VI.2]) to a unique self-adjoint bounded-from-below operator which we denote by $-\Delta_{\kappa,\alpha}$. It can be verified that $-\Delta_{\kappa,\alpha}$ acts as the classical Laplacian with the boundary conditions described in Section 12.1 provided Γ is sufficiently regular.

It follows from assumptions (12.2) that $\mathfrak{L}: I\times(-a,a)\to\Omega : \{(s,t)\mapsto\mathcal{L}(s,t)\}$ is a C^1-diffeomorphism. Consequently, Ω can be identified with the Riemannian manifold $I\times(-a,a)$ equipped with the metric $G_{ij} := (\partial_i\mathcal{L})\cdot(\partial_j\mathcal{L})$, where $i,j\in\{1,2\}$ and the *dot* denotes the scalar product in $\mathbb{R}^2$. Employing the Frenet formula $\dot N = -\kappa\dot\Gamma$, one easily finds that $(G_{ij}) = \mathrm{diag}(g_\kappa^2,1)$, where

$$g_\kappa(s,t) := 1-\kappa(s)\,t \tag{12.11}$$

is the Jacobian of $\mathfrak{L}$.

It follows that $g_\kappa(s,t)\,ds\,dt$ is the area element of the strip, $L^2(\Omega)$ can be identified with the Hilbert space

$$L^2\big(I\times(-a,a), g_\kappa(s,t)\,ds\,dt\big) \tag{12.12}$$

and $-\Delta_{\kappa,\alpha}$ is unitarily equivalent to the operator $H_{\kappa,\alpha}$ on (12.12) associated with the quadratic form

$$\begin{aligned} h_{\kappa,\alpha}[\psi] &:= \big\|g_\kappa^{-1}\partial_1\psi\big\|_\kappa^2 + \big\|\partial_2\psi\big\|_\kappa^2 + \int_{\mathbb{R}}\alpha(s)\,|\psi(s,a)|^2\,g_\kappa(s,a)\,ds\,, \\ \psi\in D(h_{\kappa,\alpha}) &:= \big\{\psi\in W^{1,2}\big(\mathbb{R}\times(-a,a)\big)\,\big|\;\psi(s,-a)=0 \quad \text{for a.e. } s\in\mathbb{R}\big\}\,. \end{aligned} \tag{12.13}$$

Here $\|\cdot\|_\kappa$ stands for the norm in (12.12) and $\psi(s,\pm a)$ means the trace of the function ψ on the part of the boundary $I\times\{\pm a\}$ (*cf* Remark 12.1 below). In fact, if the curve Γ is sufficiently smooth, then $H_{\kappa,\alpha}$ acts as the Laplace-Beltrami operator $-G^{-1/2}\partial_i G^{1/2}G^{ij}\partial_j$, where $(G^{ij})=(G_{ij})^{-1}$ and $G:=\det(G_{ij})$, but we will not use this fact. Finally, let us notice that the first assumption of (12.2) yields

$$0<1-\|\kappa\|_\infty\,a\le g_\kappa(s,t)\le 1+\|\kappa\|_\infty\,a<2$$

uniformly in $(s,t)\in I\times(-a,a)$, and that is actually why we can indeed write $W^{1,2}\big(I\times(-a,a)\big)$ instead of $W^{1,2}\big(I\times(-a,a), g_\kappa(s,t)\,ds\,dt\big)$ in (12.13).

Remark 12.1. The traces of $\psi\in W^{1,2}\big(I\times(-a,a)\big)$ on the boundary of the strip $I\times(-a,a)$ are well defined and square integrable (*cf* [1]). In particular, the boundary integral appearing in (12.13) is finite (recall that α is assumed to be bounded). To ensure that the traces and the boundary integral appearing in (12.10) are well defined too, it is sufficient to notice that one can construct traces of $u\in W^{1,2}(\Omega)$ to $\Gamma_+\cup\Gamma_-$ by means of the diffeomorphism $\mathfrak{L}$, the trace operator for the straight strip $I\times(-a,a)$ and inverses of the boundary mappings $\mathcal{L}(\cdot,\{\pm a\})$. The latter exists due to the second hypothesis in (12.2), which is in fact a bit stronger than an analogous assumption in the uniform Dirichlet case [10, 14] (there it is enough to assume that $\mathcal{L}\restriction I\times(-a,a)$ is injective). In this context, one should point out that the approach used by Daners in [6] makes it possible to deal with Robin boundary conditions with positive α on arbitrary bounded domains, without using traces.

12.4 An intermediate lower bound

In this section, we derive the central lower bound of the present paper, *i.e.* inequality (12.14) below, and explain its connection with Definition 12.1.

Neglecting in (12.13) the "longitudinal kinetic energy", *i.e.* the term $\|g_\kappa^{-1}\partial_1\psi\|_\kappa$ in the expression for $h_{\kappa,\alpha}[\psi]$, and using Fubini's theorem, one immediately gets

$$\inf\sigma(H_{\kappa,\alpha})\ge\inf_{s\in I}\lambda\big(\kappa(s),\alpha(s)\big)\,, \tag{12.14}$$

where $\lambda(\kappa,\alpha)$ denotes the first eigenvalue of the self-adjoint one-dimensional operator $B_{\kappa,\alpha}$ on

$$\mathcal{H}_\kappa := L^2\big((-a,a),(1-\kappa\,t)dt\big)$$

associated with the quadratic form

$$\begin{aligned} b_{\kappa,\alpha}[\psi] &:= \int_{-a}^{a}|\psi'(t)|^2\,(1-\kappa\,t)dt+\alpha\,|\psi(a)|^2\,(1-\kappa\,a)\,, \\ \psi\in D(b_{\kappa,\alpha}) &:= \big\{\psi\in W^{1,2}((-a,a))\;\big|\;\psi(-a)=0\big\}\,. \end{aligned} \tag{12.15}$$

With a slight abuse of notation, we denote by $\kappa\in(-1/a,1/a)$ and $\alpha\in\mathbb{R}$ given constants now. One easily verifies that

$$\begin{aligned} (B_{\kappa,\alpha}\psi)(t) &= -\psi''(t)+\frac{\kappa}{1-\kappa\,t}\,\psi'(t)\,, \\ \psi\in D(B_{\kappa,\alpha}) &= \big\{\psi\in W_0^{1,2}((-a,a))\;\big|\;\psi(-a)=0\;\;\&\;\;\psi'(a)+\alpha\,\psi(a)=0\big\}\,. \end{aligned} \tag{12.16}$$

Note that the values of ψ and ψ' at the boundary points of $(-a,a)$ are well defined due to the Sobolev embedding theorem.

$B_{\kappa,\alpha}$ is clearly a positive operator for $\alpha\ge 0$. Furthermore, using the elementary inequality $|\psi(a)|^2\le\varepsilon\int_{-a}^{a}|\psi'(t)|^2dt+\varepsilon^{-1}\int_{-a}^{a}|\psi(t)|^2dt$ with $\varepsilon>0$, it can be easily shown that

$$\lambda(\kappa,\alpha)\ge-\alpha^2\,\frac{(1+|\kappa|\,a)^2}{(1-|\kappa|\,a)^2}\,, \tag{12.17}$$

i.e., $B_{\kappa,\alpha}$ is bounded from below in any case. This and (12.14) prove that $H_{\kappa,\alpha}$ (and therefore $-\Delta_{\kappa,\alpha}$) is bounded from below *a fortiori*.

Using coordinates analogous to (12.1) and the circular (respectively straight) symmetry, it is easy to see that $B_{\kappa,\alpha}$ is nothing else than the "radial" (respectively "transversal") part of the Laplacian on $L^2(\mathcal{A}_\kappa)$ if $\kappa \neq 0$ (respectively $\kappa = 0$) in Definition 12.1. (We refer to [14, Lemma 4.1] for more details on the partial wave decomposition in the case $\alpha = +\infty$.) This shows that the geometric Definition 12.1 of λ and the definition via (12.15) are in fact equivalent.

In view of (12.14), it remains to establish the monotonicity properties of λ stated in Theorem 12.2 in order to prove Theorem 12.1. This will be done in the next section.

12.5 Dirichlet-Robin annuli

Using standard arguments (*cf* [29, Sec. 8.12]), one easily shows that $\lambda(\kappa,\alpha)$, as the lowest eigenvalue of $B_{\kappa,\alpha}$, is simple and has a positive eigenfunction. We denote the latter by $\psi_{\kappa,\alpha}$ and normalize it to have unit norm in the Hilbert space $\mathcal{H}_\kappa$.

12.5.1 Dependence on α

The first property of Theorem 12.2 follows directly from the variational definition of $\lambda(\kappa,\alpha)$. In detail, using $\psi_{\kappa,\alpha+\delta}$ with any $\delta > 0$ as a test function for $\lambda(\kappa,\alpha)$, we get

$$\lambda(\kappa,\alpha) \leq \lambda(\kappa,\alpha+\delta) - \delta\,\psi_{\kappa,\alpha+\delta}(a)^2\,(1-\kappa\,a) < \lambda(\kappa,\alpha+\delta)\,, \tag{12.18}$$

i.e. $\alpha \mapsto \lambda(\kappa,\alpha)$ is increasing. Note that the *strict* monotonicity is a consequence of the fact that $\psi_{\kappa,\alpha+\delta} \in D(B_{\kappa,\alpha+\delta})$; indeed, $\psi_{\kappa,\alpha+\delta}(a) = 0$ would imply that $\psi'_{\kappa,\alpha+\delta}(a) = 0$ also, giving a contradiction. Using now $\psi_{\kappa,\alpha}$ as a test function for $\lambda(\kappa,\alpha+\delta)$, we get

$$\lambda(\kappa,\alpha+\delta) \leq \lambda(\kappa,\alpha) + \delta\,\psi_{\kappa,\alpha}(a)^2\,(1-\kappa\,a) \xrightarrow[\delta\to 0]{} \lambda(\kappa,\alpha)\,, \tag{12.19}$$

which, together with (12.18), gives the continuity of λ in the second variable.

12.5.2 Dependence on κ

Not all of the other properties of Theorem 12.2 are so obvious from the variational definition of $\lambda(\kappa,\alpha)$ via $B_{\kappa,\alpha}$ because the Hilbert space $\mathcal{H}_\kappa$ depends on κ. To overcome this, we introduce the unitary transformation

$$U_\kappa : \mathcal{H}_\kappa \to \mathcal{H}_0 : \ \big\{\psi \mapsto (1-\kappa\,t)^{\frac{1}{2}}\,\psi\big\} \tag{12.20}$$

and the unitarily equivalent operator $\hat{B}_{\kappa,\alpha} := U_\kappa B_{\kappa,\alpha} U_\kappa^{-1}$ associated with the transformed form $\hat{b}_{\kappa,\alpha}[\cdot] := b_{\kappa,\alpha}[U_\kappa^{-1}\cdot]$. Given any $\phi \in D(\hat{b}_{\kappa,\alpha})$, we insert $\psi = U_\kappa^{-1}\phi$ into (12.15), integrate by parts and finds

$$\hat{b}_{\kappa,\alpha}[\phi] = \int_{-a}^{a} |\phi'(t)|^2 dt - \int_{-a}^{a} \frac{\kappa^2}{4(1-\kappa\,t)^2}\,|\phi(t)|^2 dt + \left(\alpha + \frac{\kappa}{2(1-\kappa\,a)}\right)|\phi(a)|^2\,. \tag{12.21}$$

We also verify that

$$\begin{aligned} (\hat{B}_{\kappa,\alpha}\phi)(t) &= -\phi''(t) - \frac{\kappa^2}{4(1-\kappa\,t)^2}\,\phi(t)\,, \\ \phi \in D(\hat{B}_{\kappa,\alpha}) &= \Big\{\phi \in W_0^{1,2}((-a,a)) \,\Big|\, \phi(-a) = 0 \\ &\qquad\qquad \&\ \phi'(a) + \left(\alpha + \tfrac{\kappa}{2(1-\kappa\,a)}\right)\phi(a) = 0\Big\}\,. \end{aligned} \tag{12.22}$$

It is important to notice that while $D(B_{\kappa,\alpha})$ is not invariant under U_κ, one still has $D(\hat{b}_{\kappa,\alpha}) = D(b_{\kappa,\alpha})$.

Continuity

Following [40, Sec. VII. 4], $\kappa \mapsto \hat{b}_{\kappa,\alpha}$ forms a holomorphic family of forms of type (a) and $\kappa \mapsto \hat{B}_{\kappa,\alpha}$ forms a self-adjoint holomorphic family of operators of type (B). In particular, $\kappa \mapsto \lambda(\kappa,\alpha)$ is continous, which proves (ii) of Theorem 12.2. Moreover, denoting by $\phi_{\kappa,\alpha} := U_\kappa\psi_{\kappa,\alpha}$ the eigenfunction of $\hat{B}_{\kappa,\alpha}$ corresponding to $\lambda(\kappa,\alpha)$, we get that $\kappa \mapsto \phi_{\kappa,\alpha}$ is continous in the norm of $\mathcal{H}_0$. In view of (12.20), it then follows that also $\kappa \mapsto \psi_{\kappa,\alpha}$ is continuous in the norm of $\mathcal{H}_0$.

Monotonicity

Since the function $f : \kappa \mapsto \frac{\kappa}{1-\kappa t}$ is increasing on $(-1/a, 1/a)$ for any $t \in [-a, a]$, one easily verifies Theorem 12.2.(iii) by means of the variational definition of $\lambda(\kappa, \alpha)$ via $\hat{B}_{\kappa,\alpha}$ and an argument similar to that used in Section 12.5.1.

However, the above argument fails to prove (iv) of Theorem 12.2 because $-f^2$ is decreasing on $[0, 1/a)$, so that one gets an interplay between the increasing boundary term and decreasing potential in (12.21) for positive curvatures. Therefore we come back to the initial operator (12.16) and calculate the derivative of $\kappa \mapsto \lambda(\kappa, \alpha)$:

Lemma 12.1. $\forall \kappa \in (-1/a, 1/a),\ \forall \alpha \in \mathbb{R}$,

$$\frac{\partial \lambda}{\partial \kappa}(\kappa, \alpha) = \int_{-a}^{a} \frac{\psi_{\kappa,\alpha}(t)\, \psi'_{\kappa,\alpha}(t)}{1-\kappa t}\, dt\,. \tag{12.23}$$

Proof. Throughout this proof, we omit the dependence of λ and the corresponding eigenfunction on α.

We write the eigenvalue equation for $B_{\kappa,\alpha}$ with ψ_κ and $\lambda(\kappa)$ as

$$-\big[\psi'_\kappa(t)\,(1-\kappa t)\big]' = \lambda(\kappa)\,\psi_\kappa(t)\,(1-\kappa t) \tag{12.24}$$

and consider the analogous equation at $\kappa+\delta$, with $\delta \in \mathbb{R}\setminus\{0\}$ so small that $|\kappa+\delta|\,a$ is less than 1. Multiplying (12.24) by $\psi_{\kappa+\delta}$, integrating by parts, combining the result with the result coming from analogous manipulations applied to the problem at $\kappa+\delta$, dividing by δ, integrating by parts once more and using the eigenvalue equation for $B_{\kappa,\alpha}$, we arrive at

$$\begin{aligned}
&\frac{\lambda(\kappa+\delta)-\lambda(\kappa)}{\delta} \int_{-a}^{a} \psi_\kappa(t)\,\psi_{\kappa+\delta}(t)\,(1-\kappa t)\,dt \\
&= \lambda(\kappa+\delta) \int_{-a}^{a} \psi_\kappa(t)\,\psi_{\kappa+\delta}(t)\,t\,dt \\
&\quad - \int_{-a}^{a} \psi'_\kappa(t)\,\psi'_{\kappa+\delta}(t)\,t\,dt - \alpha\, a\, \psi_\kappa(a)\,\psi_{\kappa+\delta}(a) \\
&= \big[\lambda(\kappa+\delta)-\lambda(\kappa)\big] \int_{-a}^{a} \psi_\kappa(t)\,\psi_{\kappa+\delta}(t)\,t\,dt + \int_{-a}^{a} \frac{\psi'_\kappa(t)\,\psi_{\kappa+\delta}(t)}{1-\kappa t}\,dt\,.
\end{aligned}$$

Letting δ go to zero yields the desired result by means of the continuity of $\kappa \mapsto \lambda(\kappa)$ and $\kappa \mapsto \psi_\kappa$ established in Section 12.5.2. □

Lemma 12.1 yields (iv) of Theorem 12.2 whenever the integral on the right hand side of (12.23) is positive. In particular, this is the case when $\psi'_{\kappa,\alpha}$ is non-negative:

Lemma 12.2. $\forall \kappa \in (-1/a, 1/a),\ \forall \alpha \in (-\infty, 0]$,

$$t \mapsto \psi_{\kappa,\alpha}(t) : (-a, a) \to \mathbb{R} \quad \textit{is increasing.}$$

Proof. Throughout this proof, we omit the dependence of λ and the corresponding eigenfunction on κ and α.

Since ψ is a positive eigenfunction and $\psi(-a) = 0$, respectively $\psi'(a) = -\alpha\psi(a)$, we know that $\psi'(-a) > 0$, respectively $\psi'(a) \geq 0$. Recall also that $\psi(a) > 0$. We claim that $\psi' > 0$ on $(-a, a)$.

Case $\lambda < 0$. The eigenvalue problem for (12.16) implies that if $\psi'(t) = 0$ for some $t \in (-a, a)$, then $\psi''(t) > 0$, *i.e.* ψ has a local minimum at t. Consequently, if there exists a $t_1 \in (-a, a)$ such that $\psi'(t_1) = 0$, then, since $\psi'(-a) > 0$, there must also be a $t_2 \in (-a, t_1)$ such that ψ has a local maximum at t_2, a contradiction.

Case $\lambda > 0$. The eigenvalue problem for (12.16) implies that if $\psi'(t) = 0$ for some $t \in (-a, a]$, then $\psi''(t) < 0$, *i.e.* ψ has a local maximum at t. Consequently, if there exists a $t_1 \in (-a, a)$ such that $\psi'(t_1) = 0$, then, since $\psi'(a) \geq 0$, there must also be a $t_2 \in (t_1, a]$ such that $\psi'(t_2) = 0$ and $\psi' < 0$ on (t_1, t_2), *i.e.* ψ does not have a local maximum at t_2, a contradiction.

Case $\lambda = 0$. Integrating (12.24), we get $\psi'(t) = -\alpha\,\frac{1-\kappa a}{1-\kappa t}\,\psi(a) > 0$ for all $t \in [-a, a]$ (the equality would imply a trivial eigenfunction). □

Boundary values

Using the geometrical meaning of $\lambda(\kappa, \alpha)$ (*cf* Definition 12.1) and since $\mathcal{A}_\kappa$ converges (*e.g.*, in the sense of metrical convergence [31]) to the disc $\mathcal{D}(2a)$ with the central point removed as $|\kappa| \to 1/a$, the limits in Theorem 12.2.(v) are natural to expect. We prove each of them separately.

The negative limit The limit value for $\lambda(\kappa,\alpha)$ as $\varepsilon := -(\kappa^{-1}+a) \to 0$ follows from Flucher's paper [18], where an approximation formula for eigenvalues in domains with spherical holes is found. The only difference is the fact that in our case the boundary of the domain also changes as ε goes to zero. We overcome this complication by transforming the eigenvalue problem for the Laplacian on $\mathcal{A}_\kappa$ into

$$\begin{cases} -\Delta u = \lambda_\varepsilon(\alpha_\varepsilon) u & \text{in} \quad \mathcal{A}\big(\varepsilon(2a+\varepsilon)^{-1},1\big)\,, \\ u = 0 & \text{on} \quad \partial\mathcal{D}\big(\varepsilon(2a+\varepsilon)^{-1}\big)\,, \\ \dfrac{\partial u}{\partial N} + \alpha_\varepsilon\, u = 0 & \text{on} \quad \partial\mathcal{D}(1)\,, \end{cases} \tag{12.25}$$

where $\lambda_\varepsilon(\alpha_\varepsilon) := (2a+\varepsilon)^2\lambda(-(a+\varepsilon)^{-1},\alpha)$, $\alpha_\varepsilon := (2a+\varepsilon)\alpha$ and N is the outward unit normal on $\partial\mathcal{D}(1)$. By the minimax principle,

$$\lambda_\varepsilon\big(\alpha_{-(\operatorname{sgn}\alpha)\varepsilon_0}\big) \le \lambda_\varepsilon(\alpha_\varepsilon) \le \lambda_\varepsilon\big(\alpha_{(\operatorname{sgn}\alpha)\varepsilon_0}\big)$$

for any fixed $\varepsilon_0 \in (\varepsilon, 2a)$, where $\lambda_\varepsilon(\alpha_{\pm\varepsilon_0})$ denotes the eigenvalue of the problem (12.25) with α_ε being replaced by $\alpha_{\pm\varepsilon_0}$. Then it is clear that $\lambda_\varepsilon(\alpha_\varepsilon) \to (2a)^2\nu(\alpha)$ as $\varepsilon \to 0$ because it is true for $\lambda_\varepsilon(\alpha_{\pm\varepsilon_0})$ by [18] and ε_0 can be chosen arbitrarily small.

The positive limit If $\alpha > 0$, the limit value for $\lambda(\kappa,\alpha)$ as $\kappa^{-1} \to a$ could be derived by means of a paper by Dancer and Daners, [5], where they study domain perturbations for elliptic equations subject to Robin boundary condtions. However, since they restrict to positive α and we do not know about a similar perturbation result for $\alpha < 0$, we establish the limit value by rather elementary considerations.

Assuming $\kappa \neq 0$, the eigenvalue problem for $B_{\kappa,\alpha}$ is explicitly solvable in terms of the Bessel functions J_0 and Y_0 (*cf* [34, Chap. 7]) and the eigenvalue $\lambda(\kappa,\alpha)$ is then determined as the smallest (in absolute value) zero λ of the implicit equation

$$\begin{aligned} J_0\big(\sqrt{\lambda}(1+\kappa a)/\kappa\big) \Big[\sqrt{\lambda} Y_1\big(\sqrt{\lambda}(1-\kappa a)/\kappa\big) + \alpha Y_0\big(\sqrt{\lambda}(1-\kappa a)/\kappa\big)\Big] \\ = Y_0\big(\sqrt{\lambda}(1+\kappa a)/\kappa\big) \Big[\sqrt{\lambda} J_1\big(\sqrt{\lambda}(1-\kappa a)/\kappa\big) + \alpha J_0\big(\sqrt{\lambda}(1-\kappa a)/\kappa\big)\Big]\,. \end{aligned} \tag{12.26}$$

Although the case $\lambda(\kappa,\alpha) = 0$ should be treated separately, a formal asymptotic expansion of (12.26) around $\sqrt{\lambda} = 0$ also gives the correct condition for a zero eigenvalue:

$$\lambda(\kappa,\alpha) = 0 \quad \Longleftrightarrow \quad \kappa = \alpha\,(1-\kappa\,a) \log\frac{1-\kappa\,a}{1+\kappa\,a}\,. \tag{12.27}$$

In particular, the condition yields that for any $\alpha < -1/(2a)$ there always exists $\kappa_0 \in (0, 1/a)$ such that $\lambda(\kappa_0,\alpha) = 0$. This and the properties (i), (ii) and (iv) of Theorem 12.1 imply that $\lim_{\kappa\to 1/a} \lambda(\kappa,\alpha) > 0$ for any $\alpha \in \mathbb{R}$. We also know that the limit is bounded because $\lambda(\kappa,\alpha) < \lambda(\kappa,+\infty)$ by the minimax principle and $\lambda(\kappa,+\infty)$ tends to the first eigenvalue of the Dirichlet Laplacian in the disc $\mathcal{D}(2a)$, *i.e.* $\nu(+\infty) \equiv j_{0,1}^2/(2a)^2$, as $\kappa \to 1/a$ by known convergence theorems (*cf* one of [31, 33, 7]). Applying the limit to (12.26), we get a bounded value on the right hand side, while the left hand side admits the asymptotic expansion $-\frac{\sqrt{\lambda}}{\pi}\, J_0(\sqrt{\lambda}\,2a)\,\big[\sqrt{\lambda}\,(1-\kappa\,a)/(2\kappa)\big]^{-1} + \mathcal{O}(\kappa^{-1}-a)$. That is, $\sqrt{\lambda}\,2a$ necessarily converges to the first zero of the Bessel function J_0 as $\kappa \to 1/a$.

12.6 Infinite strips

Let $I = \mathbb{R}$ throughout this section. The proof of Theorem 12.3 is based on the following two lemmata.

Firstly, Theorem 12.2 implies:

Lemma 12.3. *Assume the hypotheses of Theorem 12.3. Then the function $\mu : \mathbb{R} \to \mathbb{R}$ defined by*

$$s \;\mapsto\; \mu(s) := \lambda\big(\kappa(s),\alpha(s)\big) - \lambda(0,\alpha_0)$$

is continuous, non-zero and non-negative.

Hereafter we shall use the same notation μ for the function $\mu \otimes 1$ on $\mathbb{R}\times(-a,a)$.

Secondly, we shall need the following Hardy-type inequality for a Schrödinger operator in a strip with the potential being a characteristic function:

Lemma 12.4. *For any* $\psi \in W^{1,2}\big(\mathbb{R}\times(-a,a)\big)$,

$$\int_{\mathbb{R}\times(-a,a)} \rho^{-2}\,|\psi|^2 \;\leq\; 16\int_{\mathbb{R}\times(-a,a)} |\partial_1\psi|^2 + \big(2+64/|J|^2\big)\int_{J\times(-a,a)} |\psi|^2\,,$$

where $\rho(s,t) := \sqrt{1+(s-s_0)^2}$, *$J$ is any bounded subinterval of $\mathbb{R}$ and s_0 is the mid-point of J.*

This lemma can be established quite easily by means of the classical one-dimensional Hardy inequality

$$\int_{\mathbb{R}} x^{-2}|v(x)|^2\,dx \leq 4\int_{\mathbb{R}} |v'(x)|^2\,dx$$

valid for any $v\in W^{1,2}(\mathbb{R})$ with $v(0)=0$ and Fubini's theorem; we refer the reader to [17, Sec. 3.3] or [46, proof of Lem. 2] for more details.

12.6.1 Proof of Theorem 12.3

Let ψ belong to the dense subspace of $D(h_{\kappa,\alpha})$ given by C^∞-smooth functions on $\mathbb{R}\times(-a,a)$ which vanish in a neighbourhood of $\mathbb{R}\times\{-a\}$ and which are restrictions of functions from $C_0^\infty(\mathbb{R}^2)$. Assume the hypotheses of Theorem 12.3 so that the conclusions of Lemma 12.3 hold. Let J be any closed subinterval of $\mathbb{R}$ on which μ defined in Lemma 12.3 is positive.

The first step is to come back to the intermediate lower bound (12.14); we also use the definition of λ via (12.15), but we do not neglect the "longitudinal kinetic energy":

$$\begin{aligned} h_{\kappa,\alpha}[\psi] - \lambda(0,\alpha_0)\,\|\psi\|_\kappa^2 &\geq \big\|g_\kappa^{-1}\partial_1\psi\big\|_\kappa^2 + \big\|\mu^{1/2}\psi\big\|_\kappa^2 \\ &\geq \big\|g_\kappa^{-1}\partial_1\psi\big\|_\kappa^2 + \epsilon\,(1-\|\kappa\|_\infty a)\,\min_J\mu\int_{J\times(-a,a)}|\psi|^2\,. \end{aligned}$$

Here $\epsilon\in(0,1]$ is arbitrary for the time being. Applying Lemma 12.4 to the last integral, we arrive at

$$\begin{aligned} &h_{\kappa,\alpha}[\psi] - \lambda(0,\alpha_0)\,\|\psi\|_\kappa^2 \\ &\quad\geq \left(\frac{1}{1+\|\kappa\|_\infty a} - \frac{16\,\epsilon\,(1-\|\kappa\|_\infty a)\,\min_J\mu}{2+64/|J|^2}\right)\int_{\mathbb{R}\times(-a,a)}|\partial_1\psi|^2 \\ &\qquad + \frac{\epsilon\,(1-\|\kappa\|_\infty a)\,\min_J\mu}{2+64/|J|^2}\int_{\mathbb{R}\times(-a,a)}\rho^{-2}\,|\psi|^2\,. \end{aligned}$$

Choosing now ϵ as the minimum between 1 and the value such that the first term on the right hand side of the last estimate vanishes, we finally get

$$h_{\kappa,\alpha}[\psi] - \lambda(0,\alpha_0)\,\|\psi\|_\kappa^2 \;\geq\; c\,\big\|\rho^{-1}\psi\big\|_\kappa^2 \tag{12.28}$$

with

$$c \;:=\; \min\left\{\frac{(1-\|\kappa\|_\infty a)\,\min_J\mu}{\big(2+64/|J|^2\big)\,(1+\|\kappa\|_\infty a)}\,,\;\frac{1}{16\,(1+\|\kappa\|_\infty a)^2}\right\}. \tag{12.29}$$

In view of Section 12.3, we conclude that (12.28) is equivalent to

$$Q_{\kappa,\alpha}[u] - \lambda(0,\alpha_0)\,\|u\|_{L^2(\Omega)}^2 \;\geq\; c\,\big\|(\rho\circ\mathcal{L})^{-1}u\big\|_{L^2(\Omega)}^2 \tag{12.30}$$

for all $u\in D(Q_{\kappa,\alpha})$, which is the exact meaning of (12.9).

12.6.2 Proof of Corollary 12.2

Let ψ be as in the previous section. The present proof is based on an algebraic comparison of $h_{\kappa,0}[\psi] - \lambda(0,0)\,\|\psi\|_\kappa^2$ with $h_{\kappa_+,0}[\psi] - \lambda(0,0)\,\|\psi\|_{\kappa_+}^2$ and a usage of (12.28).

For every $(s,t)\in\mathbb{R}\times(-a,a)$, we have

$$1-f_\varepsilon(s) \;\leq\; \frac{g_\kappa(s,t)}{g_{\kappa_+}(s,t)} \;\leq\; 1+f_\varepsilon(s) \qquad\text{with}\qquad f_\varepsilon(s) := \frac{\varepsilon\,a\,\chi_I(s)}{1-\|\kappa_+\|_\infty a}\,,$$

where χ_I denotes the characteristic function of the set $I\times(-a,a)$. Hereafter we assume $\varepsilon\leq(1-\|\kappa_+\|_\infty a)/(2a)$ so that the lower bound is greater or equal to $1/2$. Using the same notation f_ε for the functions $f_\varepsilon\otimes 1$ on $\mathbb{R}\times(-a,a)$, we have

$$\begin{aligned}
h_{\kappa,0}[\psi]-\lambda(0,0)\,\|\psi\|^2_\kappa \\
&\geq \int_{\mathbb{R}\times(-a,a)}(1+f_\varepsilon)^{-1}\,g_{\kappa_+}^{-1}\,|\partial_1\psi|^2 \\
&\quad+\int_{\mathbb{R}}ds\,\big(1-f_\varepsilon(s)\big)\int_{-a}^{a}dt\,g_{\kappa_+}(s,t)\,\big(|\partial_2\psi(s,t)|^2-\lambda(0,0)\,|\psi(s,t)|^2\big) \\
&\quad-\lambda(0,0)\int_{\mathbb{R}\times(-a,a)}2f_\varepsilon\,g_{\kappa_+}\,|\psi|^2\,.
\end{aligned}$$

Recalling the definition of λ via (12.15) and Lemma 12.3, it is clear that the term in the second line after the inequality sign is non-negative. Consequently,

$$\begin{aligned}
h_{\kappa,0}[\psi]-\lambda(0,0)\,\|\psi\|^2_\kappa \;\geq\; &\frac12\Big(h_{\kappa_+,0}[\psi]-\lambda(0,0)\,\|\psi\|^2_{\kappa_+}\Big) \\
&-\;\lambda(0,0)\int_{\mathbb{R}\times(-a,a)}2f_\varepsilon\,g_{\kappa_+}\,|\psi|^2\,.
\end{aligned}$$

Using (12.28) with α being equal to 0, with κ being replaced by κ_+ and with s_0 being from the support of κ_+, we finally obtain

$$h_{\kappa,0}[\psi]-\lambda(0,0)\,\|\psi\|^2_\kappa\;\geq\;\big\|w^{1/2}\psi\big\|^2_\kappa\,, \tag{12.31}$$

where

$$w(s,t):=\frac{c/4}{1+(s-s_0)^2}-\lambda(0,0)\,\frac{\varepsilon\,a\,\chi_I(s)}{1-\|\kappa_+\|_\infty a}$$

is positive for all sufficiently small ε. Equivalently,

$$-\,\Delta_{\kappa,0}\;\geq\;\lambda(0,0)+w\circ\mathcal{L}^{-1} \tag{12.32}$$

in the sense of quadratic forms on $L^2(\Omega)$. This concludes the proof of Corollary 12.2.

12.7 Remarks and open questions

It follows immediately from the minimax principle that the lower bound of Theorem 12.1 also applies to other boundary conditions imposed on $L_\pm$, *e.g.*, Dirichlet, periodic, certain Robin, *etc.*

Of course, it is also possible to impose Robin boundary conditions on Γ_- instead of Dirichlet. Then the lower bound of the type (12.14) still holds and the problem is translated to the study of properties of the first eigenvalue in a Robin-Robin annulus. The techniques of the present paper will also apply to certain values of the parameters in such a case. However, we refrained from doing so to keep the statement of results as simple as possible.

It follows from Theorem 12.2 that $\nu(\alpha)$ gives a uniform lower bound to the spectral threshold of $-\Delta_{\kappa,\alpha}$ provided $\alpha\leq 0$ or $\kappa\leq 0$. We conjecture this to be always the case, but were not able to prove it in general. In this context, it would be desirable to prove that $\kappa\mapsto\lambda(\kappa,\alpha)$ does not possess local minima for any $\alpha\in\mathbb{R}$.

We proved the fact that $\kappa\mapsto\lambda(\kappa,\alpha)$ is increasing on $(0,1/a)$ only for non-positive α. It is clear from the limiting Dirichlet problem (*cf* [14]) that this property will not hold for large positive α. However, formula (12.23) suggests that this is still true for small values of α. Numerical results show (*cf* Figure 12.1) that the critical value is approximately 0.78 for $a=1$.

To transfer the numerical results of Figure 12.1 for different values of a, it is sufficient to notice that λ scales as: $\lambda(\kappa,\alpha;a)=a^{-2}\lambda(\kappa a,\alpha a;1)$.

Proposition 12.1 contains just one example of sufficient condition which guarantees the existence of discrete eigenvalues in infinite curved strips. Further results can be obtained in the spirit of [9, 25]. An open question is, *e.g.*, whether the discrete spectrum exists for certain strips with $\kappa>0$ and $\alpha>0$. Let us recall that this is always the case for $\alpha=+\infty$.

For simplicity, we assumed that κ and $\alpha-\alpha_0$ had compact support when we considered infinite strips. However, the claim of Corollary 12.1 holds whenever the essential spectrum (12.7) is preserved, and this might be checked under much less restrictive conditions about the decay of κ and $\alpha-\alpha_0$ at infinity. For instance, modifying the approach of [25], it should be enough just to require that the limits at infinity are equal to zero.

In fact, Theorem 12.3 holds without any condition about the decay of κ and $\alpha - \alpha_0$ at infinity, but it is of interest only in the case the essential spectrum does not start above $\lambda(0, \alpha_0)$. In any case, a fast decay of curvature at infinity is needed to prove Corollary 12.2; namely, $\kappa(s) = \mathcal{O}(s^{-2})$ as $|s| \to \infty$. This quadratic decay is related to the decay of the Hardy weight in Theorem 12.3, which is typical for Hardy inequalities involving the Laplacian, and cannot be therefore improved by the present method.

Under suitable global geometric conditions about the reference curve Γ, the intrinsic distance $|s - s_0|$ which appears in the function ρ of Theorem 12.3 can be estimated by an exterior one. For instance, if Γ is an embedded unit-speed curve with compactly supported curvature, then it is easy to see that there exists a positive number δ such that

$$\forall s, s' \in \mathbb{R}, \quad \delta\,|s - s'| \le |\Gamma(s) - \Gamma(s')| \le |s - s'|\,.$$

Corollary 12.2 extends the class of strips from [9] with empty discrete spectrum. An open question is to decide whether an analogous result holds for other α satisfying $\alpha_0 \le \alpha \le 0$.

Acknowledgements

This work was partially supported by FCT (Portugal) through projects POCI/MAT/60863/2004 (POCI2010) and SFRH/BPD/11457/2002. The second author (D.K.) was also supported by the Czech Academy of Sciences and its Grant Agency within the projects IRP AV0Z10480505 and A100480501, and by the project LC06002 of the Ministry of Education, Youth and Sports of the Czech Republic.

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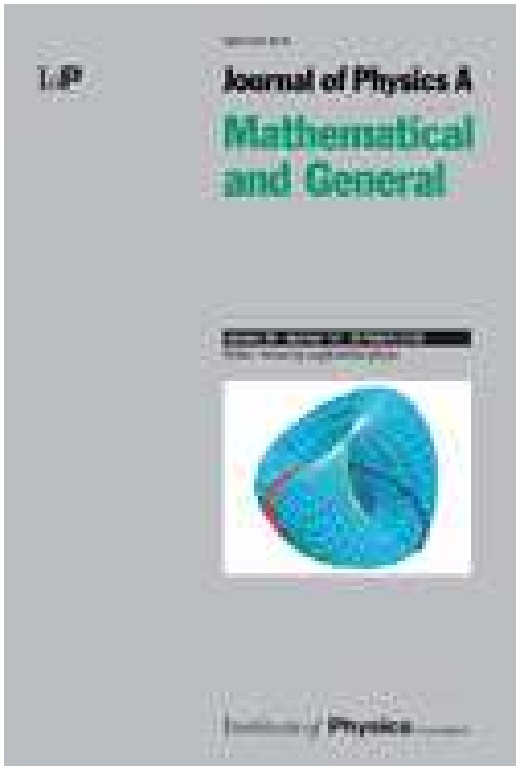
Chapter 13

The nature of the essential spectrum in curved quantum waveguides



Published in: Journal of Physics A: Mathematical and General **37** (2004), 5449–5466

Joint work with: Rafael Tiedra de Aldecoa

The nature of the essential spectrum in curved quantum waveguides

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Abstract. We study the nature of the essential spectrum of the Dirichlet Laplacian in tubes about infinite curves embedded in Euclidean spaces. Under suitable assumptions about the decay of curvatures at infinity, we prove the absence of singular continuous spectrum and state properties of possible embedded eigenvalues. The argument is based on Mourre conjugate operator method developed for acoustic multistratified domains by Benbernou in [3] and Dermenjian *et al.* in [10]. As a technical preliminary, we carry out a spectral analysis for Schrödinger-type operators in straight Dirichlet tubes. We also apply the result to the strips embedded in abstract surfaces.

13.1 Introduction

A strong physical motivation to study the Dirichlet Laplacian in infinitely stretched tubular regions comes from the fact it constitutes a reasonable model for the Hamiltonian of a non-relativistic quantum particle in mesoscopic systems called *quantum waveguides* [11, 26, 19]. Since there exists a close relation between spectral and scattering properties of Hamiltonians, one is naturally interested in carrying out the spectral analysis of the Laplacian in order to understand the quantum dynamics in waveguides. For instance, the crucial step in most proofs of asymptotic completeness is to show that the Hamiltonian has no singular continuous spectrum [28]. The Laplacian in a tube has attracted considerable attention since it was shown in [15] that there may be discrete eigenvalues in curved waveguides. However, a detailed analysis of the essential part of the spectrum has been left aside up to now. The purpose of the present paper is to fill in this gap.

The usual model for a curved quantum waveguide, which we adopt in this paper, is as follows. Let $s \mapsto p(s)$ be an infinite unit-speed smooth curve in $\mathbb{R}^d$, $d \geq 2$ (the physical cases corresponding to $d = 2, 3$). Assuming that the curve possesses an appropriate smooth Frenet frame $\{e_1, \ldots, e_d\}$ (*cf* Assumption 13.3), the i^{th} curvature κ_i of p, $i \in \{1, \ldots, d-1\}$, is a smooth function of the arc-length parameter $s \in \mathbb{R}$. Given a bounded open connected set ω in $\mathbb{R}^{d-1}$ with the centre of mass at the origin, we identify the configuration space Γ of the waveguide with a tube of cross-section ω about p, namely:

$$\Gamma := \mathfrak{L}(\mathbb{R} \times \omega), \qquad \mathfrak{L}(s, u^2, \ldots, u^d) := p(s) + u^\mu \mathcal{R}_\mu{}^\nu(s) e_\nu(s), \tag{13.1}$$

where μ, ν are summation indices taking values in $\{2, \ldots, d\}$ and $(\mathcal{R}_\mu{}^\nu)$ is a family of rotation matrices in $\mathbb{R}^{d-1}$. In this paper, we choose the rotations in such a way that (s, u), with $u := (u^2, \ldots, u^d)$, are orthogonal "co-ordinates" (*cf* Section 13.3.1) due to the technical simplicity. It should be stressed here that while the shape of the tube Γ is not influenced by a special choice of $(\mathcal{R}_\mu{}^\nu)$ provided ω is circular, this may no longer be true for a general cross-section. We make the hypotheses (Assumption 13.4) that κ_1 is bounded, $a\|\kappa_1\|_\infty < 1$, with $a := \sup_{u \in \omega} |u|$, and Γ does not overlap itself so that the tube can be globally parameterised by (s, u). Our object of interest is the Dirichlet Laplacian associated with the tube, *i.e.*,

$$-\Delta_{\mathrm{D}}^\Gamma \qquad \text{on} \qquad L^2(\Gamma). \tag{13.2}$$

If p is a straight line, *i.e.*, all $\kappa_i = 0$, then Γ may be identified with the straight tube $\Omega := \mathbb{R} \times \omega$. In that case, it is easy to see that the spectrum of (13.2) is purely *absolutely continuous* and equal to the interval $[\nu_1, \infty)$, where ν_1 denotes the first eigenvalue of the Dirichlet Laplacian in the cross-section ω.

On the other hand, if p is non-trivially curved and straight asymptotically, in the sense that the curvature κ_1 vanishes at infinity, then the essential spectrum of (13.2) remains equal to $[\nu_1, \infty)$. However, there are always *discrete eigenvalues* below ν_1. When $d = 2$, the latter was proved for the first time in [15] for a rapidly decaying curvature and sufficiently small a. Numerous subsequent studies improved and generalised this initial result [17, 30, 11, 23, 24, 8]. The generalisation to tubes of circular cross-section in $\mathbb{R}^3$ was done in [17] (see also [11])

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and the case of any dimension $d \geq 2$ and arbitrary cross-section can be found in [8]. Let us also mention that the discrete spectrum may be generated by other local perturbations of the straight tube Ω (see, *e.g.*, [7, 16, 4]), but in the bent-tube case the phenomenon is of a purely quantum origin because there are no classical closed trajectories, apart from those given by a zero measure set of initial conditions in the phase space.

The main goal of the present work is a thorough analysis of the *essential spectrum* of (13.2). In particular, we find sufficient conditions which guarantee that the essential spectrum of a curved tube "does not differ too much" from the straight case (for simplicity, we present here our results only for $d = 2$, see Theorem 13.4 for the d-dimensional case):

Theorem 13.1 (***d***=2)**.** *Let Γ be as above for $d = 2$ ($\kappa := \kappa_1$) and $\mathcal{T} := \{n^2\nu_1\}_{n=1}^{\infty}$ with $\nu_1 := \pi^2/(2a)^2$ (the set of eigenvalues of the Dirichlet Laplacian in the 1-dimensional cross-section ω). Suppose*

1. $\kappa(s),\ \ddot{\kappa}(s) \longrightarrow 0$ *as* $|s| \to \infty$,
2. $\exists \vartheta \in (0,1]$ *s.t.* $\dot{\kappa}(s),\ \dddot{\kappa}(s) = \mathcal{O}\left(|s|^{-(1+\vartheta)}\right)$.

Then

(i) $\sigma_{\mathrm{ess}}\left(-\Delta_{\mathrm{D}}^{\Gamma}\right) = [\nu_1, \infty)$,

(ii) $\sigma_{\mathrm{sc}}\left(-\Delta_{\mathrm{D}}^{\Gamma}\right) = \varnothing$,

(iii) $\sigma_{\mathrm{p}}\left(-\Delta_{\mathrm{D}}^{\Gamma}\right) \cup \mathcal{T}$ *is closed and countable,*

(iv) $\sigma_{\mathrm{p}}\left(-\Delta_{\mathrm{D}}^{\Gamma}\right) \setminus \mathcal{T}$ *is composed of finitely degenerated eigenvalues which can accumulate at points of $\mathcal{T}$ only.*

To prove this theorem (and the general Theorem 13.4), we use the conjugate operator method introduced by [27] E. Mourre and lastly developed by [2] W. Amrein *et al.* Notice that the set $\mathcal{T}$ plays a role analogous to the set of *thresholds* in the Mourre theory of N-body Schrödinger operators [9].

Actually, the property (i) holds true whenever the first curvature vanishes at infinity, without assuming any decay of the derivatives (they may not even exist), see [24] for $d = 2$ and [8] for the general case. Our second result (ii) can be compared only with [13] (see also [12]), where the problem of resonances is investigated for $d = 2$. Assuming that there exists $\vartheta \in (0,1]$ such that $\kappa(s), \dot{\kappa}(s)^2, \ddot{\kappa}(s) = \mathcal{O}\big(|s|^{-(1+\vartheta)}\big)$, the authors proved the absence of singular continuous spectrum as a consequence of the completeness of wave operators obtained by standard smooth perturbation methods of scattering theory. Notice that our and their results are independent. Indeed, while we need to require a faster decay of $\dot{\kappa}$ and also impose a condition on $\dddot{\kappa}$, our decay assumptions on κ and $\ddot{\kappa}$ are on the contrary much weaker. Our other spectral results (iii) and (iv) (and (ii) for $d \geq 3$) are new.

The organisation of this paper is as follows. In Section 13.2, we consider the Schrödinger-type operator

$$H := -\partial_i G^{ij} \partial_j + V \qquad \text{on} \qquad \mathcal{H}(\Omega) := L^2(\Omega), \tag{13.3}$$

subject to Dirichlet boundary conditions, i and j being summation indices taking values in $\{1, \ldots, d\}$, $G \equiv (G^{ij})$ a real symmetric matrix-valued measurable function on Ω and V the multiplication operator by a real-valued measurable function on Ω. We make Assumption 13.1 and Assumption 13.2 stated below. Adapting the approach of [3, 10] to non-zero V and G different from a multiple of the identity, we study the nature of the essential spectrum of the operator H. In particular, we prove the absence of singular continuous spectrum and state properties of possible embedded eigenvalues. The result is contained in Theorem 13.3 and is of independent interest. In Section 13.3, we apply it to the case of curved tubes (13.1). Using the diffeomorphism $\mathfrak{L} : \Omega \to \Gamma$ and a unitary transformation (ideas which go back to [15]), we cast the Laplacian (13.2) into a unitarily equivalent operator of the form (13.3) for which Theorem 13.3 can be used. The obtained spectral results can be found in Theorem 13.4 (the general version of Theorem 13.1 above). Finally, in Section 13.4, we similarly investigate the essential spectrum of the Dirichlet Laplacian in an infinite strip in an abstract two-dimensional Riemannian manifold of curvature K. The general result is contained in Theorem 13.5, while the case of flat strips, *i.e.*, with $K = 0$, is summarised in Theorem 13.6 (the latter involves the curved strips in $\mathbb{R}^2$ as a special case).

For the conjugate operator method and notation used in Section 13.2, the reader is referred to [2] and particularly to short well-arranged reviews of the abstract theory in [2, Sec. 2] or [11, Sec. 1]. A more detailed geometric background for Section 13.3 and Section 13.4 can be found in [22, 8] and [18, 23], respectively.

We use the standard component notation of tensor analysis throughout the paper. In particular, the repeated indices convention is adopted henceforth, the range of indices being $1, \ldots, d$ for Latin and $2, \ldots, d$ for Greek. The indices are associated in a natural way with the components of $x \in \mathbb{R} \times \omega$. The partial derivative w.r.t. x^i is often denoted by a comma with the index i. The brackets $(\cdot)$ are used in order to distinguish a matrix from its coefficients. The symbols δ_{ij} and δ^{ij} are reserved for the components of the identity matrix 1.

13.2 Schrödinger-type operators in straight tubes

13.2.1 Preliminaries

Let ω be an (arbitrary) bounded open connected set in $\mathbb{R}^{d-1}$, $d \geq 2$, and consider the straight tube $\Omega := \mathbb{R} \times \omega$. Our object of interest in this section is the operator given formally by (13.3), subject to Dirichlet boundary conditions. In addition to the basic properties required for the matrix G and function V, we make the following assumptions.

Assumption 13.1.

1. $\exists C_\pm \in (0,\infty)$ s.t. $C_- 1 \leq G(x) \leq C_+ 1$ for a.e. $x \in \Omega$,
2. $\forall i,j \in \{1,\dots,d\}$, $\lim\limits_{R\to\infty} \operatorname*{ess\,sup}\limits_{x\in(\mathbb{R}\setminus[-R,R])\times\omega} \left|G^{ij}(x) - \delta^{ij}\right| = 0$,
3. $\exists \vartheta_1 \in (0,1]$, $C \in (0,\infty)$ s.t. $\left(\left|G^{ij}_{,1}(x)\right|\right) \leq C\langle x^1\rangle^{-(1+\vartheta_1)} 1$ for a.e. $x \in \Omega$,
4. $G^{1i}_{,i} \in L^\infty(\Omega)$.

Here $\langle\cdot\rangle := \left(1 + |\cdot|^2\right)^{1/2}$ and the inequalities must be understood in the sense of matrices.

Assumption 13.2.

1. $V \in L^\infty(\Omega)$,
2. $\lim\limits_{R\to\infty} \operatorname*{ess\,sup}\limits_{x\in(\mathbb{R}\setminus[-R,R])\times\omega} |V(x)| = 0$,
3. $\exists \vartheta_2 \in (0,1]$, $C \in (0,\infty)$ s.t. $|V_{,1}(x)| \leq C\langle x^1\rangle^{-(1+\vartheta_2)}$ for a.e. $x \in \Omega$.

Let us fix some notations. We write $\mathcal{H}^\nu(\Omega)$ and $\mathcal{H}^\nu_0(\Omega)$, $\nu \in \mathbb{R}$, for the usual Sobolev spaces [1]. Given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we denote by $\mathfrak{B}(\mathcal{H}_1,\mathcal{H}_2)$, respectively $\mathcal{K}(\mathcal{H}_1,\mathcal{H}_2)$, the set of bounded, respectively compact, operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. We also define $\mathfrak{B}(\mathcal{H}_1) := \mathfrak{B}(\mathcal{H}_1,\mathcal{H}_1)$ and $\mathcal{K}(\mathcal{H}_1) := \mathcal{K}(\mathcal{H}_1,\mathcal{H}_1)$. We denote by $\mathcal{H}_1^*$ the topological antidual of $\mathcal{H}_1$. We write $(\cdot,\cdot)$ for the inner product in $\mathcal{H}(\Omega)$ and $\|\cdot\|$ for the norm in $\mathcal{H}(\Omega)$ and $\mathfrak{B}(\mathcal{H}(\Omega))$.

We now give a meaning to the formal expression (13.3). We start by introducing the sesquilinear form Q_0 on $\mathcal{H}(\Omega)$ defined by

$$Q_0(\varphi,\psi) := \left(\varphi_{,i}, \delta^{ij}\psi_{,j}\right), \qquad \varphi,\psi \in \mathfrak{D}(Q_0) := \mathcal{H}^1_0(\Omega), \tag{13.4}$$

which is densely defined, symmetric, non-negative and closed. Consequently, there exists a unique self-adjoint operator H_0 associated with it, which is just the Dirichlet Laplacian $-\Delta^\Omega_{\mathrm{D}}$ on $L^2(\Omega)$. We have

$$H_0\psi = -\Delta\psi, \qquad \psi \in \mathfrak{D}(H_0) = \left\{\psi \in \mathcal{H}^1_0(\Omega) \,:\, \Delta\psi \in \mathcal{H}(\Omega)\right\}.$$

We consider H as an operator obtained by perturbing the free Hamiltonian H_0. Since the matrix G is uniformly positive and bounded by Assumption 13.1.1, the sesquilinear form $(\varphi,\psi) \mapsto \left(\varphi_{,i}, G^{ij}\psi_{,j}\right)$ defined on $\mathfrak{D}(Q_0) \times \mathfrak{D}(Q_0)$ is also densely defined, symmetric, non-negative and closed. At the same time, the potential V is supposed to be bounded by Assumption 13.2.1, which means that the sesquilinear form Q defined by

$$Q(\varphi,\psi) := \left(\varphi_{,i}, G^{ij}\psi_{,j}\right) + (\varphi, V\psi), \qquad \varphi,\psi \in \mathfrak{D}(Q) := \mathcal{H}^1_0(\Omega), \tag{13.5}$$

gives rise to a semi-bounded self-adjoint operator H. Using the representation theorem [40, Chap. VI, Thm. 2.1] and the fact that V is bounded (recall also Assumption 13.1.1), one may check that

$$\mathfrak{D}(H) = \left\{\psi \in \mathcal{H}^1_0(\Omega) \,:\, \partial_i G^{ij}\partial_j\psi \in \mathcal{H}(\Omega)\right\},$$

where the derivatives must be interpreted in the distributional sense, and that H is acting as in (13.3) on its domain.

For any $z \in \mathbb{C}\setminus\sigma(H_0)$, respectively $z \in \mathbb{C}\setminus\sigma(H)$, let $R_0(z) := (H_0 - z)^{-1}$, respectively $R(z) := (H - z)^{-1}$.

13.2.2 Localisation of the essential spectrum

The Dirichlet Laplacian $-\Delta_{\mathrm{D}}^{\omega}$ on $L^2(\omega)$, *i.e.*, the operator associated with

$$q(\varphi,\psi) := (\varphi_{,\mu}, \delta^{\mu\nu}\psi_{,\nu})\,, \qquad \varphi,\psi \in \mathfrak{D}(q) := \mathcal{H}_0^1(\omega),$$

has a purely discrete spectrum consisting of eigenvalues $\nu_1 < \nu_2 \leq \nu_3 \leq \dots$ with $\nu_1 > 0$. We set $\mathcal{T} := \{\nu_n\}_{n=1}^{\infty}$. Since H_0 is naturally decoupled in the following way:

$$H_0 = -\Delta^{\mathbb{R}} \otimes 1 + 1 \otimes (-\Delta_{\mathrm{D}}^{\omega}) \qquad \text{on} \qquad L^2(\mathbb{R}) \otimes L^2(\omega),$$

where "$\otimes$" denote the closed tensor product, 1 the identity operators on appropriate spaces and $-\Delta^{\mathbb{R}}$ the Laplacian on $L^2(\mathbb{R})$, one has

$$\sigma(H_0) = \sigma_{\mathrm{ess}}(H_0) = [\nu_1, \infty). \tag{13.6}$$

In order to prove that (under our assumptions) H possesses the same essential spectrum, we need the following lemma.

Lemma 13.1. *Let $\varphi \in C_0^{\infty}(\mathbb{R})$ and set $\phi := \varphi \otimes 1$ on Ω. Then, as a multiplication operator,*

$$\phi \in \mathcal{K}\left(\mathfrak{D}(H_0), \mathcal{H}_0^1(\Omega)\right).$$

Proof. Since

$$\phi = H_0^{-1/2} H_0^{1/2} \phi H_0^{-1} H_0$$

in $\mathfrak{B}\left(\mathfrak{D}(H_0), \mathcal{H}_0^1(\Omega)\right)$, $H_0 \in \mathfrak{B}\left(\mathfrak{D}(H_0), \mathcal{H}(\Omega)\right)$ and $H_0^{-1/2} \in \mathfrak{B}\left(\mathcal{H}(\Omega), \mathcal{H}_0^1(\Omega)\right)$, it is enough to prove that $H_0^{1/2}\phi H_0^{-1} \in \mathcal{K}\left(\mathcal{H}(\Omega)\right)$. However,

$$\begin{aligned} H_0^{1/2}\phi H_0^{-1} &= H_0^{-1/2}\left[H_0, \phi\right] H_0^{-1} + H_0^{-1/2}\phi \\ &= -H_0^{-1/2}\left(2\phi_{,1}\partial_1 + \phi_{,11}\right) H_0^{-1} + H_0^{-1/2}\phi, \end{aligned} \tag{13.7}$$

where each term on the r.h.s. is in $\mathcal{K}\left(\mathcal{H}(\Omega)\right)$. Let us demonstrate it for the first term. Since $\partial_1 H_0^{-1} \in \mathfrak{B}\left(\mathcal{H}(\Omega)\right)$, it is sufficient to prove that $H_0^{-1/2}\phi_{,1} \in \mathcal{K}\left(\mathcal{H}(\Omega)\right)$. Let $z_1 \in (-\infty, 0)$ and $z_2 \in (-\infty, \nu_1)$ be such that $z_1 + z_2 = 0$. Define $R_{\shortparallel}(z_1) := \left(-\Delta^{\mathbb{R}} - z_1\right)^{-1}$ and $R_{\perp}(z_2) := \left(-\Delta_{\mathrm{D}}^{\omega} - z_2\right)^{-1}$. Then, using some standard results on tensor products of operators [20, Chap. 11], one can write

$$H_0^{-1/2}\phi_{,1} = H_0^{-1/2}\left[R_{\shortparallel}^{-1/4}(z_1) \otimes R_{\perp}^{-1/4}(z_2)\right]\left[R_{\shortparallel}^{1/4}(z_1)\varphi_{,1} \otimes R_{\perp}^{1/4}(z_2)\right]$$

where $\varphi_{,1}$ is viewed as a multiplication operator in $L^2(\mathbb{R})$. The third factor on the r.h.s. is in $\mathcal{K}\left(\mathcal{H}(\Omega)\right)$ because $-\Delta_{\mathrm{D}}^{\omega}$ has a compact resolvent and $R_{\shortparallel}^{1/4}(z_1)\varphi_{,1} \in \mathcal{K}\left(L^2(\mathbb{R})\right)$ by [2, Thm. 4.1.3]. The remaining factors can be rewritten as

$$\Psi(X_1, X_2) := (X_1 + X_2)^{-1/2} X_1^{1/4} X_2^{1/4}$$

with $X_1 := \left(-\Delta^{\mathbb{R}} - z_1\right) \otimes 1$ and $X_2 := 1 \otimes (-\Delta_{\mathrm{D}}^{\omega} - z_2)$ (both self-adjoint and mutually commuting). So, one can estimate

$$\|\Psi(X_1, X_2)\| \leq \sup_{x_1, x_2 \in (0,\infty)} (x_1 + x_2)^{-1/2}(x_1 x_2)^{1/4} < \infty.$$

Hence, the first term on the r.h.s. of (13.7) is in $\mathcal{K}\left(\mathcal{H}(\Omega)\right)$. The argument is similar for the remaining terms. □

Proposition 13.1. *One has*

(i) $\forall z \in \mathbb{C} \setminus (\sigma(H) \cup \sigma(H_0))$, $R(z) - R_0(z) \in \mathcal{K}\big(\mathcal{H}(\Omega)\big)$,

(ii) $\sigma_{\mathrm{ess}}(H) = [\nu_1, \infty)$.

Proof. We prove (i) for some (and hence for all) value of $z \in \mathbb{C} \setminus (\sigma(H) \cup \sigma(H_0))$. Let $z \in \mathbb{C} \setminus \mathbb{R}$. Define $R_1(z) := (H_0 + V - z)^{-1}$. Then, one has

$$R(z) - R_0(z) = R(z) - R_1(z) - R_1(z) V R_0(z).$$

Let us first consider $R(z) - R_1(z)$. Knowing that H and $H_0 + V$ have the same form domain, the identity

$$R(z) - R_1(z) = -R(z)(H - H_0 - V)R_1(z)$$

holds in $\mathfrak{B}\left(\mathcal{H}^{-1}(\Omega),\mathcal{H}_0^1(\Omega)\right)$. But, one has the following sequence of continuous and dense imbeddings of Hilbert spaces

$$\mathfrak{D}(H)\subset\mathcal{H}_0^1(\Omega)\subset\mathcal{H}(\Omega)\subset\mathcal{H}^{-1}(\Omega)\subset\mathfrak{D}(H)^*$$

which implies that $R(z)$ extends (by duality) to a homeomorphism of $\mathfrak{D}(H)^*$ onto $\mathcal{H}(\Omega)$. Thus, since $R_1(z)$ is also a homeomorphism from $\mathcal{H}(\Omega)$ onto $\mathfrak{D}(H_0)$, $R(z)-R_1(z)\in\mathcal{K}\left(\mathcal{H}(\Omega)\right)$ if and only if $H-H_0-V\in\mathcal{K}\left(\mathfrak{D}(H_0),\mathfrak{D}(H)^*\right)$. For all $n\in\mathbb{N}\setminus\{0\}$, let $\varphi_n\in C_0^\infty(\mathbb{R})$ be such that $0\leq\varphi_n\leq 1$ and

$$\varphi_n(x^1)=\begin{cases}1 & \text{if} \quad |x^1|\leq n\\ 0 & \text{if} \quad |x^1|\geq n+1.\end{cases}$$

Set $\phi_n:=\varphi_n\otimes 1$ on Ω and

$$K_n\psi:=-\partial_i F^{ij}\,\phi_n\partial_j\psi,\qquad \psi\in\mathfrak{D}(H_0),$$

where $(F^{ij}):=(G^{ij}-\delta^{ij})$. Clearly, $H-H_0-V,K_n\in\mathfrak{B}\left(\mathfrak{D}(H_0),\mathfrak{D}(H)^*\right)$ and

$$\begin{aligned}&\|K_n-(H-H_0-V)\|_{\mathfrak{B}(\mathfrak{D}(H_0),\mathfrak{D}(H)^*)}\\ &\equiv\sup_{\psi\in\mathfrak{D}(H_0),\|\psi\|_{\mathfrak{D}(H_0)}=1}\left\|\left(1+H^2\right)^{-1/2}\left[-\partial_i F^{ij}\,(\phi_n-1)\partial_j\right]\psi\right\|\\ &\leq\sup_{\psi\in\mathfrak{D}(H_0),\|\psi\|_{\mathfrak{D}(H_0)}=1}\sum_{j=1}^{d}\left\|\left(1+H^2\right)^{-1/2}\partial_i\right\|\left\|F^{ij}(\phi_n-1)\right\|_\infty\|\psi\|_{\mathcal{H}_0^1(\Omega)}\xrightarrow[n\to\infty]{}0,\end{aligned}$$

where we have used the fact that $\mathfrak{D}(H_0)\subset\mathcal{H}_0^1(\Omega)$ continuously and Assumption 13.1.2 in the final step. So, it only remains to show that $K_n\in\mathcal{K}\left(\mathfrak{D}(H_0),\mathfrak{D}(H)^*\right)$. After a commutation, one gets in $\mathfrak{B}\left(\mathfrak{D}(H_0),\mathfrak{D}(H)^*\right)$

$$K_n=-\partial_i F^{ij}\,\partial_j\phi_n+\partial_i F^{i1}\,\phi_{n,1}$$

where ϕ_n, $\phi_{n,1}$ are seen as multiplication operators in $\mathcal{H}(\Omega)$. It is clear that both $\partial_i F^{ij}\partial_j$ and $\partial_i F^{i1}$ are in $\mathfrak{B}\left(\mathcal{H}_0^1(\Omega),\mathfrak{D}(H)^*\right)$. Moreover, ϕ_n and $\phi_{n,1}$ are in $\mathcal{K}\left(\mathfrak{D}(H_0),\mathcal{H}_0^1(\Omega)\right)$ by Lemma 13.1. Thus, $K_n\in\mathcal{K}\left(\mathfrak{D}(H_0),\mathfrak{D}(H)^*\right)$ so that $R(z)-R_1(z)\in\mathcal{K}\left(\mathcal{H}(\Omega)\right)$. Using similar arguments, one can also prove that the $R_1(z)V\phi_nR_0(z)$ is compact and converges to $R_1(z)VR_0(z)$ in $\mathfrak{B}(\mathcal{H}(\Omega))$ due to Assumption 13.2.2. This implies that $R_1(z)VR_0(z)\in\mathcal{K}\left(\mathcal{H}(\Omega)\right)$.
(ii) It is a direct consequence of (i), (13.6) and Weyl's theorem [59, Thm. XIII.14]. □

Remark 13.1. Notice that *Assumptions 13.1.3*, *13.1.4* and *13.2.3* are not used in the proof of *Proposition 13.1*.

13.2.3 Nature of the essential spectrum

This part is devoted to a more detailed analysis of the essential spectrum of H. In particular, we show that the singular continuous spectrum is empty. The strategy adapted from [3] is the following. Firstly, we construct a dilation operator A such that $H_0\in C^\infty(A)$ and $H\in C^{1+\vartheta}(A)$ with $\vartheta:=\min\{\vartheta_1,\vartheta_2\}\in(0,1]$ (see [2], [3, Sec. 2] or [10, Sec. 1] for definitions of the spaces involved here and in the sequel). Secondly, we prove that A is *strictly conjugate* (in Mourre's sense) to H_0 on $\mathbb{R}\setminus\mathcal{T}$. Finally, since $R(i)-R_0(i)$ is compact by the first claim of Proposition 13.1 and both H and H_0 are of class $C_u^1(A)\supseteq C^{1+\vartheta}(A)\supseteq C^\infty(A)$, it follows that A is conjugate to H on $\mathbb{R}\setminus\mathcal{T}$ as well.

The dilation operator

Let q^1 be the multiplication operator by the coordinate x^1 in $\mathcal{H}(\Omega)$. Let

$$A:=\tfrac{1}{2}\left(q^1p_1+p_1q^1\right)\qquad\text{with}\qquad p_1:=-i\partial_1 \tag{13.8}$$

be the dilation operator in $\mathcal{H}(\Omega)$ w.r.t. x^1, *i.e.*, the self-adjoint extension of the operator defined by the expression (13.8) with $C_0^\infty(\Omega)$ as initial domain. Define $A_{\shortparallel}$ as the self-adjoint operator in $L^2(\mathbb{R})$ such that $A=A_{\shortparallel}\otimes 1$.

Remark 13.2. The group $\left\{\mathrm{e}^{iAt}\right\}_{t\in\mathbb{R}}$ leaves invariant $\mathcal{H}_0^1(\Omega)$. Indeed, using the natural isomorphism $\mathcal{H}_0^1(\Omega)\simeq\mathcal{H}^1(\mathbb{R})\otimes\mathcal{H}_0^1(\omega)$, one can write

$$\forall t\in\mathbb{R},\qquad \mathrm{e}^{iAt}\mathcal{H}_0^1(\Omega)=\left(\mathrm{e}^{iA_{\shortparallel}t}\mathcal{H}^1(\mathbb{R})\right)\otimes\mathcal{H}_0^1(\omega).$$

Then, the affirmation follows from the fact *[2, Prop. 4.2.4]* that $\mathcal{H}^1(\mathbb{R})$ is stable under $\left\{\mathrm{e}^{iA_{\shortparallel}t}\right\}_{t\in\mathbb{R}}$.

In order to deal with the commutator $i\,[H,A]$, we need the following family of operators

$$\big\{p_1(\varepsilon) := p_1(1+i\varepsilon p_1)^{-1}\big\}_{\varepsilon>0}\,, \tag{13.9}$$

which regularises the momentum operator p_1:

Lemma 13.2. *One has*

(i) $\{p_1(\varepsilon)\}_{\varepsilon>0} \subset \mathfrak{B}\big(\mathcal{H}(\Omega)\big)$,

(ii) $\{p_1(\varepsilon)\}_{\varepsilon>0}$ *is uniformly bounded in* $\mathfrak{B}\big(\mathcal{H}^1(\Omega),\mathcal{H}(\Omega)\big)$ *and* $\text{s-}\lim_{\varepsilon\to 0} p_1(\varepsilon) = p_1$ *in* $\mathfrak{B}\big(\mathcal{H}^1(\Omega),\mathcal{H}(\Omega)\big)$,

(iii) $\forall\varepsilon>0,\ \ [p_1(\varepsilon),q_1] = -i(1+i\varepsilon p_1)^{-2}$ *in* $\mathfrak{B}(\mathcal{H}(\Omega))$,

(iv) $\forall\varepsilon>0,\ \ p_1(\varepsilon)\mathcal{H}_0^1(\Omega)\subset\mathcal{H}_0^1(\Omega)$.

Proof. The first three assertions are established in [2, Lemma 4.1]. Consequently, it only remains to prove the last statement. Using the isomorphism mentioned in Remark 13.2, one can write

$$\forall\varepsilon>0,\qquad p_1(\varepsilon)\mathcal{H}_0^1(\Omega) = -i\varepsilon^{-1}\big\{\big[1+i\varepsilon^{-1}(p_1-i\varepsilon^{-1})^{-1}\big]\mathcal{H}^1(\mathbb{R})\big\}\otimes\mathcal{H}_0^1(\omega),$$

where p_1 on the r.h.s. must be viewed as an operator acting in $L^2(\mathbb{R})$. With this last relation, it is clear that $\mathcal{H}_0^1(\Omega)$ is left invariant by the family $\{p_1(\varepsilon)\}_{\varepsilon>0}$. □

We also need the following density result for the set $\mathfrak{D}(H)_c := \{\psi\in\mathfrak{D}(H)\,:\,\text{supp}(\psi)\text{ is compact}\}$.

Lemma 13.3. *One has*

(i) $\mathfrak{D}(H)_c$ *is dense in* $\mathfrak{D}(H)$,

(ii) $\mathfrak{D}(H)_c$ *is dense in* $\mathcal{H}_0^1(\Omega)$.

Proof. (i) We are inspired by [11, Lemma 2.1]. Let $\psi\in\mathfrak{D}(H)$. Define $\varphi_0\in C_0^\infty(\mathbb{R})$ such that

$$\varphi_0(x^1) = \begin{cases} 1 & \text{if} \quad |x^1|\le 1 \\ 0 & \text{if} \quad |x^1|\ge 2. \end{cases}$$

Let $n\in\mathbb{N}$. Set $\varphi_n(x^1) := \varphi_0(x^1/(n+1))$ for $x^1\in\mathbb{R}$ and $\phi_n := \varphi_n\otimes 1$ on Ω. Then, $\phi_n\psi\in\mathcal{H}_0^1(\Omega)$, $\lim_{n\to\infty}\phi_n\psi=\psi$ in $\mathcal{H}(\Omega)$ and

$$H\phi_n\psi = \phi_n H\psi - 2\phi_{n,1}G^{1j}\psi_{,j} - \phi_{n,11}G^{11}\psi - \phi_{,1}G^{1i}_{\ \ ,i}\psi \tag{13.10}$$

in the sense of distributions. Using the fact that $\text{supp}(\phi_n)$ is compact, Assumption 13.1.1 and Assumption 13.1.4, one has $\phi_n\psi\in\mathfrak{D}(H)_{\rm c}$. Moreover, as a consequence of (13.10) and the property

$$\forall k\in\mathbb{N},\ \forall x\in\Omega,\qquad \partial_1^k\phi_n(x) = (n+1)^{-k}\varphi_0^{(k)}(x^1/(n+1)),$$

one also has $\lim_{n\to\infty} H\phi_n\psi = H\psi$ in $\mathcal{H}(\Omega)$.
(ii) Using point (i) and the fact that $\mathfrak{D}(H)\subset\mathcal{H}_0^1(\Omega)$ continuously and densely, one gets the following embeddings

$$\mathcal{H}_0^1(\Omega) = \overline{\mathfrak{D}(H)_{\rm c}}^{\mathfrak{D}(H)}{}^{\mathcal{H}_0^1(\Omega)} \subseteq \overline{\mathfrak{D}(H)_{\rm c}}^{\mathcal{H}_0^1(\Omega)}{}^{\mathcal{H}_0^1(\Omega)} = \overline{\mathfrak{D}(H)_{\rm c}}^{\mathcal{H}_0^1(\Omega)} \subseteq \mathcal{H}_0^1(\Omega)$$

which, in particular, imply that $\mathfrak{D}(H)_{\rm c}$ is dense in $\mathcal{H}_0^1(\Omega)$. □

Now, we can compute the commutator $i\,[H,A]$.

Proposition 13.2. *The sesquilinear form* $\mathcal{Q}$ *on* $\mathcal{H}(\Omega)$ *defined by*

$$\mathcal{Q}(\varphi,\psi) := i\,[(H\varphi,A\psi)-(A\varphi,H\psi)]\,,\qquad \varphi,\psi\in\mathfrak{D}(\mathcal{Q}) := \mathfrak{D}(H)\cap\mathfrak{D}(A),$$

is continuous on $\mathfrak{D}(H)_{\rm c}$ *for the topology induced by* $\mathcal{H}_0^1(\Omega)$. *Moreover,*

$$i\,[H,A] = -\partial_j G^{1j}\partial_1 - \partial_1 G^{1j}\partial_j + \partial_i q^1 G^{ij}_{\ \ ,1}\partial_j - q^1 V_{,1} \tag{13.11}$$

as operators in $\mathfrak{B}\left(\mathcal{H}_0^1(\Omega),\mathcal{H}^{-1}(\Omega)\right)$.

Proof. Let $\varphi, \psi \in \mathfrak{D}(H)_c$. Using the identity $A = q^1 p_1 - \frac{i}{2}$ valid on $\mathfrak{D}(H)_c \subset \mathfrak{D}(A)$, we have

$$\begin{aligned}\mathcal{Q}(\varphi,\psi) &= i\left[(H\varphi, A\psi) - (A\varphi, H\psi)\right] \\ &= (\varphi, H\psi) + i\left[\left(-\partial_i G^{ij}\,\partial_j\varphi, q^1 p_1\psi\right) - \left(q^1 p_1\varphi, -\partial_i G^{ij}\,\partial_j\psi\right)\right] \\ &\quad + \left(V\varphi, q^1\psi_{,1}\right) + \left(q^1\varphi_{,1}, V\psi\right).\end{aligned}$$

In order to justify the subsequent integration by parts, we employ the family (13.9). Since ψ has a compact support and belongs to $\mathcal{H}^1_0(\Omega)$, it follows by using properties (iii) and (iv) of Lemma 13.2 that $q^1 p_1(\varepsilon)\psi \in \mathcal{H}^1_0(\Omega)$ for all $\varepsilon > 0$. So, we can write

$$\begin{aligned}\left(-\partial_i G^{ij}\,\partial_j\varphi, q^1 p_1\psi\right) &= \lim_{\varepsilon\to 0}\left(-\partial_i G^{ij}\,\partial_j\varphi, q^1 p_1(\varepsilon)\psi\right) \\ &= \lim_{\varepsilon\to 0}\left(\varphi_{,j}, G^{ij}\,\partial_i q^1 p_1(\varepsilon)\psi\right) \\ &= -i\left(\varphi_{,j}, G^{1j}\psi_{,1}\right) + \lim_{\varepsilon\to 0}\left(\varphi_{,i}, G^{ij} q^1 p_1(\varepsilon)\psi_{,j}\right)\end{aligned}$$

and similarly for the integral

$$\left(q^1 p_1\varphi, -\partial_i G^{ij}\,\partial_j\psi\right) = i\left(\varphi_{,1}, G^{1j}\psi_{,j}\right) + \lim_{\varepsilon\to 0}\left(p_1(\varepsilon)^*\varphi_{,i}, q^1 G^{ij}\psi_{,j}\right).$$

Since

$$\begin{aligned}\lim_{\varepsilon\to 0}\left(p_1(\varepsilon)^*\varphi_{,i}, q^1 G^{ij}\psi_{,j}\right) &= \lim_{\varepsilon\to 0}\left(\varphi_{,i}, p_1(\varepsilon) q^1 G^{ij}\psi_{,j}\right) \\ &= -i\big[\left(\varphi_{,i}, G^{ij}\psi_{,j}\right) + \left(\varphi_{,i}, q^1 G^{ij}_{\ ,1}\psi_{,j}\right)\big] \\ &\quad + \lim_{\varepsilon\to 0}\left(\varphi_{,i}, q^1 G^{ij} p_1(\varepsilon)\psi_{,j}\right),\end{aligned}$$

and

$$\left(q^1\varphi_{,1}, V\psi\right) = -\left(\varphi, \partial_1 q^1 V\psi\right) = -\left(\varphi, V\psi\right) - \left(\varphi, q^1 V_{,1}\psi\right) - \left(\varphi, q^1 V\psi_{,1}\right),$$

we finally obtain that

$$\mathcal{Q}(\varphi,\psi) = \left(\varphi_{,j}, G^{1j}\psi_{,1}\right) + \left(\varphi_{,1}, G^{1j}\psi_{,j}\right) - \left(\varphi_{,i}, q^1 G^{ij}_{\ ,1}\psi_{,j}\right) - \left(\varphi, q^1 V_{,1}\psi\right). \tag{13.12}$$

This implies that $\mathcal{Q}$ restricted to $\mathfrak{D}(H)_{\rm c}$ is continuous for the topology induced by $\mathcal{H}^1_0(\Omega)$. Now, $\mathfrak{D}(H)_{\rm c}$ is dense in $\mathcal{H}^1_0(\Omega)$ by Lemma 13.3.(ii). Thus, $\mathcal{Q}$ defines (by continuous extension) an operator in $\mathfrak{B}(\mathcal{H}^1_0(\Omega), \mathcal{H}^{-1}(\Omega))$, which we shall denote $i\,[H, A]$. Furthermore, using (13.12), we obtain (13.11) in $\mathfrak{B}(\mathcal{H}^1_0(\Omega), \mathcal{H}^{-1}(\Omega))$. □

Strict Mourre estimate for the free Hamiltonian

Now we prove that H_0 is of class $C^\infty(A)$ and A strictly conjugate to it on $\mathbb{R}\setminus\mathcal{T}$. So, let us first recall the following definition [2, Sec. 7.2.1 & 7.2.2]:

Definition 13.1. Let A, H be self-adjoint operators in a Hilbert space $\mathcal{H}$ with H of class $C^1(A)$. Furthermore, if $S, T \in \mathfrak{B}(\mathcal{H})$, we write $S \gtrsim T$ if there exists $K \in \mathcal{K}(\mathcal{H})$ so that $S \geq T + K$. Then, $\forall\lambda \in \mathbb{R}$,

$$\begin{aligned}\varrho^A_H(\lambda) &:= \sup\left\{a \in \mathbb{R} \,:\, \exists\varepsilon > 0 \ \text{ s.t. } \ E^H(\lambda;\varepsilon)\,i\,[H,A]\,E^H(\lambda;\varepsilon) \geq a E^H(\lambda;\varepsilon)\right\}, \\ \widetilde{\varrho}^A_H(\lambda) &:= \sup\left\{a \in \mathbb{R} \,:\, \exists\varepsilon > 0 \ \text{ s.t. } \ E^H(\lambda;\varepsilon)\,i\,[H,A]\,E^H(\lambda;\varepsilon) \gtrsim a E^H(\lambda;\varepsilon)\right\}\end{aligned}$$

where $E^H(\lambda;\varepsilon) := E^H((\lambda-\varepsilon, \lambda+\varepsilon))$ designates the spectral projection of H for the interval $(\lambda-\varepsilon, \lambda+\varepsilon)$.

We also need the following natural generalisation of [5, Thm. 3.4].

Theorem 13.2. *Let H_1, H_2 be two self-adjoint, bounded from below operators in the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. Assume that $A_\jmath$, $\jmath = 1, 2$, is a self-adjoint operator in $\mathcal{H}_\jmath$ such that $H_\jmath$ is of class $C^k(A_\jmath)$, $k \in (\mathbb{N}\setminus\{0\})\cup\{+\infty\}$. Let $H := H_1 \otimes 1 + 1 \otimes H_2$ and $A := A_1 \otimes 1 + 1 \otimes A_2$, which are self-adjoint operators in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then H is of class $C^k(A)$ and $\forall\lambda \in \mathbb{R}$:*

$$\varrho^A_H(\lambda) = \inf_{\lambda=\lambda_1+\lambda_2}\left[\varrho^{A_1}_{H_1}(\lambda_1) + \varrho^{A_2}_{H_2}(\lambda_2)\right].$$

Corollary 13.1. *$H_0 \in C^\infty(A)$ and*

$$\forall\lambda \in \mathbb{R}, \qquad \varrho^A_{H_0}(\lambda) = \begin{cases} 2\rho(\lambda) & \text{if} \quad \lambda \geq \nu_1 \\ +\infty & \text{if} \quad \lambda < \nu_1, \end{cases} \tag{13.13}$$

where $\rho(\lambda) := \lambda - \sup\{\zeta \in \mathcal{T} \,:\, \zeta \leq \lambda\}$ is strictly positive on $\mathbb{R}\setminus\mathcal{T}$.

Proof. $A_1 := A_{\shortparallel}$, $A_2 := 0$ are self-adjoint in $L^2(\mathbb{R})$, respectively $L^2(\omega)$. $H_1 := p_1^2$, $H_2 := -\Delta_{\mathrm{D}}^{\omega}$ are self-adjoint, bounded from below in $L^2(\mathbb{R})$, respectively $L^2(\omega)$. Clearly, [2, Ex. 6.2.8] $p_1^2 \in C^\infty(A_{\shortparallel})$ and $-\Delta_{\mathrm{D}}^{\omega} \in C^\infty(0)$. The first part of the claim and (13.13) then follows from Theorem 13.2. The expression for $\rho(\lambda)$ is a direct consequence of the respective behaviours of [2, Sec. 7.2.1] $\varrho_{p_1^2}^{A_{\shortparallel}}$ and $\varrho_{-\Delta_{\mathrm{D}}^{\omega}}^{0}$:

$$\begin{bmatrix} \varrho_{p_1^2}^{A_{\shortparallel}}(\lambda_1) \\ \varrho_{-\Delta_{\mathrm{D}}^{\omega}}^{0}(\lambda_2) \end{bmatrix} = \begin{cases} \begin{bmatrix} 2\lambda_1 \\ +\infty \end{bmatrix} & \text{if} \quad \begin{bmatrix} \lambda_1 \geq 0 \\ \lambda_1 < 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ +\infty \end{bmatrix} & \text{if} \quad \begin{bmatrix} \lambda_2 \in \mathcal{T} \\ \lambda_2 \in \mathbb{R} \setminus \mathcal{T} \end{bmatrix}. \end{cases}$$

□

Regularity of the Hamiltonian

In order to prove the regularity of H, we need two technical lemmas.

Lemma 13.4. *$\forall z \in \mathbb{R} \setminus \sigma(H)$, $\forall \vartheta \leq 1$, one has*

(i) $\big[R(z), \langle q^1 \rangle^{\vartheta}\big] \in \mathfrak{B}\left(\mathcal{H}(\Omega), \mathcal{H}_0^1(\Omega)\right)$,

(ii) $\forall i \in \{1, \ldots, d\}$, $\big[R(z), \langle q^1 \rangle^{\vartheta}\big]\, \partial_i \in \mathfrak{B}\left(\mathcal{H}(\Omega), \mathcal{H}_0^1(\Omega)\right)$.

This is established by adapting the proof of [2, Lemma 4.3] while next Lemma follows from the use of [2, Proof of Prop. 4.2].

Lemma 13.5. *Let $S \in \mathfrak{B}\left(\mathcal{H}(\Omega)\right)$ be self-adjoint and $\vartheta \in (0, 1]$, then*

$$\langle q^1 \rangle^{\vartheta} S \in \mathfrak{B}\left(\mathcal{H}(\Omega), \mathcal{H}^{\vartheta}(\mathbb{R}) \otimes L^2(\omega)\right) \implies S \in C^{\vartheta}(A).$$

(Note that the proof involves principally two facts. First, $S \in \mathfrak{B}\left(\mathcal{H}(\Omega), \mathfrak{D}(|A|^{\vartheta})\right)$ implies $S \in C^{\vartheta}(A)$. Second, the continuous imbedding $\mathcal{H}_{\vartheta}^{\vartheta}(\mathbb{R}) \subseteq \mathfrak{D}(|A_{\shortparallel}|^{\vartheta})$, which follows by real interpolation [2, Sec. 2.7] from the continuous imbedding $\mathcal{H}_1^1(\mathbb{R}) \subseteq \mathfrak{D}(|A_{\shortparallel}|)$.)

Remark 13.3. The facts that $i\,[H, A] \in \mathfrak{B}\left(\mathcal{H}_0^1(\Omega), \mathcal{H}^{-1}(\Omega)\right)$ and that $\mathcal{H}_0^1(\Omega)$ is stable under $\{\mathrm{e}^{iAt}\}_{t \in \mathbb{R}}$ imply *[2, Sec. 6.3]* that $H \in C^1(A)$.

Proposition 13.3. *$\exists \vartheta \in (0, 1]$ such that $H \in C^{1+\vartheta}(A)$.*

Proof. We show that each term appearing in the expression for $B := i\,[H, A]$ is at least of class $C^{\gamma}(A)$ for a certain $\gamma \in (0, 1]$.

Consider first $B_1 := -\partial_j\, G^{j1}\, \partial_1 - \partial_1\, G^{1j}\, \partial_j$. An explicit calculation (analogous to that of the proof of Proposition 13.2) implies that

$$i\,[B_1, A] = -2\partial_1 G^{11} \partial_1 - \partial_1 G^{1j} \partial_j - \partial_j G^{j1} \partial_1 + \partial_j q^1 G^{j1}_{\ ,1} \partial_1 + \partial_1 q^1 G^{1j}_{\ ,1} \partial_j$$

as operators in $\mathfrak{B}(\mathcal{H}_0^1(\Omega), \mathcal{H}^{-1}(\Omega))$. Thus, $B_1 \in C^1(A)$ by Remark 13.3.

Let $z \in \mathbb{R} \setminus \sigma(H)$. As a consequence of the fact that $H \in C^1(A)$, one can interpret $i\,[A, R(z)]$ as the product of [2, Sec. 6.2.2] three bounded operators, *viz.* $R(z) : \mathcal{H}(\Omega) \to \mathfrak{D}(H)$, $B : \mathfrak{D}(H) \to \mathfrak{D}(H)^*$ and $R(z) : \mathfrak{D}(H)^* \to \mathcal{H}(\Omega)$. Thus, using Proposition 13.2, one can write as an operator identity in $\mathfrak{B}\left(\mathcal{H}(\Omega)\right)$

$$\begin{aligned} i\,[A, R(z)] = R(z) B R(z) = {} & R(z) B_1 R(z) + R(z) \partial_i q^1 G^{ij}_{\ ,1} \partial_j R(z) \\ & - R(z) q^1 V_{,1} R(z). \end{aligned}$$

Since the first term has already been shown to be bounded, it is enough to prove that the second and third terms on the r.h.s. are of class $C^{\gamma}(A)$ for some $\gamma \in (0, 1]$.

We employ Lemma 13.5 with $\vartheta := \min\{\vartheta_1, \vartheta_2\}$ in order to deal with both terms. Using some commutation relations, we get

$$\begin{aligned} \langle q^1 \rangle^{\vartheta} R(z) \partial_i q^1 G^{ij}_{\ ,1} \partial_j\, R(z) = {} & R(z) \partial_i \langle q^1 \rangle^{\vartheta} q^1 G^{ij}_{\ ,1} \partial_j\, R(z) \\ & - \big[R(z), \langle q^1 \rangle^{\vartheta}\big]\, \partial_i q^1 G^{ij}_{\ ,1} \partial_j\, R(z) \\ & - R(z) \left[\partial_i, \langle q^1 \rangle^{\vartheta}\right] q^1 G^{ij}_{\ ,1} \partial_j\, R(z). \end{aligned}$$

Under Assumption 13.1.3, the first term on the r.h.s. is in $\mathfrak{B}(\mathcal{H}(\Omega), \mathcal{H}^1_0(\Omega))$. The second and the last one are in $\mathfrak{B}(\mathcal{H}(\Omega), \mathcal{H}^1_0(\Omega))$ by Lemma 13.4.(ii) and the boundedness of $\langle q^1\rangle^\vartheta_{,1}$, respectively. Moreover,

$$\langle q^1\rangle^\vartheta R(z) q^1 V_{,1} R(z) = R(z)\langle q^1\rangle^\vartheta q^1 V_{,1} R(z) + \left[\langle q^1\rangle^\vartheta, R(z)\right] q^1 V_{,1} R(z)$$

is in $\mathfrak{B}\left(\mathcal{H}(\Omega), \mathcal{H}^1_0(\Omega)\right)$ by Assumption 13.2.3 and Lemma 13.4.(i). Thus, all the terms in the expression of B are at least of class $C^\vartheta(A)$. This implies the claim. □

The main result

Proposition 13.4. $\forall \lambda \in \mathbb{R} \setminus \mathcal{T},\ \widetilde{\varrho}^A_H(\lambda) > 0$.

Proof. Corollary 13.1 and Proposition 13.3 imply that both H_0 and H are of class $C^1_u(A)$. Furthermore, $R(i) - R_0(i)$ is compact by Proposition 13.1, with the result that $\widetilde{\varrho}^A_H = \widetilde{\varrho}^A_{H_0}$ due to [2, Thm. 7.2.9]. Finally, since [2, Prop. 7.2.6] $\widetilde{\varrho}^A_{H_0} \geq \varrho^A_{H_0}$, we can conclude using Corollary 13.1. □

Summing up, we result in the following spectral properties of H.

Theorem 13.3. *Let ω be a bounded open connected set in $\mathbb{R}^{d-1}$, $d \geq 2$, and denote by $\mathcal{T}$ the set of eigenvalues of $-\Delta^\omega_{\mathrm{D}}$. Let H be the operator* (13.3) *with $\Omega := \mathbb{R} \times \omega$, subject to Dirichlet boundary conditions, and satisfying* Assumptions 13.1 *and* 13.2. *Then*

(i) $\sigma_{\mathrm{ess}}(H) = [\kappa, \infty)$, *where* $\kappa := \inf \mathcal{T}$,

(ii) $\sigma_{\mathrm{sc}}(H) = \varnothing$,

(iii) $\sigma_{\mathrm{p}}(H) \cup \mathcal{T}$ *is closed and countable,*

(iv) $\sigma_{\mathrm{p}}(H) \setminus \mathcal{T}$ *is composed of finitely degenerated eigenvalues, which can accumulate at the points of $\mathcal{T}$ only,*

Proof. The claim (i) is included in Proposition 13.1. Since A is conjugate to H on $\mathbb{R} \setminus \mathcal{T}$ by Proposition 13.4, the assertions (ii)–(iv) follow by the abstract conjugate operator method [2, Thm. 7.4.2]. □

To conclude this section, let us remark that Assumptions 13.1.3 and 13.2.3 could be weakened. Firstly, we recall that the situation with $V = 0$ and $G = \rho 1$, ρ being a real-valued function greater than a strictly positive constant, is investigated in [3, 10] where the authors admit local singularities of ρ. More specifically, one assumes that $\rho = \rho_{\mathrm{s}} + \rho_\ell$, where ρ_ℓ is the part satisfying a condition analogous to Assumption 13.1.3, while ρ_{s} need not be differentiable. (In [3], $\mathrm{supp}(\rho_{\mathrm{s}})$ is assumed to be compact. The result of [10] is better in the sense that ρ_s is only supposed to be a short-range perturbation there. However, this requires strengthening of the condition analogous to Assumption 13.1.2 about the decay of ρ at infinity.) Secondly, the optimal conditions one has to impose on the potential of a Schrödinger operator are known [6, 2].

13.3 Curved tubes

In this part, we use Theorem 13.3 in order to find geometric sufficient conditions which guarantee that the spectral results of the theorem hold true for curved tubes.

13.3.1 Geometric preliminaries

The reference curve

Given $d \geq 2$, let $p : \mathbb{R} \to \mathbb{R}^d$ be a regular unit-speed smooth (*i.e.*, C^∞-smooth) curve satisfying the following hypothesis.

Assumption 13.3. There exists a collection of d smooth mappings $e_i : \mathbb{R} \to \mathbb{R}^d$ with the following properties:

1. $\forall i, j \in \{1, \ldots, d\}, \forall s \in \mathbb{R},\ e_i(s) \cdot e_j(s) = \delta_{ij}$,
2. $\forall i \in \{1, \ldots, d-1\}, \forall s \in \mathbb{R},$ the i^{th} derivative $p^{(i)}(s)$ of $p(s)$ lies in the span of $e_1(s), \ldots, e_i(s)$,
3. $e_1 = \dot{p}$,
4. $\forall s \in \mathbb{R},\ \{e_1(s), \ldots, e_d(s)\}$ has the positive orientation,
5. $\forall i \in \{1, \ldots, d-1\}, \forall s \in \mathbb{R},\ \dot{e}_i(s)$ lies in the span of $e_1(s), \ldots, e_{i+1}(s)$.

Here and in the sequel, "$\cdot$" denotes the inner product in $\mathbb{R}^d$.

Remark 13.4. A vector field with the property 1 is called a *moving frame* along p and it is a *Frenet frame* if it satisfies 2 in addition, *cf [42, Sec. 1.2]*. A sufficient condition to ensure the existence of the frame of *Assumption 13.3* is to require that *[42, Prop. 1.2.2]*, for all $s \in \mathbb{R}$, the vectors $\dot{p}(s), p^{(2)}(s), \ldots, p^{(d-1)}(s)$ are linearly independent. This is always satisfied if $d = 2$. However, we do not assume *a priori* the above non-degeneracy condition for $d \geq 3$ because it excludes the curves such that, for some open $I \subseteq \mathbb{R}$, $p \restriction I$ lies in a lower-dimensional subspace of $\mathbb{R}^d$.

The properties of $\{e_1, \ldots, e_d\}$ summarised in Assumption 13.3 yield [42, Sec. 1.3] the Serret-Frenet formulae,

$$\dot{e}_i = \mathcal{K}_i{}^j e_j \tag{13.14}$$

with $\mathcal{K} \equiv \big(\mathcal{K}_i{}^j\big)$ being a skew-symmetric $d \times d$ matrix defined by

$$\mathcal{K} := \begin{pmatrix} 0 & \kappa_1 & & 0 \\ -\kappa_1 & \ddots & \ddots & \\ & \ddots & \ddots & \kappa_{d-1} \\ 0 & & -\kappa_{d-1} & 0 \end{pmatrix}. \tag{13.15}$$

Here κ_i is called the i^{th} *curvature* of p. Under our Assumption 13.3, the curvatures are smooth functions of the arc-length parameter $s \in \mathbb{R}$.

The appropriate moving frame

In this subsection, we introduce another moving frame along p, which better reflects the geometry of the curve, and will be used later to define a tube about it. We shall refer to it as the *Tang frame* because it is a natural generalisation of the Tang frame known from the theory of 3-dimensional waveguides [31, 17, 11]. Our construction follows the generalisation introduced in [8].

Let the $(d-1) \times (d-1)$ matrix $(\mathcal{R}_\mu{}^\nu)$ be defined by the system of differential equations

$$\dot{\mathcal{R}}_\mu{}^\nu + \mathcal{R}_\mu{}^\alpha \mathcal{K}_\alpha{}^\nu = 0 \tag{13.16}$$

with $(\mathcal{R}_\mu{}^\nu(s_0))$ being a rotation matrix in $\mathbb{R}^{d-1}$ for some $s_0 \in \mathbb{R}$ as initial condition, *i.e.*,

$$\det\big(\mathcal{R}_\mu{}^\nu(s_0)\big) = 1 \qquad \text{and} \qquad \delta_{\alpha\beta}\mathcal{R}_\mu{}^\alpha(s_0)\mathcal{R}_\nu{}^\beta(s_0) = \delta_{\mu\nu}. \tag{13.17}$$

The solution of (13.16) exists and is smooth by standard arguments in the theory of differential equations (*cf* [25, Sec. 4]). Furthermore, the conditions (13.17) are satisfied for *all* $s_0 \in \mathbb{R}$. Indeed, by means of Liouville's formula [25, Thm. 4.7.1] and $\mathrm{tr}(\mathcal{K}) = 0$, one checks that $\det\big(\mathcal{R}_\mu{}^\nu\big) = 1$ identically, while the validity of the second condition for all $s_0 \in \mathbb{R}$ is obtained via the skew-symmetry of $\mathcal{K}$:

$$\big(\delta_{\alpha\beta}\mathcal{R}_\mu{}^\alpha\mathcal{R}_\nu{}^\beta\big)^{\cdot} = -\mathcal{R}_\mu{}^\alpha\big(\delta_{\gamma\beta}\mathcal{K}_\alpha{}^\gamma + \delta_{\alpha\gamma}\mathcal{K}_\beta{}^\gamma\big)\mathcal{R}_\nu{}^\beta = 0.$$

We set

$$\mathcal{R} \equiv \big(\mathcal{R}_i{}^j\big) := \begin{pmatrix} 1 & 0 \\ 0 & \big(\mathcal{R}_\mu{}^\nu\big) \end{pmatrix}$$

and introduce the Tang frame as the moving frame $\{\tilde{e}_1, \ldots, \tilde{e}_d\}$ along p defined by

$$\tilde{e}_i := \mathcal{R}_i{}^j e_j. \tag{13.18}$$

Combining (13.14) with (13.16), one easily finds

$$\dot{\tilde{e}}_1 = \kappa_1 e_2 \qquad \text{and} \qquad \dot{\tilde{e}}_\mu = \mathcal{R}_\mu{}^\alpha \mathcal{K}_\alpha{}^1 e_1 = -\kappa_1 \mathcal{R}_\mu{}^2 e_1. \tag{13.19}$$

The interest of the Tang frame will appear in the following subsection.

The tube

Let ω be a bounded open connected set in $\mathbb{R}^{d-1}$. Without loss of generality, we assume that ω is translated so that its centre of mass is at the origin. Let $\Omega := \mathbb{R} \times \omega$ be a straight tube. We define the curved tube Γ of the same cross-section ω about p as the image of the mapping

$$\mathfrak{L} : \Omega \to \mathbb{R}^d, \qquad \big(s, u^2, \dots, u^d\big) \mapsto p(s) + \tilde{e}_\mu(s) u^\mu, \tag{13.20}$$

i.e., $\Gamma := \mathfrak{L}(\Omega)$.

As already mentioned in Introduction, the shape of the curved tube Γ of cross-section ω about p depends on the choice of rotations $(\mathcal{R}_\mu^{\ \nu})$ in (13.18), unless ω is rotation invariant. As usual in the theory of quantum waveguides (see, *e.g.*, [11, 8]), we restrict ourselves to the technically most advantageous choice determined by (13.16), *i.e.*, when the cross-section ω rotates along p w.r.t. the Tang frame (another choice can be found in [14]).

We write $u \equiv (u^2, \dots, u^d)$, define $a := \sup_{u \in \omega} |u|$ and always assume

Assumption 13.4.

1. $\kappa_1 \in L^\infty(\mathbb{R})$ and $a\|\kappa_1\|_\infty < 1$,
2. Γ does not overlap itself.

Then, the mapping $\mathfrak{L} : \Omega \to \Gamma$ is a diffeomorphism. Indeed, by virtue of the inverse function theorem, the first condition guarantees that it is a local diffeomorphism which is global through the injectivity induced by the second condition. Consequently, $\mathfrak{L}^{-1}$ determines a system of global (*geodesic* or *Fermi*) "coordinates" (s, u). At the same time, the tube Γ can be identified with the Riemannian manifold (Ω, g), where $g \equiv (g_{ij})$ is the metric tensor induced by the immersion (13.20), that is $g_{ij} := \mathfrak{L}_{,i} \cdot \mathfrak{L}_{,j}$. The formulae (13.19) yield

$$g = \mathrm{diag}\left(h^2, 1, \dots, 1\right) \qquad \text{with} \qquad h(s,u) := 1 + u^\mu \mathcal{R}_\mu^{\ \alpha}(s) \mathcal{K}_\alpha^{\ 1}(s). \tag{13.21}$$

Note that the metric tensor (13.21) is diagonal due to our special choice of the "transverse" frame $\{\tilde{e}_2, \dots, \tilde{e}_d\}$, which is the advantage of the Tang frame.

We set $|g| := \det(g) = h^2$, which defines through $\mathrm{d}v := h(s,u)\,\mathrm{d}s\,\mathrm{d}u$ the volume element of Γ; here $\mathrm{d}u$ denotes the $(d-1)$-dimensional Lebesgue measure in ω.

Remark 13.5 (Low-dimensional examples)**.** When $d = 2$, the cross-section ω is just the interval $(-a, a)$, the curve p has only one curvature $\kappa := \kappa_1$, the rotation matrix $(\mathcal{R}_\mu^{\ \nu})$ equals (the scalar) 1 and

$$h(s,u) = 1 - \kappa(s) u.$$

If $d = 3$, it is convenient to make the Ansatz

$$(\mathcal{R}_\mu^{\ \nu}) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

where α is a real-valued differentiable function. Then, it is easy to see that *(13.16)* reduces to the differential equation $\dot{\alpha} = \tau$, where τ is the torsion of p, *i.e.*, one puts $\kappa := \kappa_1$ and $\tau := \kappa_2$. Choosing α as an integral of τ, we can write

$$h(s,u) = 1 - \kappa(s) \left[u^2 \cos\alpha(s) + u^3 \sin\alpha(s)\right].$$

Remark 13.6 (On Assumption 13.4)**.** If p were a compact embedded curve, then *Assumption 13.4* could always be achieved for sufficiently small a. In general, however, one cannot exclude self-intersections of the tube using the local geometry of an embedded curve p only. One way to avoid this disadvantage would be to consider (Ω, g) as an abstract Riemannian manifold where only the curve p is embedded in $\mathbb{R}^d$. Nonetheless, in the present paper, we prefer to assume *Assumption 13.4.2 a priori* because Γ does not have a physical meaning if it is self-intersecting. Finding global geometric conditions on p ensuring the validity of *Assumption 13.4.2* is an interesting question, which is beyond the scope of the present paper, however.

13.3.2 The Laplacian

Our object of interest is the Dirichlet Laplacian (13.2), with Γ defined by (13.20). We construct it as follows. Using the diffeomorphism (13.20), we identify the Hilbert space $L^2(\Gamma)$ with $L^2(\Omega, \mathrm{d}v)$ and consider on the latter the Dirichlet form

$$\widetilde{Q}(\varphi, \psi) := \int_\Omega \overline{\varphi_{,i}} g^{ij} \psi_{,j} \,\mathrm{d}v, \qquad \varphi, \psi \in \mathfrak{D}(\widetilde{Q}) := \mathcal{H}_0^1(\Omega, \mathrm{d}v), \tag{13.22}$$

where $(g^{ij}) := g^{-1}$. The form $\widetilde{Q}$ is clearly densely defined, non-negative, symmetric and closed on its domain. Consequently, there exists a unique non-negative self-adjoint operator $\widetilde{H}$ satisfying $\mathfrak{D}(\widetilde{H}) \subset \mathfrak{D}(\widetilde{Q})$ associated with $\widetilde{Q}$.We have

$$\widetilde{H}\psi = -|g|^{-1/2}\partial_i|g|^{1/2}g^{ij}\partial_j\psi, \tag{13.23}$$

$$\psi \in \mathfrak{D}(\widetilde{H}) = \big\{\psi \in \mathcal{H}^1_0(\Omega, \mathrm{d}v) \,:\, \partial_i|g|^{1/2}g^{ij}\partial_j\psi \in L^2(\Omega, \mathrm{d}v)\big\}. \tag{13.24}$$

That is, $\widetilde{H}$ is the Laplacian (13.2) expressed in the coordinates (s, u).

In order to apply Theorem 13.3, we transform $\widetilde{H}$ into a unitarily equivalent operator H of the form (13.3) acting on the Hilbert space $\mathcal{H}(\Omega) := L^2(\Omega)$, without the additional weight $|g|^{1/2}$ in the volume element. This is achieved by means of the unitary mapping $\mathcal{U} : L^2(\Omega, \mathrm{d}v) \to \mathcal{H}(\Omega), \psi \mapsto |g|^{1/4}\psi$. Defining $H := \mathcal{U}\widetilde{H}\mathcal{U}^{-1}$, one has

$$H\psi = -|g|^{-1/4}\partial_i|g|^{1/2}g^{ij}\partial_j|g|^{-1/4}\psi, \tag{13.25}$$

$$\psi \in \mathfrak{D}(H) = \big\{\psi \in \mathcal{H}^1_0(\Omega) \,:\, \partial_i|g|^{1/2}g^{ij}\partial_j|g|^{-1/4}\psi \in L^2(\Omega)\big\}. \tag{13.26}$$

Commuting $|g|^{-1/4}$ with the gradient components in the expression for H, we obtain on $\mathfrak{D}(H)$

$$H = -\partial_i g^{ij}\partial_j + V, \tag{13.27}$$

where

$$V := -\frac{5}{4}\frac{(h_{,1})^2}{h^4} + \frac{1}{2}\frac{h_{,11}}{h^3} - \frac{1}{4}\frac{\delta^{\mu\nu}h_{,\mu}h_{,\nu}}{h^2} + \frac{1}{2}\frac{\delta^{\mu\nu}h_{,\mu\nu}}{h}. \tag{13.28}$$

Actually, (13.27) with (13.28) is a general formula valid for any smooth metric of the form $g = \mathrm{diag}(h^2, 1, \ldots, 1)$. In our special case with h given by (13.21), we find that $h_{,\mu\nu} = 0$, $\delta^{\mu\nu}h_{,\mu}h_{,\nu} = \delta^{\alpha\beta}\mathcal{K}_\alpha{}^1\mathcal{K}_\beta{}^1$ by (13.17), while (13.16) gives

$$\begin{aligned} h_{,1}(\cdot, u) &= u^\mu \mathcal{R}_\mu{}^\alpha\big(\dot{\mathcal{K}}_\alpha{}^1 - \mathcal{K}_\alpha{}^\beta\mathcal{K}_\beta{}^1\big), \\ h_{,11}(\cdot, u) &= u^\mu \mathcal{R}_\mu{}^\alpha\big(\ddot{\mathcal{K}}_\alpha{}^1 - \dot{\mathcal{K}}_\alpha{}^\beta\mathcal{K}_\beta{}^1 - 2\mathcal{K}_\alpha{}^\beta\dot{\mathcal{K}}_\beta{}^1 + \mathcal{K}_\alpha{}^\beta\mathcal{K}_\beta{}^\gamma\mathcal{K}_\gamma{}^1\big). \end{aligned} \tag{13.29}$$

13.3.3 Results

It remains to impose decay conditions on the curvatures of p (and their derivatives) in order that the operator (13.27) satisfies Assumption 13.1 and Assumption 13.2.

Let us first consider the more general situation where the matrix (g^{ij}) is equal to $\mathrm{diag}(h^{-2}, 1, \ldots, 1)$ with the explicit dependence of h on s and u not specified. One shows that it is sufficient to impose the following hypotheses.

Assumption 13.5. Uniformly for $u \in \omega$,

1. $h(s, u) \longrightarrow 1$ as $|s| \to \infty$,
2. $h_{,11}(s, u)$, $(\delta^{\mu\nu}h_{,\mu}h_{,\nu})(s, u)$, $\delta^{\mu\nu}h_{,\mu\nu}(s, u) \longrightarrow 0$ as $|s| \to \infty$,
3. $\exists \vartheta \in (0, 1]$ s.t.

$$h_{,1}(s, u),\ h_{,111}(s, u),\ (\delta^{\mu\nu}h_{,\mu}h_{,\nu})_{,1}(s, u),\ \delta^{\mu\nu}h_{,1\mu\nu}(s, u) = \mathcal{O}\big(|s|^{-(1+\vartheta)}\big).$$

Indeed, the first hypothesis supplies Assumption 13.1.2, while Assumption 13.1.1 is fulfilled due to basic Assumption 13.4. Next, since h is a smooth function, Assumption 13.5.2 together with the behaviour of $h_{,1}$ in Assumption 13.5.3 are sufficient to ensure both Assumption 13.2.1 and Assumption 13.2.2. It is also clear that the asymptotic behaviour of $h_{,1}$ in Assumption 13.5.3 supplies Assumption 13.1.3. Assumption 13.1.4 holds true due to Assumption 13.1.3 and the particular form of (g^{ij}). It remains to check Assumption 13.2.3. This is easily done by calculating the derivative of the potential (13.28):

$$\begin{aligned} V_{,1} = {} & 5\frac{(h_{,1})^3}{h^5} - 4\frac{h_{,1}h_{,11}}{h^4} + \frac{h_{,111}}{2h^3} \\ & + \frac{\delta^{\mu\nu}}{2}\left(\frac{h_{,1}h_{,\mu}h_{,\nu}}{h^3} - \frac{h_{,1}h_{,\mu\nu} + h_{,1\mu}h_{,\nu}}{h^2} + \frac{h_{,1\mu\nu}}{h}\right). \end{aligned}$$

With h given by (13.21), we find in addition to (13.29) that $h_{,1\mu\nu}=0$ and

$$\begin{aligned}(\delta^{\mu\nu}h_{,\mu}h_{,\nu})_{,1} &= 2\delta^{\alpha\beta}\dot{\mathcal{K}}_\alpha{}^1\mathcal{K}_\beta{}^1\\ h_{,111}(\cdot,u) &= u^\mu\mathcal{R}_\mu{}^\alpha\big(\dddot{\mathcal{K}}_\alpha{}^1-\ddot{\mathcal{K}}_\alpha{}^\beta\mathcal{K}_\beta{}^1-3\mathcal{K}_\alpha{}^\beta\ddot{\mathcal{K}}_\beta{}^1-3\dot{\mathcal{K}}_\alpha{}^\beta\dot{\mathcal{K}}_\beta{}^1+\dot{\mathcal{K}}_\alpha{}^\beta\mathcal{K}_\beta{}^\gamma\mathcal{K}_\gamma{}^1\\ &\qquad +2\mathcal{K}_\alpha{}^\beta\dot{\mathcal{K}}_\beta{}^\gamma\mathcal{K}_\gamma{}^1+3\mathcal{K}_\alpha{}^\beta\mathcal{K}_\beta{}^\gamma\dot{\mathcal{K}}_\gamma{}^1-\mathcal{K}_\alpha{}^\beta\mathcal{K}_\beta{}^\gamma\mathcal{K}_\gamma{}^\delta\mathcal{K}_\delta{}^1\big).\end{aligned}$$

Since $|u^\mu\mathcal{R}_\mu{}^\alpha|<a$, Assumption 13.5 holds true provided we impose the following conditions on the curvatures

Assumption 13.6.

1. $\forall\alpha\in\{2,\dots,d\},\ \mathcal{K}_\alpha{}^1(s),\ \ddot{\mathcal{K}}_\alpha{}^1(s)\longrightarrow 0$ as $|s|\to\infty$,
2. $\forall\alpha,\beta\in\{2,\dots,d\},\ \mathcal{K}_\alpha{}^\beta,\ \dot{\mathcal{K}}_\alpha{}^2\in L^\infty(\mathbb{R})$,
3. $\exists\vartheta\in(0,1]$ s.t. $\forall\alpha\in\{2,\dots,d\}$,

$$\dot{\mathcal{K}}_\alpha{}^1(s),\ \dddot{\mathcal{K}}_\alpha{}^1(s),\ \mathcal{K}_\alpha{}^2(s),\ \ddot{\mathcal{K}}_\alpha{}^2(s),\ \big(\dot{\mathcal{K}}_\alpha{}^\beta\mathcal{K}_\beta{}^2\big)(s),\ \big(\mathcal{K}_\alpha{}^\beta\dot{\mathcal{K}}_\beta{}^2\big)(s)=\mathcal{O}\big(|s|^{-(1+\vartheta)}\big).$$

Remark 13.7. These conditions reduce to those of *Theorem 13.1* provided $d=2$. When $d=3$, it is sufficient to assume the conditions of *Theorem 13.1* for the first curvature, and $\dot\kappa_2\in L^\infty(\mathbb{R})$ and $\kappa_2(s),\ddot\kappa_2(s)=\mathcal{O}\left(|s|^{-(1+\vartheta)}\right)$ for some $\vartheta\in(0,1]$.

We conclude this section by applying Theorem 13.3.

Theorem 13.4. *Let Γ be a tube defined via* (13.20) *about a smooth infinite curve embedded in $\mathbb{R}^d$. Suppose* Assumptions 13.3*,* 13.4 *and* 13.6*. Then all the spectral results* (i)–(iv) *of* Theorem 13.3 *hold true for the Dirichlet Laplacian on $L^2(\Gamma)$.*

13.4 Curved strips on surfaces

In this final section, we investigate the situation where the ambient space is a general Riemannian manifold instead of the Euclidean space $\mathbb{R}^d$. We restrict ourselves to $d=2$, *i.e.*, Γ is a strip around an infinite curve in an (abstract) two-dimensional surface. We refer to [23] for basic spectral properties of $-\Delta_{\mathrm{D}}^\Gamma$ and geometric details.

13.4.1 Preliminaries

Let $\mathcal{A}$ be a smooth connected complete non-compact two-dimensional Riemannian manifold of bounded Gauss curvature K. Let $p:\mathbb{R}\to\mathcal{A}$ be a smooth unit-speed curve embedded in $\mathcal{A}$ with (geodesic) curvature κ and denote by $n:\mathbb{R}\to T_{p(\cdot)}\mathcal{A}$ a smooth unit normal vector field along p. Given $a>0$, we consider the straight strip $\Omega:=\mathbb{R}\times(-a,a)$ and define a curved strip Γ of same width over p as a a-tubular neighbourhood of p in $\mathcal{A}$ by

$$\Gamma:=\mathfrak{L}(\Omega),\qquad\text{where}\qquad\mathfrak{L}:(s,u)\mapsto\exp_{p(s)}(un(s)).\tag{13.30}$$

Note that $s\mapsto\mathfrak{L}(s,u)$ traces the curves parallel to p at a fixed distance $|u|$, while the curve $u\mapsto\mathfrak{L}(s,u)$ is a unit-speed geodesic orthogonal to p for any fixed s. We always assume

Assumption 13.7. $\mathfrak{L}:\Omega\to\Gamma$ is a diffeomorphism,

Then $\mathfrak{L}^{-1}$ determines a system of Fermi "coordinates" (s,u), *i.e.*, the geodesic coordinates based on p. The metric tensor of Γ in these coordinates acquires [32, Sec. 2.4] the diagonal form

$$g(s,u)=\mathrm{diag}\left(h^2(s,u),1\right),\tag{13.31}$$

where h is a smooth function satisfying the Jacobi equation

$$h_{,22}+Kh=0\qquad\text{with}\qquad\begin{cases}h(\cdot,0)=1\\ h_{,2}(\cdot,0)=-\kappa.\end{cases}\tag{13.32}$$

Here K and κ are considered as functions of the Fermi coordinates (the sign of κ being uniquely determined up to the re-parameterisation $s\mapsto -s$ or the choice of n). The determinant of the metric tensor, $|g|:=\det(g)=h^2$, defines through $\mathrm{d}v:=h(s,u)\mathrm{d}s\mathrm{d}u$ the area element of the strip.

Assuming that the metric g is uniformly elliptic in the sense that

Assumption 13.8. $\exists c_\pm \in (0,\infty)$ s.t. $\forall (s,u) \in \Omega,\ c_- \leq h(s,u) \leq c_+$

holds true, the Dirichlet Laplacian corresponding to Γ can be defined in the same way as in Section 13.3.2, *i.e.*, as the operator $\widetilde{H}$ associated with the form (13.22), satisfying (13.23). At the same time, we may introduce the unitarily equivalent operator H on $L^2(\Omega)$ given by (13.25) and satisfying (13.27) with (13.28).

Remark 13.8. If *Assumption 13.8* holds true, then the inverse function theorem together with *(13.32)* yield that *Assumption 13.7* is satisfied for all sufficiently small a provided the strip Γ does not overlap itself. *Assumption 13.8* is satisfied, for instance, if Γ is a sufficiently thin strip on a ruled surface, *cf [45, Sec. 7].*

13.4.2 Results

In view of the more general approach in the beginning of Section 13.3.3, we see that Assumption 13.5 (with $d = 2$) guarantees Assumptions 13.1 and 13.2 also in the present case. Applying Theorem 13.3, we obtain, with $\mathcal{T} = \{n^2\nu_1\}_{n=1}^\infty$ where $\nu_1 := \pi^2/(2a)^2$, the following result

Theorem 13.5. *Let Γ be a tubular neighbourhood of radius $a > 0$ about a smooth infinite curve, which is embedded in a smooth connected complete non-compact surface of bounded curvature. Suppose* Assumptions 13.7, 13.8 *and* 13.5. *Then all the spectral results* (i)–(iv) *of* Theorem 13.3 *hold true for the Dirichlet Laplacian on $L^2(\Gamma)$.*

Assume now that the strip is *flat* in the sense of [23], *i.e.*, the curvature K is equal to zero everywhere on Γ. Then the Jacobi equation (13.32) has the explicit solution (*cf* (13.21) for $d = 2$)

$$h(s,u) = 1 - \kappa(s)u \tag{13.33}$$

and Assumption 13.5 can be replaced by some conditions on the decay of the curvature κ at infinity, namely, we adopt Assumption 13.6 with $\kappa_1 \equiv \kappa$ and $\mathcal{K}_\mu^{\ \nu} = 0$ (*cf* the assumptions of Theorem 13.1). At the same time, it easy to see that Assumption 13.7 and 13.8 are satisfied if Assumption 13.4 holds true.

Theorem 13.6 (Flat strips)**.** *Let Γ be a tubular neighbourhood of radius $a > 0$ about a smooth infinite curve of curvature κ, which is embedded in a smooth connected complete non-compact surface of bounded curvature K such that $K \restriction \Gamma = 0$. Suppose* Assumption 13.4 *and*

1. $\kappa(s),\ \ddot{\kappa}(s) \longrightarrow 0$ *as* $|s| \to \infty$,
2. $\exists \vartheta \in (0,1]$ *s.t.* $\dot{\kappa}(s),\ \dddot{\kappa}(s) = \mathcal{O}\left(|s|^{-(1+\vartheta)}\right)$.

Then, all the spectral results (i)–(iv) *of* Theorem 13.3 *hold true for the Dirichlet Laplacian on $L^2(\Gamma)$.*

Acknowledgements

One of the authors (D.K.) would like to thank M. Măntoiu and R. Purice for proposing to him to use the conjugate operator method in the theory of quantum waveguides and for many discussions on the topic. We are very grateful to W. Amrein for his careful reading of the present paper and S. Richard for his useful suggestions and comments. This work was partially supported by FCT/POCTI/FEDER, Portugal, GA AS CR grant IAA 1048101 and the Fonds National Suisse.

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Chapter 14

The Hardy inequality and the heat equation in twisted tubes

Published in: Journal de Mathématiques Pures et Appliquées **94** (2010), 277–303

Joint work with: Enrique Zuazua

The Hardy inequality and the heat equation in twisted tubes

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Abstract. We show that a twist of a three-dimensional tube of uniform cross-section yields an improved decay rate for the heat semigroup associated with the Dirichlet Laplacian in the tube. The proof employs Hardy inequalities for the Dirichlet Laplacian in twisted tubes and the method of self-similar variables and weighted Sobolev spaces for the heat equation.

Résumé. Nous montrons que la torsion d'un tube non-borné à section transversale constante dans l'espace euclidien trois-dimensionnel induit une amélioration du taux de décroissance pour le semigroupe associé à l'equation de la chaleur avec des conditions aux limites de Dirichlet dans le tube. La preuve utilise des inégalités de type Hardy pour le laplacien Dirichlet dans les tubes torsadés et la méthode de variables de similarité et les espaces de Sobolev à poids gaussiens pour l'équation de la chaleur.

14.1 Introduction

It has been shown recently in [7] that a local twist of a straight three-dimensional tube $\Omega_0 := \mathbb{R} \times \omega$ of non-circular cross-section $\omega \subset \mathbb{R}^2$ leads to an effective repulsive interaction in the Schrödinger equation of a quantum particle constrained to the twisted tube Ω_θ. More precisely, there is a Hardy-type inequality for the particle Hamiltonian modelled by the Dirichlet Laplacian $-\Delta_D^{\Omega_\theta}$ at its threshold energy E_1 if, and only if, the tube is twisted (*cf* Figure 14.1). That is, the inequality

$$-\Delta_D^{\Omega_\theta} - E_1 \geq \varrho \tag{14.1}$$

holds true, in the sense of quadratic forms in $L^2(\Omega_\theta)$, with a positive function ϱ provided that the tube is twisted, while ϱ is necessarily zero for Ω_0. Here E_1 coincides with the first eigenvalue of the Dirichlet Laplacian $-\Delta_D^\omega$ in the cross-section ω.

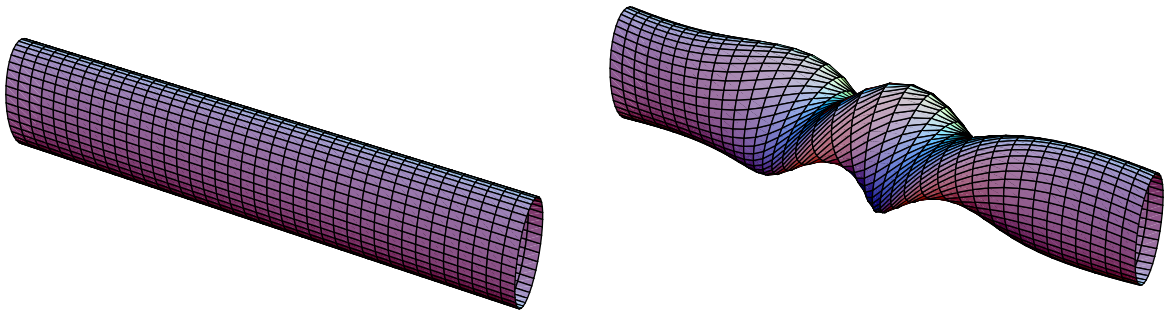

Figure 14.1: Untwisted and twisted tubes of elliptical cross-section.

The inequality (14.1) has important consequences for conductance properties of quantum waveguides. It clearly implies the absence of bound states (*i.e.*, stationary solutions to the Schrödinger equation) below the energy E_1 even if the particle is subjected to a small attractive interaction, which can be either of potential or geometric origin (*cf* [7] for more details). At the same time, a repulsive effect of twisting on eigenvalues embedded in the essential spectrum has been demonstrated in [14]. Hence, roughly speaking, the twist prevents the particle to be trapped in the waveguide. Additional spectral properties of twisted tubes have been studied in [9, 18, 2].

It is natural to ask whether the repulsive effect of twisting demonstrated in [7] in the quantum context has its counterpart in other areas of physics, too. The present paper gives an affirmative answer to this question for systems modelled by the diffusion equation in the tube Ω_θ:

$$u_t - \Delta u = 0\,, \tag{14.2}$$

subject to Dirichlet boundary conditions on $\partial\Omega_\theta$. Indeed, we show that the twist is responsible for a faster convergence of the solutions of (14.2) to the (zero) stable equilibrium. The second objective of the paper is to give a new (simpler and more direct) proof of the Hardy inequality (14.1) under weaker conditions than those in [7].

14.1.1 The main result

Before stating the main result about the large time behaviour of the solutions to (14.2), let us make some comments on the subtleties arising with the study of the heat equation in Ω_θ.

The specific deformation Ω_θ of Ω_0 via twisting we consider can be visualized as follows: instead of simply translating ω along $\mathbb{R}$ we also allow the (non-circular) cross-section ω to rotate with respect to a (non-constant) angle $x_1 \mapsto \theta(x_1)$. See Figure 14.1 (the precise definition is postponed until Section 14.2, *cf* Definition 14.1). We assume that the deformation is local, *i.e.*,

$$\dot\theta \text{ has compact support in } \mathbb{R}. \tag{14.3}$$

Then the straight and twisted tubes have the same spectrum (*cf* [17, Sec. 4]):

$$\sigma(-\Delta_D^{\Omega_\theta}) = \sigma_{\rm ess}(-\Delta_D^{\Omega_\theta}) = [E_1, \infty)\,. \tag{14.4}$$

The fine difference between twisted and untwisted tubes in the spectral setting is reflected in the existence of (14.1) for the former.

In view of the spectral mapping theorem, the indifference (14.4) transfers to the following identity for the heat semigroup:

$$\forall t \geq 0\,, \qquad \left\|e^{\Delta_D^{\Omega_\theta} t}\right\|_{L^2(\Omega_\theta)\to L^2(\Omega_\theta)} = e^{-E_1 t}\,, \tag{14.5}$$

irrespectively whether the tube Ω_θ is twisted or not. That is, we clearly have the exponential decay

$$\|u(t)\|_{L^2(\Omega_\theta)} \leq e^{-E_1 t}\,\|u_0\|_{L^2(\Omega_\theta)} \tag{14.6}$$

for each time $t \geq 0$ and any initial datum u_0 of (14.2). To obtain some finer differences as regards the time-decay of solutions, it is therefore natural to consider rather the *"shifted"* semigroup

$$S(t) := e^{(\Delta_D^\Omega + E_1)t} \tag{14.7}$$

as an operator from a *subspace* of $L^2(\Omega_\theta)$ to $L^2(\Omega_\theta)$.

In this paper we mainly (but not exclusively) consider the subspace of initial data given by the weighted space

$$L^2(\Omega_\theta, K) \qquad \text{with} \qquad K(x) := e^{x_1^2/4}\,, \tag{14.8}$$

and study the asymptotic properties of the semigroup via the *decay rate* defined by

$$\Gamma(\Omega_\theta) := \sup\left\{\Gamma \;\middle|\; \exists C_\Gamma > 0,\, \forall t \geq 0,\; \|S(t)\|_{L^2(\Omega_\theta,K)\to L^2(\Omega_\theta)} \leq C_\Gamma\,(1+t)^{-\Gamma}\right\}.$$

Our main result reads as follows:

Theorem 14.1. *Let $\theta \in C^1(\mathbb{R})$ satisfy* (14.3)*. We have*

$$\Gamma(\Omega_\theta) \begin{cases} = 1/4 & \text{if } \Omega_\theta \text{ is untwisted,} \\ \geq 3/4 & \text{if } \Omega_\theta \text{ is twisted.} \end{cases}$$

The statement of the theorem for solutions u of (14.2) in Ω_θ can be reformulated as follows. For every $\Gamma < \Gamma(\Omega_\theta)$, there exists a positive constant C_Γ such that

$$\|u(t)\|_{L^2(\Omega_\theta)} \leq C_\Gamma\,(1+t)^{-\Gamma}\,e^{-E_1 t}\,\|u_0\|_{L^2(\Omega_\theta,K)} \tag{14.9}$$

for each time $t \geq 0$ and any initial datum $u_0 \in L^2(\Omega_\theta, K)$. This should be compared with the inequality (14.6) which is sharp in the sense that it does not allow for any extra polynomial-type decay rate due to (14.5). On the other hand, we see that the decay rate is at least three times better in a twisted tube provided that the initial data are restricted to the weighted space.

A type of the estimate (14.9) in an untwisted tube can be obtained in a less restrictive weighted space (*cf* Theorem 14.4). The power $1/4$ actually reflects the quasi-one-dimensional nature of our model. Indeed, in the whole Euclidean space one has the well known d-dimensional bound

$$\forall t \geq 0\,, \qquad \left\|e^{\Delta_D^{\mathbb{R}^d} t}\right\|_{L^2(\mathbb{R}^d,K_d)\to L^2(\mathbb{R}^d)} \;\leq\; (1+t)^{-d/4}\,, \tag{14.10}$$

where $K_d(x) := e^{|x|^2/4} = K(x_1)\dots K(x_d)$ with K being given by (14.8). The fact that the power $1/4$ is optimal for untwisted tubes can be established quite easily by a "separation of variables" (*cf* Proposition 14.5). The fine effect of twisting is then reflected in the positivity of $\Gamma(\Omega_\theta) - 1/4$; in view of (14.10), it can be interpreted as "enlarging the dimension" of the tube.

14.1.2 The idea of the proof

The principal idea behind the main result of Theorem 14.1, *i.e.* the better decay rate in twisted tubes, is the positivity of the function ϱ in (14.1). In fact, Hardy inequalities have already been used as an essential tool to study the asymptotic behaviour of the heat equation in other situations [3, 22]. However, it should be stressed that Theorem 14.1 does not follow as a direct consequence of (14.1) by some energy estimates (*cf* Section 14.4.3) but that important and further technical developments that we explain now are needed. Nevertheless, overall, the main result of the paper confirms that the Hardy inequalities end up enhancing the decay rate of solutions..

Let us now briefly describe our proof (as given in Section 14.5) that there is the extra decay rate if the tube is twisted.

I. First, we map the twisted tube Ω_θ to the straight one Ω_0 by a change of variables, and consider rather the transformed (and shifted by E_1) equation

$$u_t - (\partial_1 - \dot\theta\,\partial_\tau)^2 u - \Delta' u - E_1 u = 0 \tag{14.11}$$

in Ω_0 instead of (14.2). Here $-\Delta' := -\partial_2^2 - \partial_3^2$ and $\partial_\tau := x_3\partial_2 - x_2\partial_3$, with $x = (x_1, x_2, x_3) \in \Omega_0$, denote the "transverse" Laplace and angular-derivative operators, respectively.

II. The main ingredient in the subsequent analysis is the method of self-similar solutions developed in the whole Euclidean space by Escobedo and Kavian [8]. Writing

$$\tilde u(y_1, y_2, y_3, s) = e^{s/4} u(e^{s/2} y_1, y_2, y_3, e^s - 1)\,, \tag{14.12}$$

the equation (14.11) is transformed to

$$\tilde u_s - \tfrac12\, y_1\,\partial_1\tilde u - (\partial_1 - \sigma_s\,\partial_\tau)^2\tilde u - e^s\,\Delta'\tilde u - E_1\,e^s\,\tilde u - \tfrac14\,\tilde u = 0 \tag{14.13}$$

in self-similarity variables $(y, s) \in \Omega_0 \times (0, \infty)$, where

$$\sigma_s(y_1) := e^{s/2}\dot\theta(e^{s/2} y_1)\,. \tag{14.14}$$

Note that (14.13) is a parabolic equation with *time-dependent* coefficients. This non-autonomous feature is a consequence of the non-trivial geometry we deal with and represents thus the main difficulty in our study. We note that an analogous difficulty has been encountered previously for a convection-diffusion equation in the whole space but with a variable diffusion coefficient [5].

III. We reconsider (14.13) in the weighted space (14.8) and show that the associated generator has purely discrete spectrum then. Now a difference with respect to the self-similarity transformation in the whole Euclidean space is that the generator is not a symmetric operator if the tube is twisted. However, this is not a significant obstacle since only the real part of the corresponding quadratic form is relevant for subsequent energy estimates (*cf* (14.61)).

IV. Finally, we look at the asymptotic behaviour of (14.13) as the self-similar time s tends to infinity. Assume that the tube is twisted. The scaling coming from the self-similarity transformation is such that the function (14.14) converges in a distributional sense to a multiple of the delta function supported at zero as $s \to \infty$. The square of σ_s becomes therefore extremely singular at the section $\{0\} \times \omega$ of the tube for large times. At the same time, the prefactors e^s in (14.13) diverge exactly as if the cross-section of the tube shrunk to zero as $s \to \infty$. Taking these two simultaneous limits into account, it is expectable that (14.13) will be approximated for large times by the essentially one-dimensional problem

$$\varphi_s - \tfrac12\, y_1\,\varphi_{y_1} - \varphi_{y_1 y_1} - \tfrac14\,\varphi = 0\,, \qquad s \in (0,\infty),\ y_1 \in \mathbb{R}\,, \tag{14.15}$$

with an extra Dirichlet boundary condition at $y_1 = 0$. This evolution equation is explicitly solvable in $L^2(\mathbb{R}, K)$ and it is easy to see that

$$\|\varphi\|_{L^2(\mathbb{R},K)} \le e^{-\frac34 s}\,\|\varphi_0\|_{L^2(\mathbb{R},K)}\,, \tag{14.16}$$

for any initial datum φ_0. Here the exponential decay rate transfers to a polynomial one after returning to the original time t, and the number $3/4$ gives rise to that of the bound of Theorem 14.1 in the twisted case.

On the other hand, we get just $e^{-\frac{1}{4}s}$ in (14.16) provided that the tube is untwisted (which corresponds to imposing no extra condition at $y_1 = 0$).

Two comments are in order. First, we do not establish any theorem that solutions of (14.13) can be approximated by those of (14.15) as $s \to \infty$. We only show a strong-resolvent convergence for operators related to their generators (Proposition 14.9). This is, however, sufficient to prove Theorem 14.1 with help of energy estimates. Proposition 14.9 is probably the most significant auxiliary result of the paper and we believe it is interesting in its own right.

Second, in the proof of Proposition 14.9 we essentially use the existence of the Hardy inequality (14.1) in twisted tubes. In fact, the positivity of ϱ is directly responsible for the extra Dirichlet boundary condition of (14.15). Since the Hardy inequality holds in the Hilbert space $L^2(\Omega_0)$ (no weight), Proposition 14.9 is stated for operators transformed to it from (14.8) by an obvious unitary transform. In particular, the asymptotic operator h_D of Proposition 14.9 acts in a different space, $L^2(\mathbb{R})$, but it is unitarily equivalent to the generator of (14.15).

14.1.3 The content of the paper

The organization of this paper is as follows.

In the following Section 14.2 we give a precise definition of twisted tubes Ω_θ and the corresponding Dirichlet Laplacian $-\Delta_D^{\Omega_\theta}$.

Section 14.3 is mainly devoted to a new proof of the Hardy inequality (Theorem 14.2) as announced in [18]. We mention its consequences on the stability of the spectrum of the Laplacian (Proposition 14.2) and emphasize that the Hardy weight cannot be made arbitrarily large by increasing the twisting (Proposition 14.3). Finally, we establish there a new Nash-type inequality in twisted tubes (Theorem 14.3).

The heat equation in twisted tubes is considered in Section 14.4. Using some energy-type estimates, we prove in Theorems 14.4 and 14.5 polynomial-type decay results for the heat semigroup as a consequence of the Nash and Hardy inequalities, respectively. Unfortunately, Theorem 14.5 does not represent any improvement upon the 1/4-decay rate of Theorem 14.4 which is valid in untwisted tubes as well.

The main body of the paper is therefore represented by Section 14.5 where we develop the method of self-similar solutions to get the improved decay rate of Theorem 14.1 as described above. Furthermore, in Section 14.5.9 we establish an alternative version of Theorem 14.1.

The paper is concluded in Section 14.6 by referring to physical interpretations of the result and to some open problems.

14.2 Preliminaries

In this section we introduce some basic definitions and notations we shall use throughout the paper. All functional spaces are assumed to be over the complex field.

14.2.1 The geometry of a twisted tube

Given a bounded open connected set $\omega \subset \mathbb{R}^2$, let $\Omega_0 := \mathbb{R} \times \omega$ be a straight tube of cross-section ω. We assume no regularity hypotheses about ω. Let $\theta : \mathbb{R} \to \mathbb{R}$ be a C^1-smooth function with bounded derivative (occasionally we will denote by the same symbol θ the function $\theta \otimes 1$ on Ω_0). We introduce another tube of the same cross-section ω as the image

$$\Omega_\theta := \mathcal{L}_\theta(\Omega_0)\,,$$

where the mapping $\mathcal{L}_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\mathcal{L}_\theta(x) := \big(x_1, x_2 \cos\theta(x_1) + x_3 \sin\theta(x_1), -x_2 \sin\theta(x_1) + x_3 \cos\theta(x_1)\big)\,. \tag{14.17}$$

Definition 14.1 (Twisted and untwisted tubes)**.** We say that the tube Ω_θ is *twisted* if the following two conditions are satisfied:

1. θ is not constant,

2. ω is not rotationally symmetric with respect to the origin in $\mathbb{R}^2$.

Otherwise we say that Ω_θ is *untwisted*.

Here the precise meaning of ω being "rotationally symmetric with respect to the origin in $\mathbb{R}^2$" is that, for every $\vartheta \in (0, 2\pi)$,

$$\omega_\vartheta := \big\{ x_2 \cos\vartheta + x_3 \sin\vartheta, -x_2 \sin\vartheta + x_3 \cos\vartheta \,\big|\, (x_2, x_3) \in \omega \big\} = \omega \,,$$

with the natural convention that we identify ω and ω_ϑ (and other open sets) provided that they differ on a set of zero capacity. Hence, modulus a set of zero capacity, ω is rotationally symmetric with respect to the origin in $\mathbb{R}^2$ if, and only if, it is a disc or an annulus centered at the origin of $\mathbb{R}^2$.

In view of the above convention, any untwisted Ω_θ can be identified with the straight tube Ω_0 by an isometry of the Euclidean space. On the other hand, the shape of a twisted tube Ω_θ is not preserved by isometries of the cross-section ω in $\mathbb{R}^2$; this makes Definition 14.1 most general through the position of ω in $\mathbb{R}^2$.

We write $x = (x_1, x_2, x_3)$ for a point/vector in $\mathbb{R}^3$. If x is used to denote a point in Ω_0 or Ω_θ, we refer to x_1 and $x' := (x_2, x_3)$ as "longitudinal" and "transverse" variables in the tube, respectively.

It is easy to check that the mapping $\mathcal{L}_\theta$ is injective and that its Jacobian is identically equal to 1. Consequently, $\mathcal{L}_\theta$ induces a (global) diffeomorphism between Ω_0 and Ω_θ.

14.2.2 The Dirichlet Laplacian in a twisted tube

It follows from the last result that Ω_θ is an open set. The corresponding Dirichlet Laplacian in $L^2(\Omega_\theta)$ can be therefore introduced in a standard way as the self-adjoint operator $-\Delta_D^{\Omega_\theta}$ associated with the quadratic form

$$Q_D^{\Omega_\theta}[\Psi] := \|\nabla\Psi\|^2_{L^2(\Omega_\theta)}, \qquad \Psi \in \mathfrak{D}(Q_D^{\Omega_\theta}) := H_0^1(\Omega_\theta)\,.$$

By the representation theorem, $-\Delta_D^{\Omega_\theta}\Psi = -\Delta\Psi$ for $\Psi \in \mathfrak{D}(-\Delta_D^{\Omega_\theta}) := \{\Psi \in H_0^1(\Omega_\theta) \,|\, \Delta\Psi \in L^2(\Omega_\theta)\}$, where the Laplacian $\Delta\Psi$ should be understood in the distributional sense.

Moreover, using the diffeomorphism induced by $\mathcal{L}_\theta$, we can "untwist" the tube by expressing the Laplacian $-\Delta_D^{\Omega_\theta}$ in the curvilinear coordinates determined by (14.17). More precisely, let U_θ be the unitary transformation from $L^2(\Omega_\theta)$ to $L^2(\Omega_0)$ defined by

$$U_\theta\Psi := \Psi \circ \mathcal{L}_\theta\,. \tag{14.18}$$

It is easy to check that $H_\theta := U_\theta(-\Delta_D^{\Omega_\theta})U_\theta^{-1}$ is the self-adjoint operator in $L^2(\Omega_0)$ associated with the quadratic form

$$Q_\theta[\psi] := \|\partial_1\psi - \dot\theta\,\partial_\tau\psi\|^2_{L^2(\Omega_0)} + \|\nabla'\psi\|^2_{L^2(\Omega_0)}\,, \qquad \psi \in \mathfrak{D}(Q_\theta) := H_0^1(\Omega_0)\,. \tag{14.19}$$

Here $\nabla' := (\partial_2, \partial_3)$ denotes the transverse gradient and ∂_τ is a shorthand for the transverse angular-derivative operator

$$\partial_\tau := \tau\cdot\nabla' = x_3\partial_2 - x_2\partial_3\,, \qquad \text{where} \qquad \tau(x_2, x_3) := (x_3, -x_2)\,.$$

We have the point-wise estimate

$$|\partial_\tau\psi| \le a\,|\nabla'\psi|\,, \qquad \text{where} \qquad a := \sup_{x'\in\omega}|x'|\,. \tag{14.20}$$

The sesquilinear form associated with $Q_\theta[\cdot]$ will be denoted by $Q_\theta(\cdot,\cdot)$. In the distributional sense, we can write

$$H_\theta\psi = -(\partial_1 - \dot\theta\,\partial_\tau)^2\psi - \Delta'\psi\,, \tag{14.21}$$

where $-\Delta' := -\partial_2^2 - \partial_3^2$ denotes the transverse Laplacian.

14.3 The Hardy and Nash inequalities

In this section we summarize basic spectral results about the Laplacian $-\Delta_D^{\Omega_\theta}$ we shall need later to study the asymptotic behaviour of the associated semigroup.

14.3.1 The Poincaré inequality

Let E_1 be the first eigenvalue of the Dirichlet Laplacian in ω. Using the Poincaré-type inequality in the cross-section

$$\|\nabla f\|^2_{L^2(\omega)} \ge E_1\|f\|^2_{L^2(\omega)}\,, \qquad \forall f \in H_0^1(\omega)\,, \tag{14.22}$$

and Fubini's theorem, it readily follows that $Q_\theta[\psi] \ge E_1\|\psi\|^2_{L^2(\Omega_0)}$ for every $\psi \in H_0^1(\Omega_0)$. Or, equivalently,

$$-\Delta_D^{\Omega_\theta} \ge E_1 \tag{14.23}$$

in the form sense in $L^2(\Omega_\theta)$. Consequently, the spectrum of $-\Delta_D^{\Omega_\theta}$ does not start below E_1. The result (14.23) can be interpreted as a Poincaré-type inequality and it holds for any tube Ω_θ.

The inequality (14.23) is clearly sharp for an untwisted tube, since (14.4) holds in that case trivially by separation of variables. In general, the spectrum of $-\Delta_D^{\Omega_\theta}$ can start strictly above E_1 if the twisting is effective at infinity (*cf* [18, Corol. 6.6]). In this paper, however, we focus on tubes for which the energy E_1 coincides with the spectral threshold of $-\Delta_D^{\Omega_\theta}$. This is typically the case if the twisting vanishes at infinity (*cf* [17, Sec. 4]). More restrictively, we assume (14.3). Under this hypothesis, (14.4) holds and (14.23) is sharp in the twisted case too.

14.3.2 The Poincaré inequality in a bounded tube

For our further purposes, it is important that a better result than (14.23) holds in bounded tubes.

Given a bounded open interval $I \subset \mathbb{R}$, let H_θ^I be the "restriction" of H_θ to the tube $I \times \omega$ determined by the conditions $\partial_1\psi - \dot\theta\partial_\tau\psi = 0$ on the new boundary $(\partial I) \times \omega$. More precisely, H_θ^I is introduced as the self-adjoint operator in $L^2(I \times \omega)$ associated with the quadratic form

$$\begin{aligned} Q_\theta^I[\psi] &:= \|\partial_1\psi - \dot\theta\,\partial_\tau\psi\|^2_{L^2(I\times\omega)} + \|\nabla'\psi\|^2_{L^2(I\times\omega)}\,, \\ \psi \in \mathfrak{D}(Q_\theta^I) &:= \left\{\psi\restriction(I\times\Omega) \mid \psi \in H_0^1(\Omega_0)\right\}\,. \end{aligned}$$

That is, we impose no additional boundary conditions in the form setting.

Contrary to H_θ, H_θ^I is an operator with compact resolvent. Let $\lambda(\dot\theta, I)$ denote the lowest eigenvalue of the shifted operator $H_\theta^I - E_1$. We have the following variational characterization:

$$\lambda(\dot\theta, I) = \min_{\psi\in\mathfrak{D}(Q_\theta^I)\setminus\{0\}} \frac{Q_\theta^I[\psi] - E_1\|\psi\|^2_{L^2(I\times\omega)}}{\|\psi\|^2_{L^2(I\times\omega)}}\,. \tag{14.24}$$

As in the unbounded case, (14.22) yields that $\lambda(\dot\theta, I)$ is non-negative (it is zero if the tube is untwisted). However, thanks to the compactness, now we have that $H_\theta^I - E_1$ is a positive operator whenever the tube is twisted.

Lemma 14.1. *Let $\theta \in C^1(\mathbb{R})$. Let $I \subset \mathbb{R}$ be a bounded open interval such that $\theta\restriction I$ is not constant. Let ω be not rotationally invariant with respect to the origin in $\mathbb{R}^2$. Then*

$$\lambda(\dot\theta, I) > 0\,.$$

Proof. We proceed by contradiction and assume that $\lambda(\dot\theta, I) = 0$. Then the minimum (14.24) is attained by a (smooth) function $\psi \in \mathfrak{D}(Q_\theta^I)$ satisfying (recall (14.22))

$$\|\partial_1\psi - \dot\theta\,\partial_\tau\psi\|^2_{L^2(I\times\omega)} = 0 \quad \text{and} \quad \|\nabla'\psi\|^2_{L^2(I\times\omega)} - E_1\|\psi\|^2_{L^2(I\times\omega)} = 0\,. \tag{14.25}$$

Writing $\psi(x) = \varphi(x_1)\mathcal{J}_1(x') + \phi(x)$, where $\mathcal{J}_1$ is the positive eigenfunction corresponding to E_1 of the Dirichlet Laplacian in $L^2(\omega)$ and $(\mathcal{J}_1, \phi(x_1,\cdot))_{L^2(\omega)} = 0$ for every $x_1 \in I$, we deduce from the second equality in (14.25) that $\phi = 0$. The first identity is then equivalent to

$$\|\dot\varphi\|^2_{L^2(I)}\|\mathcal{J}_1\|^2_{L^2(\omega)} + \|\dot\theta\,\varphi\|^2_{L^2(I)}\|\partial_\tau\mathcal{J}_1\|^2_{L^2(\omega)} - 2(\mathcal{J}_1, \partial_\tau\mathcal{J}_1)_{L^2(\omega)}\Re(\dot\varphi, \dot\theta\,\varphi)_{L^2(I)} = 0\,.$$

Since $(\mathcal{J}_1, \partial_\tau\mathcal{J}_1)_{L^2(\omega)} = 0$ by an integration by parts, it follows that φ must be constant and that

$$\|\dot\theta\|_{L^2(I)} = 0 \qquad \text{or} \qquad \|\partial_\tau\mathcal{J}_1\|_{L^2(\omega)} = 0\,.$$

However, this is impossible under the stated assumptions because $\|\dot\theta\|_{L^2(I)}$ vanishes if and only if θ is constant on I, and $\partial_\tau\mathcal{J}_1 = 0$ identically in ω if and only if ω is rotationally invariant with respect to the origin. □

Lemma 14.1 was the cornerstone of the method of [18] to establish the existence of Hardy inequalities in twisted tubes (see also the proof of Theorem 14.2 below).

14.3.3 Infinitesimally thin tubes

It is clear that $\lambda(\dot\theta, \mathbb{R}) := \inf\sigma(H_\theta) = 0$ whenever (14.4) holds (*e.g.*, if (14.3) is satisfied). It turns out that the shifted spectral threshold diminishes also in the opposite asymptotic regime, *i.e.* when the interval $I_\epsilon := (-\epsilon, \epsilon)$ shrinks, and this irrespectively of the properties of ω and $\dot\theta$.

Proposition 14.1. *Let $\theta \in C^1(\mathbb{R})$. We have*

$$\lim_{\epsilon\to 0} \lambda\big(\dot\theta, I_\epsilon\big) = 0\,.$$

Proof. Let $\{\omega_k\}_{k=0}^\infty$ be an exhaustion sequence of ω, *i.e.*, each ω_k is an open set with smooth boundaries satisfying $\omega_k \Subset \omega_{k+1}$ and $\cup_{k=0}^\infty \omega_k = \omega$. Let $\mathcal{J}_1^k$ denote the first eigenfunction of the Dirichlet Laplacian in $L^2(\omega_k)$; we extend it by zero to the whole $\mathbb{R}^2$. Finally, set $\psi^k := (1 \otimes \mathcal{J}_1^k) \circ \mathcal{L}_{\theta_0}$ with $\theta_0(x_1) := \dot\theta(0)\, x_1$, *i.e.*,

$$\psi^k(x) = \mathcal{J}_1^k \Big(x_2 \cos(\dot\theta_0 x_1) + x_3 \sin(\dot\theta_0 x_1), -x_2 \sin(\dot\theta_0 x_1) + x_3 \cos(\dot\theta_0 x_1)\Big)\,,$$

where $\dot\theta_0 = \dot\theta(0)$.

For any (large) $k \in \mathbb{N}$ there exists (small) positive ϵ_k such that ψ^k belongs to $\mathfrak{D}(Q_\theta^{I_\epsilon})$ for all $\epsilon \le \epsilon_k$. Hence it is an admissible trial function for (14.24). Now, fix $k \in \mathbb{N}$ and assume that $\epsilon \le \epsilon_k$. Then we have

$$\|\psi^k\|^2_{L^2(I_\epsilon\times\omega)} = |I_\epsilon|\, \|\mathcal{J}_1^k\|^2_{L^2(\omega_k)}\,,$$

where we have used the change of variables $y = \mathcal{L}_{\theta_0}(x)$. At the same time, employing consecutively the identity $\partial_1\psi^k - \dot\theta\,\partial_\tau\psi^k = (\dot\theta_0 - \dot\theta)\,\partial_\tau\psi^k$, the bound (14.20), the identity $|\nabla'\psi^k(x)| = |\nabla\mathcal{J}_1^k(y_2,y_3)|$ and the same change of variables, we get the estimate

$$\|\partial_1\psi^k - \dot\theta\,\partial_\tau\psi^k\|^2_{L^2(I_\epsilon\times\omega)} \le \|(\dot\theta_0 - \dot\theta)\|^2_{L^\infty(I_\epsilon)}\, |I_\epsilon|\, a^2\, \|\nabla\mathcal{J}_1^k\|^2_{L^2(\omega_k)}\,,$$

where the supremum norm clearly tends to zero as $\epsilon \to 0$. Finally,

$$\|\nabla'\psi^k\|^2_{L^2(I_\epsilon\times\omega)} - E_1\|\psi^k\|^2_{L^2(I_\epsilon\times\omega)} = (E_1^k - E_1)\, |I_\epsilon|\, \|\mathcal{J}_1^k\|^2_{L^2(\omega_k)}\,,$$

where E_1^k denotes the first eigenvalue of the Dirichlet Laplacian in $L^2(\omega_k)$. Sending ϵ to zero, the trial-function argument therefore yields

$$\lim_{\epsilon\to 0} \lambda(\dot\theta, I_\epsilon) \le E_1^k - E_1\,.$$

Since k can be made arbitrarily large and $E_1^k \to E_1$ as $k \to \infty$ by standard approximation arguments (see, *e.g.*, [4]), we conclude with the desired result. □

Remark 14.1 (An erratum to [17])**.** The study of the infinitesimally thin tubes played a crucial role in the proof of Hardy inequalities given in [17]. According to Lemma 6.3 in [17], $\lambda\big(\dot\theta, I_\epsilon\big)$, with constant $\dot\theta$, is independent of $\epsilon > 0$ (and therefore remains positive for a twisted tube even if $\epsilon \to 0$). However, in view Proposition 14.1, this is false. Consequently, Lemmata 6.3 and 6.5 and Theorem 6.6 in [17] cannot hold. The proof of Hardy inequalities presented in [17] is incorrect. A corrected version of the paper [17] can be found in [18].

14.3.4 The Hardy inequality

Now we come back to unbounded tubes Ω_θ. Although (14.23) represents a sharp Poincaré-type inequality both for twisted and untwisted tubes (if (14.4) holds), there is a fine difference in the spectral setting. Whenever the tube Ω_θ is non-trivially twisted (*cf* Definition 14.1), there exists a positive function ϱ (necessarily vanishing at infinity if (14.4) holds) such that (14.23) is improved to (14.1). A variant of the Hardy inequality is represented by the following theorem:

Theorem 14.2 (Hardy inequality)**.** *Let $\theta \in C^1(\mathbb{R})$ and suppose that $\dot\theta$ has compact support. Then for every $\Psi \in H_0^1(\Omega_\theta)$ we have*

$$\|\nabla\Psi\|^2_{L^2(\Omega_\theta)} - E_1\,\|\Psi\|^2_{L^2(\Omega_\theta)} \;\ge\; c_H\,\|\rho\Psi\|^2_{L^2(\Omega_\theta)}\,, \tag{14.26}$$

where $\rho(x) := 1/\sqrt{1+x_1^2}$ and c_H is a non-negative constant depending on $\dot\theta$ and ω. Moreover, c_H is positive if, and only if, Ω_θ is twisted.

Proof. It is clear that the left hand side of (14.26) is non-negative due to (14.23). The fact that $c_H = 0$ if the tube is untwisted follows from the more general result included in Proposition 14.2.2 below. We divide the proof of the converse fact (*i.e.* that twisting implies $c_H > 0$) into several steps. Recall the identification of $\Psi \in L^2(\Omega_\theta)$ with $\psi := U_\theta\Psi \in L^2(\Omega_0)$ via (14.18).

1. Let us first assume that the interval $I := (\inf\operatorname{supp}\dot\theta, \sup\operatorname{supp}\dot\theta)$ is symmetric with respect to the origin of $\mathbb{R}$.

2. The main ingredient in the proof is the following Hardy-type inequality for a Schrödinger operator in $\mathbb{R}\times\omega$ with a characteristic-function potential:

$$\|\rho\psi\|^2_{L^2(\Omega_0)} \le 16\,\|\partial_1\psi\|^2_{L^2(\Omega_0)} + (2 + 64/|I|^2)\,\|\psi\|^2_{L^2(I\times\omega)} \tag{14.27}$$

for every $\psi \in H^1_0(\Omega_0)$. This inequality is a consequence of the classical one-dimensional Hardy inequality $\int_{\mathbb{R}} x_1^{-2}|\varphi(x_1)|^2 dx_1 \leq 4\int_{\mathbb{R}}|\dot\varphi(x_1)|^2 dx_1$ valid for any $\varphi \in H^1_0(\mathbb{R}\setminus\{0\})$. Indeed, following [17, Sec. 3.3], let η be the Lipschitz function on $\mathbb{R}$ defined by $\eta(x_1) := 2|x_1|/|I|$ for $|x_1| \leq |I|/2$ and 1 otherwise in $\mathbb{R}$ (we shall denote by the same symbol the function $\eta \otimes 1$ on $\mathbb{R}\times\omega$). For any $\psi \in C^\infty_0(\Omega_0)$, let us write $\psi = \eta\psi + (1-\eta)\psi$, so that $(\eta\psi)(\cdot, x') \in H^1_0(\mathbb{R}\setminus\{0\})$ for every $x' \in \omega$. Then, employing Fubini's theorem, we can estimate as follows:

$$\begin{aligned}\|\rho\psi\|^2_{L^2(\Omega_0)} &\leq 2\int_{\Omega_0} x_1^{-2}\,|(\eta\psi)(x)|^2\,dx + 2\,\|(1-\eta)\psi\|^2_{L^2(\Omega_0)}\\ &\leq 8\,\|\partial_1(\eta\psi)\|^2_{L^2(\Omega_0)} + 2\,\|\psi\|^2_{L^2(I\times\omega)}\\ &\leq 16\,\|\eta\partial_1\psi\|^2_{L^2(\Omega_0)} + 16\,\|(\partial_1\eta)\psi\|^2_{L^2(\Omega_0)} + 2\,\|\psi\|^2_{L^2(I\times\omega)}\\ &\leq 16\,\|\partial_1\psi\|^2_{L^2(\Omega_0)} + (2+64/|I|^2)\,\|\psi\|^2_{L^2(I\times\omega)}\,.\end{aligned}$$

By density, this result extends to all $\psi \in H^1_0(\Omega_0) = \mathfrak{D}(Q_\theta)$.

3. By the definition (14.24), we have

$$Q_\theta[\psi] - E_1\,\|\psi\|^2_{L^2(\Omega_0)} \;\geq\; Q^I_\theta[\psi] - E_1\,\|\psi\|^2_{L^2(I\times\omega)} \;\geq\; \lambda(\dot\theta, I)\,\|\psi\|^2_{L^2(I\times\omega)} \tag{14.28}$$

for every $\psi \in \mathfrak{D}(Q_\theta)$. Here the first inequality employs the trivial fact that the restriction to $I\times\omega$ of a function from $\mathfrak{D}(Q_\theta)$ belongs to $\mathfrak{D}(Q^I_\theta)$. Under the stated hypotheses, we know from Lemma 14.1 that $\lambda(\dot\theta, I)$ is a positive number.

4. At the same time, for every $\psi \in \mathfrak{D}(Q_\theta)$,

$$\begin{aligned}&Q_\theta[\psi] - E_1\,\|\psi\|^2_{L^2(\Omega_0)}\\ &\quad\geq \epsilon\,\|\partial_1\psi\|^2_{L^2(\Omega_0)} + \int_{\Omega_0}\left\{\left[1 - \frac{\epsilon}{1-\epsilon}\,a^2\,\dot\theta^2(x_1)\right]|\nabla'\psi(x)|^2 - E_1\,|\psi(x)|^2\right\}dx\\ &\quad\geq \epsilon\,\|\partial_1\psi\|^2_{L^2(\Omega_0)} - \frac{\epsilon}{1-\epsilon}\,a^2E_1\,\|\dot\theta\psi\|^2_{L^2(\Omega_0)}\\ &\quad\geq \epsilon\,\|\partial_1\psi\|^2_{L^2(\Omega_0)} - \frac{\epsilon}{1-\epsilon}\,\|\dot\theta\|^2_{L^\infty(I)}\,a^2E_1\,\|\psi\|^2_{L^2(I\times\omega)}\end{aligned} \tag{14.29}$$

for sufficiently small positive ϵ. Here the first estimate is an elementary Young-type inequality employing (14.20) and valid for all $\epsilon \in (0,1)$. The second inequality in (14.29) follows from (14.22) with help of Fubini's theorem provided that ϵ is sufficiently small, namely if $\epsilon < \big(1 + a^2\|\dot\theta\|^2_{L^\infty(\mathbb{R})}\big)^{-1}$.

5. Interpolating between the bounds (14.28) and (14.29), and using (14.27) in the latter, we finally arrive at

$$\begin{aligned}Q_\theta[\psi] - E_1\,\|\psi\|^2_{L^2(\Omega_0)} \geq \frac{1}{2}\frac{\epsilon}{16}\,\|\rho\psi\|^2_{L^2(\Omega_0)}&\\ + \frac{1}{2}\left[\lambda(\dot\theta, I) - \epsilon\left(\frac{1}{8} + \frac{4}{|I|^2}\right) - \frac{\epsilon}{1-\epsilon}\,\|\dot\theta\|^2_{L^\infty(I)}\,a^2E_1\right]\|\psi\|^2_{L^2(I\times\omega)}&\end{aligned}$$

for every $\psi \in \mathfrak{D}(Q_\theta)$. It is clear that the last line on the right hand side of this inequality can be made non-negative by choosing ϵ sufficiently small. Such an ϵ then determines the Hardy constant $c'_H := \epsilon/32$.

6. The previous bound can be transferred to $L^2(\Omega_\theta)$ via (14.18). In general, if the centre of I is an arbitrary point $x^0_1 \in \mathbb{R}$, the obtained result is equivalent to

$$\forall\Psi \in \mathfrak{D}(Q^{\Omega_\theta}_D)\,, \qquad \|\nabla\Psi\|^2_{L^2(\Omega_\theta)} - E_1\,\|\Psi\|^2_{L^2(\Omega_\theta)} \;\geq\; c'_H\,\|\rho_{x^0_1}\,\Psi\|^2_{L^2(\Omega_\theta)}\,,$$

where $\rho_{x^0_1}(x) := 1/\sqrt{1+(x_1-x^0_1)^2}$. This yields (14.26) with

$$c_H := c'_H \min_{x_1\in\mathbb{R}} \frac{1+x_1^2}{1+(x_1-x^0_1)^2}\,,$$

where the minimum is a positive constant depending on x^0_1. □

The Hardy inequality of Theorem 14.2 was first established [7] under additional hypotheses. The present version is adopted from [18], where other variants of the inequality can be found, too.

14.3.5 The spectral stability

Theorem 14.2 provides certain stability properties of the spectrum for twisted tubes, while the untwisted case is always unstable, in the following sense:

Proposition 14.2. *Let V be the multiplication operator in $L^2(\Omega_\theta)$ by a bounded non-zero non-negative function v such that $v(x) \sim |x_1|^{-2}$ as $|x_1| \to \infty$. Then*

1. *if Ω_θ is twisted with $\theta \in C^1(\mathbb{R})$ and $\dot\theta$ has compact support, then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$,*
$$\inf \sigma(-\Delta_D^{\Omega_\theta} - \varepsilon V) \geq E_1 \,;$$
2. *if Ω_θ is untwisted, then for all $\varepsilon > 0$,*
$$\inf \sigma(-\Delta_D^{\Omega_\theta} - \varepsilon V) < E_1 \,.$$

Proof. The first statement follows readily from (14.26) in Theorem 14.2, since in this case $c_H > 0$. To prove the second property (and therefore the other part of Theorem 14.2 stating that $c_H = 0$ if the tube is untwisted), it is enough to consider the case $\theta = 0$ and construct a test function ψ from $H_0^1(\Omega_0)$ such that

$$P_0[\psi] := \|\nabla\psi\|^2_{L^2(\Omega_0)} - E_1\|\psi\|^2_{L^2(\Omega_0)} - \varepsilon \left\|v^{1/2}\psi\right\|^2_{L^2(\Omega_0)} < 0$$

for all positive ε. For every $n \geq 1$, we define

$$\psi_n(x) := \varphi_n(x_1)\mathcal{J}_1(x_2, x_3)\,, \tag{14.30}$$

where $\mathcal{J}_1$ is the positive eigenfunction corresponding to E_1 of the Dirichlet Laplacian in the cross-section ω, normalized to 1 in $L^2(\omega)$, and

$$\varphi_n(x_1) := \exp\left(-\frac{x_1^2}{n}\right). \tag{14.31}$$

In view of the separation of variables and the normalization of $\mathcal{J}_1$, we have

$$P_0[\psi_n] = \|\dot\varphi_n\|^2_{L^2(\mathbb{R})} - \varepsilon \left\|v_1^{1/2}\varphi_n\right\|^2_{L^2(\mathbb{R})}\,,$$

where $v_1(x_1) := \|v(x_1,\cdot)^{1/2}\mathcal{J}_1\|^2_{L^2(\omega)}$. By hypothesis, $v_1 \in L^1(\mathbb{R})$ and the integral $\|v_1\|_{L^1(\mathbb{R})}$ is positive. Finally, an explicit calculation yields $\|\dot\varphi_n\|_{L^2(\mathbb{R})} \sim n^{-1/4}$. By the dominated convergence theorem, we therefore have

$$P_0[\psi_n] \xrightarrow[n\to\infty]{} -\varepsilon\,\|v_1\|_{L^1(\mathbb{R})}\,.$$

Consequently, taking n sufficiently large and ε positive, we can make the form $P_0[\psi_n]$ negative. □

Since the potential V in Proposition 14.2 is bounded and vanishes at infinity, it is easy to see that the essential spectrum is not changed, *i.e.*, $\sigma_{\rm ess}(-\Delta_D^{\Omega_\theta} - \varepsilon V) = [E_1, \infty)$, independently of the value of ε and irrespectively of whether the tube is twisted or not. As a consequence of Proposition 14.2, we have that an arbitrarily small attractive potential $-\varepsilon V$ added to the shifted operator $-\Delta_D^{\Omega_\theta} - E_1$ in the untwisted tube would generate negative discrete eigenvalues, however, a certain critical value of ε is needed in order to generate the negative spectrum in the twisted case. In the language of [21], the operator $-\Delta_D^{\Omega_\theta} - E_1$ is therefore subcritical (respectively critical) if Ω_θ is twisted (respectively untwisted).

14.3.6 An upper bound to the Hardy constant

Now we come back to Theorem 14.2 and show that the Hardy weight at the right hand side of (14.26) cannot be made arbitrarily large by increasing $\dot\theta$ or making the cross-section ω more eccentric.

Proposition 14.3. *Let $\theta \in C^1(\mathbb{R})$ and suppose that $\dot\theta$ has compact support. Then*

$$c_H \leq 1/2\,,$$

where c_H is the constant of Theorem 14.2.

Proof. Recall the unitary equivalence of $-\Delta_D^{\Omega_\theta}$ and H_θ given by (14.18). We proceed by contradiction and show that the operator $H_\theta - E_1 - c\rho^2$ is not non-negative if $c > 1/2$, irrespectively of properties of θ and ω. (Recall that ρ was initially introduced in Theorem 14.2 as a function on Ω_θ. In this proof, with an abuse of notation, we denote by the same symbol analogous functions on Ω_0 and $\mathbb{R}$.) It is enough to construct a test function ψ from $\mathfrak{D}(Q_\theta)$ such that

$$P_\theta^c[\psi] := Q_\theta[\psi] - E_1\|\psi\|^2_{L^2(\Omega_0)} - c\,\|\rho\psi\|^2_{L^2(\Omega_0)} < 0\,.$$

As in the proof of Proposition 14.2, we use the decomposition (14.30), but now the sequence of functions φ_n is defined as follows:

$$\varphi_n(x_1) := \begin{cases} \frac{x_1 - b_n^1}{b_n^2 - b_n^1} & \text{if} \quad x_1 \in [b_n^1, b_n^2)\,, \\ \frac{b_n^3 - x_1}{b_n^3 - b_n^2} & \text{if} \quad x_1 \in [b_n^2, b_n^3)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Here $\{b_n^j\}_{n\in\mathbb{N}}$, with $j = 1, 2, 3$, are numerical sequences such that $\sup\operatorname{supp}\dot\theta < b_n^1 < b_n^2 < b_n^3$ for each $n \in \mathbb{N}$ and $b_n^1 \to \infty$ as $n \to \infty$; further requirements will be imposed later on. Since φ_n and $\dot\theta$ have disjoint supports, and $\mathcal{J}_1$ is supposed to be normalized to 1 in $L^2(\omega)$, it easily follows that

$$P_\theta^c[\psi_n] = \|\dot\varphi_n\|^2_{L^2(\mathbb{R})} - c\,\|\rho\varphi_n\|^2_{L^2(\mathbb{R})}\,.$$

Note that the right hand side is independent of θ and ω. An explicit calculation yields

$$\begin{aligned} \|\dot\varphi_n\|^2_{L^2(\mathbb{R})} &= \frac{1}{b_n^2 - b_n^1} + \frac{1}{b_n^3 - b_n^2}\,, \\ \|\rho\varphi_n\|^2_{L^2(\mathbb{R})} &= \frac{b_n^2 - b_n^1 + [(b_n^1)^2 - 1](\arctan b_n^2 - \arctan b_n^1) - b_n^1 \log\frac{1+(b_n^2)^2}{1+(b_n^1)^2}}{(b_n^2 - b_n^1)^2} \\ &\quad + \frac{b_n^3 - b_n^2 + [(b_n^3)^2 - 1](\arctan b_n^3 - \arctan b_n^2) - b_n^3 \log\frac{1+(b_n^3)^2}{1+(b_n^2)^2}}{(b_n^3 - b_n^2)^2}\,. \end{aligned}$$

Specifying the numerical sequences in such a way that also the quotients b_n^2/b_n^1 and b_n^3/b_n^2 tend to infinity as $n \to \infty$, it is then straightforward to check that

$$b_n^2\, P_\theta^c[\psi_n] \xrightarrow[n\to\infty]{} 1 - 2c\,.$$

Since the limit is negative for $c > 1/2$, it follows that $P_\theta^c[\psi_n]$ can be made negative by choosing n sufficiently large. □

The proposition shows that the effect of twisting is limited in its nature, at least if (14.3) holds. This will have important consequences for the usage of energy methods when studying the heat semigroup below.

14.3.7 The Nash inequality

Regardless of whether the tube is twisted or not, the operator $-\Delta_D^{\Omega_\theta} - E_1$ satisfies the following Nash-type inequality.

Theorem 14.3 (Nash inequality)**.** *Let $\theta \in C^1(\mathbb{R})$ and suppose that $\dot\theta$ has compact support. Then for every $\Psi \in H_0^1(\Omega_\theta) \cap L^2(\Omega_\theta, \rho^{-2})$ we have*

$$\|\nabla\Psi\|^2_{L^2(\Omega_\theta)} - E_1\|\Psi\|^2_{L^2(\Omega_\theta)} \;\geq\; c_N\,\frac{\|\Psi\|^6_{L^2(\Omega_\theta)}}{\|\Psi\|_1^4}\,, \tag{14.32}$$

where $\|\Psi\|_1 := \sqrt{\int_\omega dx_2 dx_3\,\big(\int_{\mathbb{R}} dx_1 |(\Psi\circ\mathcal{L}_\theta)(x)|\big)^2}$, ρ is introduced in Theorem 14.2 and c_N is a positive constant depending on $\dot\theta$ and ω.

Proof. Recall that $\Psi \circ \mathcal{L}_\theta = U_\theta\Psi =: \psi$ belongs to $L^2(\Omega_0)$. First of all, let us notice that $\|\Psi\|_1$ is well defined for $\Psi \in L^2(\Omega_\theta, \rho^{-2})$. Indeed, the Schwarz inequality together with Fubini's theorem yields

$$\|\Psi\|_1^2 \leq \|\rho^{-1}\psi\|^2_{L^2(\Omega_0)} \int_{\mathbb{R}} \frac{dx_1}{1 + x_1^2} = \|\rho^{-1}\Psi\|^2_{L^2(\Omega_\theta)}\,\pi < \infty\,. \tag{14.33}$$

Here the equality of the norms is obvious from the facts that the mapping $\mathcal{L}_\theta$ leaves invariant the first coordinate in $\mathbb{R}^3$ and that its Jacobian is one. We also remark that, by density, it is enough to prove the theorem for $\Psi \in C_0^\infty(\Omega_\theta)$.

The inequality (14.32) is a consequence of the one-dimensional Nash inequality

$$\forall \varphi \in H^1(\mathbb{R}) \cap L^1(\mathbb{R}), \qquad \|\dot\varphi\|^2_{L^2(\mathbb{R})} \geq \frac{1}{4} \frac{\|\varphi\|^6_{L^2(\mathbb{R})}}{\|\varphi\|^4_{L^1(\mathbb{R})}}, \tag{14.34}$$

which is established quite easily by combining elementary estimates

$$\|\varphi\|^2_{L^2(\mathbb{R})} \leq \|\varphi\|_{L^1(\mathbb{R})} \|\varphi\|_{L^\infty(\mathbb{R})} \qquad \text{and} \qquad \|\varphi\|^2_{L^\infty(\mathbb{R})} \leq 2\, \|\varphi\|_{L^2(\mathbb{R})} \|\dot\varphi\|_{L^2(\mathbb{R})}\,.$$

In order to apply (14.34), we need to estimate the left hand side of (14.32) from below by $\|\partial_1 \psi\|^2_{L^2(\Omega_0)}$, *i.e.*,

$$\|\nabla \Psi\|^2_{L^2(\Omega_\theta)} - E_1 \|\Psi\|^2_{L^2(\Omega_\theta)} \geq c'_N \, \|\partial_1 \psi\|^2_{L^2(\Omega_0)}\,, \tag{14.35}$$

where c'_N is a positive constant to be specified later. We distinguish two cases:

1. Such an estimate is straightforward if Ω_θ is untwisted, since then we can assume $\theta = 0$ (consequently, $\psi = \Psi$) and the Poincaré inequality (14.22) in the cross-section ω, employing the decoupling $\Omega_0 = \mathbb{R} \times \omega$, immediately yields (14.35) with $c'_N = 1$.

2. On the other hand, if Ω_θ is twisted, we can proceed as in the proof of Theorem 14.2. Interpolating between the bounds (14.28) and (14.29), we get (14.35) with $c'_N = \epsilon/2$, where ϵ is a positive constant depending on $\dot\theta$ and ω.

Using now (14.34) with help of Fubini's theorem, we obtain

$$\|\partial_1 \psi\|^2_{L^2(\Omega_0)} \geq \frac{1}{4} \int_\omega \frac{\|\psi(\cdot, x_2, x_3)\|^6_{L^2(\mathbb{R})}}{\|\psi(\cdot, x_2, x_3)\|^4_{L^1(\mathbb{R})}}\, dx_2 dx_3 \geq \frac{1}{4} \frac{\|\Psi\|^6_{L^2(\Omega_\theta)}}{\|\Psi\|^4_1}\,,$$

where the second inequality follows by the Hölder inequality with properly chosen conjugate exponents (recall also that $\|\psi\|_{L^2(\Omega_0)} = \|\Psi\|_{L^2(\Omega_\theta)}$). This together with (14.35) concludes the proof of (14.32) with $c_N := c'_N/4$. □

14.4 The energy estimates

14.4.1 The heat equation

Having the replacement $u(x,t) \mapsto e^{-E_1 t}\, u(x,t)$ for (14.2) in mind, let us consider the following t-time evolution problem in the tube Ω_θ:

$$\begin{cases} u_t - \Delta u - E_1 u = 0 & \text{in} \quad \Omega_\theta \times (0,\infty)\,, \\ u = u_0 & \text{in} \quad \Omega_\theta \times \{0\}\,, \\ u = 0 & \text{in} \quad (\partial\Omega_\theta) \times (0,\infty)\,, \end{cases} \tag{14.36}$$

where $u_0 \in L^2(\Omega_\theta)$.

As usual, we consider the weak formulation of the problem, *i.e.*, we say a Hilbert space-valued function $u \in L^2_{\mathrm{loc}}\big((0,\infty); H^1_0(\Omega_\theta)\big)$, with the weak derivative $u' \in L^2_{\mathrm{loc}}\big((0,\infty); H^{-1}(\Omega_\theta)\big)$, is a (global) solution of (14.36) provided that

$$\big\langle v, u'(t) \big\rangle + \big(\nabla v, \nabla u(t)\big)_{L^2(\Omega_\theta)} - E_1 \big(v, u(t)\big)_{L^2(\Omega_\theta)} = 0 \tag{14.37}$$

for each $v \in H^1_0(\Omega_\theta)$ and a.e. $t \in [0,\infty)$, and $u(0) = u_0$. Here $\langle \cdot, \cdot \rangle$ denotes the pairing of $H^1_0(\Omega_\theta)$ and $H^{-1}(\Omega_\theta)$. With an abuse of notation, we denote by the same symbol u both the function on $\Omega_\theta \times (0,\infty)$ and the mapping $(0,\infty) \to H^1_0(\Omega_\theta)$.

Standard semigroup theory implies that there indeed exists a unique solution of (14.36) that belongs to $C^0\big([0,\infty), L^2(\Omega_\theta)\big)$. More precisely, the solution is given by $u(t) = S(t)u_0$, where $S(t)$ is the heat semigroup (14.7) associated with $-\Delta_D^{\Omega_\theta} - E_1$. By the Beurling-Deny criterion, $S(t)$ is positivity-preserving for all $t \geq 0$.

Since E_1 corresponds to the threshold of the spectrum of $-\Delta_D^{\Omega_\theta}$ if (14.3) holds, we cannot expect a uniform decay of solutions of (14.36) as $t \to \infty$ in this case. More precisely, the spectral mapping theorem together with (14.4) yields:

Proposition 14.4. *Let $\theta \in C^1(\mathbb{R})$ and suppose that $\dot\theta$ has compact support. Then for each time $t \geq 0$ we have*

$$\|S(t)\|_{L^2(\Omega_\theta)\to L^2(\Omega_\theta)} \;=\; 1\,.$$

Consequently, for each $t > 0$ and each $\varepsilon \in (0,1)$ we can find an initial datum $u_0 \in H^1_0(\Omega_\theta)$ such that $\|u_0\|_{L^2(\Omega_\theta)} = 1$ and such that the solution of (14.36) satisfies

$$\|u(t)\|_{L^2(\Omega_\theta)} \;\geq\; 1-\varepsilon\,.$$

14.4.2 The dimensional decay rate

However, if we restrict ourselves to initial data decaying sufficiently fast at the infinity of the tube, it is possible to obtain a polynomial decay rate for the solutions of (14.36). In particular, we have the following result based on Theorem 14.3:

Theorem 14.4. *Let $\theta \in C^1(\mathbb{R})$ and suppose that $\dot\theta$ has compact support. Then for each time $t \geq 0$ we have*

$$\|S(t)\|_{L^2(\Omega_\theta,\rho^{-2})\to L^2(\Omega_\theta)} \;\leq\; \left(1+\frac{4\,c_N}{\pi^2}\,t\right)^{-1/4}\,,$$

where c_N is the positive constant of Theorem 14.3 and ρ is introduced in Theorem 14.2.

Proof. The statement is equivalent to the following bound for the solution u of (14.36):

$$\forall t\in[0,\infty)\,,\qquad \|u(t)\|_{L^2(\Omega_\theta)} \;\leq\; \|\rho^{-1}u_0\|_{L^2(\Omega_\theta)}\left(1+\frac{4\,c_N}{\pi^2}\,t\right)^{-1/4}\,, \tag{14.38}$$

where $u_0 \in L^2(\Omega_\theta,\rho^{-2})$ is any non-trivial datum. It is easy to see that the real and imaginary parts of the solution of (14.36) evolve separately. Furthermore, since $S(t)$ is positivity-preserving, given a non-negative datum u_0, the solution $u(t)$ remains non-negative for all $t \geq 0$. Consequently, establishing the bound for positive and negative parts of $u(t)$ separately, it is enough to prove (14.38) for non-negative (and non-trivial) initial data only. Without loss of generality, we therefore assume in the proof below that $u(t) \geq 0$ for all $t \geq 0$.

Let $\{\varphi_n\}_{n=1}^\infty$ be the family of mollifiers on $\mathbb{R}$ given by (14.31); we denote by the same symbol the functions $\varphi_n \otimes 1$ on $\mathbb{R}\times\mathbb{R}^2 \supset \Omega_\theta$. Inserting the trial function

$$v_n(x;t) := \varphi_n(x_1)\,\bar u_n(x_2,x_3;t)\,,\qquad \bar u_n(x_2,x_3;t) := \big\|\varphi_n u(\cdot,x_2,x_3;t)\big\|_{L^1(\mathbb{R})}\,,$$

into (14.37), we arrive at (recall the definition of $\|\cdot\|_1$ from Theorem 14.3)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\varphi_n u(t)\|_1^2 &= -\|\nabla\bar u_n(t)\|^2_{L^2(\omega)} + E_1\|\bar u_n(t)\|^2_{L^2(\omega)} - \big(\partial_1 v_n(t),\partial_1 u(t)\big)_{L^2(\Omega_\theta)}\\ &\leq \big(\partial_1 v_n(t),\partial_1 u(t)\big)_{L^2(\Omega_\theta)}\\ &\leq \|\partial_1 v_n(t)\|_{L^2(\Omega_\theta)}\|\nabla u(t)\|_{L^2(\Omega_\theta)}\,.\end{aligned}$$

Here the first inequality is due to the Poincaré-type inequality in the cross-section (14.22). We clearly have

$$\|\partial_1 v_n(t)\|_{L^2(\Omega_\theta)} = \|\dot\varphi_n\|_{L^2(\mathbb{R})}\,\|\bar u_n(t)\|_{L^2(\omega)} = \|\dot\varphi_n\|_{L^2(\mathbb{R})}\,\|\varphi_n u(t)\|_1\,.$$

Integrating the differential inequality, we therefore get

$$\|\varphi_n u(t)\|_1 - \|\varphi_n u_0\|_1 \leq \|\dot\varphi_n\|_{L^2(\mathbb{R})}\int_0^t \|\nabla u(t')\|^2_{L^2(\Omega_\theta)}\,dt'\,.$$

Since $\|\dot\varphi_n\|_{L^2(\mathbb{R})}\to 0$ and $\{\varphi_n\}_{n=1}^\infty$ is an increasing sequence of functions converging pointwise to 1 as $n\to\infty$, we conclude from this inequality that

$$\forall t\in[0,\infty)\,,\qquad \|u(t)\|_1 \leq \|u_0\|_1\,, \tag{14.39}$$

where $\|u_0\|_1$ is finite due to (14.33).

Now, substituting u for the trial function v in (14.37), applying Theorem 14.3 and using (14.39), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega_\theta)} &= -\Big(\|\nabla u(t)\|^2_{L^2(\Omega_\theta)} - E_1\|u(t)\|^2_{L^2(\Omega_\theta)}\Big)\\ &\leq -c_N\,\frac{\|u(t)\|^6_{L^2(\Omega_\theta)}}{\|u(t)\|_1^4}\\ &\leq -c_N\,\frac{\|u(t)\|^6_{L^2(\Omega_\theta)}}{\|u_0\|_1^4}\,.\end{aligned}$$

An integration of this differential inequality leads to

$$\forall t \in [0,\infty)\,, \qquad \|u(t)\|_{L^2(\Omega_\theta)} \;\le\; \|u_0\|_{L^2(\Omega_\theta)} \left(1 + 4\,c_N\,\frac{\|u_0\|^4_{L^2(\Omega_\theta)}}{\|u_0\|^4_1}\;t\right)^{-1/4}.$$

Dividing the last inequality by $\|\rho^{-1}u_0\|_{L^2(\Omega_\theta)}$ and replacing $\|u_0\|_1$ with $\|\rho^{-1}u_0\|_{L^2(\Omega_\theta)}$ using (14.33), we get

$$\frac{\|u(t)\|_{L^2(\Omega_\theta)}}{\|\rho^{-1}u_0\|_{L^2(\Omega_\theta)}} \;\le\; \xi\left(1 + \frac{4\,c_N}{\pi^2}\,\xi^4\;t\right)^{-1/4} \le \left(1 + \frac{4\,c_N}{\pi^2}\;t\right)^{-1/4},$$

where $\xi := \|u_0\|_{L^2(\Omega_\theta)}/\|\rho^{-1}u_0\|_{L^2(\Omega_\theta)} \in (0,1)$. This establishes (14.38). □

As a direct consequence of the theorem, we get:

Corollary 14.1. *Under the hypotheses of Theorem 14.4,* $\Gamma(\Omega_\theta) \ge 1/4$.

Proof. It is enough to realize that $L^2(\Omega_\theta, K)$ is embedded in $L^2(\Omega_\theta, \rho^{-2})$. □

The following proposition shows that the decay rate of Theorem 14.4 is optimal for untwisted tubes.

Proposition 14.5. *Let Ω_θ be untwisted. Then for each time $t \ge 0$ we have*

$$\|S(t)\|_{L^2(\Omega_\theta,K)\to L^2(\Omega_\theta)} \;\ge\; \frac{1}{\sqrt{2}}\,(1+t)^{-1/4}\,,$$

where K is introduced in (14.8).

Proof. Without loss of generality, we may assume $\theta = 0$. It is enough to find an initial datum $u_0 \in L^2(\Omega_0, K)$ such that the solution u of (14.36) satisfies

$$\forall t \in [0,\infty)\,, \qquad \frac{\|u(t)\|_{L^2(\Omega_0)}}{\|u_0\|_{L^2(\Omega_0,K)}} \;\ge\; \frac{1}{\sqrt{2}}\,(1+t)^{-1/4}\,. \tag{14.40}$$

The idea is to take $u_0 := \psi_n$, where $\{\psi_n\}_{n=1}^\infty$ is the sequence (14.30) approximating a generalized eigenfunction of $-\Delta_D^{\Omega_0}$ corresponding to the threshold energy E_1. Using the fact that Ω_0 is a cross-product of $\mathbb{R}$ and ω, (14.36) can be solved explicitly in terms of an expansion into the eigenbasis of the Dirichlet Laplacian in the cross-section and a partial Fourier transform in the longitudinal variable. In particular, for our initial data we get

$$\|u(t)\|^2_{L^2(\Omega_0)} = \int_{\mathbb{R}} |\hat\varphi_n(\xi)|^2\,\exp\left(-2\xi^2 t\right)d\xi = \sqrt{\frac{n}{n+4t}}\sqrt{\frac{\pi n}{2}}\,,$$

where the second equality is a result of an explicit calculation enabled due to the special form of φ_n given by (14.31). At the same time, for every $n < 8$ ψ_n belongs to $L^2(\Omega_0, K)$ and an explicit calculation yields

$$\|u_0\|^2_{L^2(\Omega_0,K)} = 2\sqrt{\frac{\pi n}{8-n}}\,.$$

For the special choice $n = 6$ we get that the left hand side of (14.40) actually equals the right hand side with t being replaced by $2t/3 < t$. □

The power $1/4$ in the decay bounds of Theorem 14.4 and Proposition 14.5 reflects the quasi-one-dimensional nature of Ω_θ (*cf* (14.10)), at least if the tube is untwisted. More precisely, Proposition 14.5 readily implies that the inequality of Corollary 14.1 is sharp for untwisted tubes.

Corollary 14.2. *Let Ω_θ be untwisted. Then* $\Gamma(\Omega_\theta) = 1/4$.

This result establishes one part of Theorem 14.1. The much more difficult part is to show that the decay rate is improved whenever the tube is twisted.

14.4.3 The failure of the energy method

As a consequence of combination of direct energy arguments with Theorem 14.2, we get the following result. In Remark 14.3 below we explain why it is useless.

Theorem 14.5. *Let Ω_θ be twisted with $\theta \in C^1(\mathbb{R})$. Suppose that $\dot\theta$ has compact support. Then for each time $t \geq 0$ we have*

$$\|S(t)\|_{L^2(\Omega_\theta,\rho^{-2})\to L^2(\Omega_\theta)} \leq (1+2\,t)^{-\min\{1/2,c_H/2\}} \quad , \tag{14.41}$$

where c_H is the positive constant of Theorem 14.2.

Proof. For any positive integer n and $x \in \Omega_\theta$, let us set $\rho_n(x) := \max\{\rho(x), n^{-1}\}$. Then $\{\rho_n^{-1}\}_{n=1}^\infty$ is a non-decreasing sequence of bounded functions converging pointwise to ρ^{-1} as $n \to \infty$. Recalling the definition of ρ from Theorem 14.2, it is clear that $x \mapsto \rho_n(x)$ is in fact independent of the transverse variables x'. Moreover, $\rho_n^{-\gamma} u$ belongs to $H_0^1(\Omega_\theta)$ for every $\gamma \in \mathbb{R}$ provided $u \in H_0^1(\Omega_\theta)$.

Choosing $v := \rho_n^{-2} u$ in (14.37) (and possibly combining with the conjugate version of the equation if we allow non-real initial data), we get the identity

$$\frac{1}{2}\frac{d}{dt}\|\rho_n^{-1}u(t)\|^2 = -\|\rho_n^{-1}\nabla u(t)\|^2 + E_1\|\rho_n^{-1}u(t)\|^2 - \Re\Big(u(t)\nabla\rho_n^{-2}, \nabla u(t)\Big)\,. \tag{14.42}$$

Here and in the rest of the proof, $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and inner product in $L^2(\Omega_\theta)$ (we suppress the subscripts in this proof). Since ρ_n depends on the first variable only, we clearly have $\nabla(\rho_n^{-2}) = (-2\rho^{-3}\partial_1\rho, 0, 0)$. Introducing an auxiliary function $v_n(t) := \rho_n^{-1}u(t)$, one finds

$$\begin{aligned}
\|\rho_n^{-1}\nabla u(t)\|^2 &= \|\nabla v_n(t)\|^2 + \|(\partial_1\rho_n/\rho_n)\,v_n(t)\|^2 + 2\Re\Big(v_n(t), (\partial_1\rho_n/\rho_n)\,\partial_1 v_n(t)\Big)\,,\\
\Re\Big(u(t)\nabla\rho_n^{-2}, \nabla u(t)\Big) &= -2\|(\partial_1\rho_n/\rho_n)\,v_n(t)\|^2 - 2\Re\Big(v_n(t), (\partial_1\rho_n/\rho_n)\,\partial_1 v_n(t)\Big)\,.
\end{aligned}$$

Combining these two identities and substituting the explicit expression for ρ, we see that the right hand side of (14.42) equals

$$\begin{aligned}
&-\|\nabla v_n(t)\|^2 + E_1\|v_n(t)\|^2 + \|(\partial_1\rho_n/\rho_n)\,v_n(t)\|^2\\
&\quad = -\|\nabla v_n(t)\|^2 + E_1\|v_n(t)\|^2 + \|\chi_n\rho v_n(t)\|^2 - \|\chi_n\rho^2 v_n(t)\|^2\\
&\quad \leq (1-c_H)\,\|\chi_n\rho v_n(t)\|^2 - \|\chi_n\rho^2 v_n(t)\|^2\,.
\end{aligned} \tag{14.43}$$

Here χ_n denotes the characteristic function of the set $\Omega_\theta^n := \Omega_\theta \cap \{\mathrm{supp}(\partial_1\rho_n)\}$, and the inequality follows from Theorem 14.2 and an obvious inclusion $\Omega_\theta^n \subset \Omega_\theta$. Substituting back the solution $u(t)$, we finally arrive at

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\rho_n^{-1}u(t)\|^2 &\leq (1-c_H)\,\|\chi_n\rho v_n(t)\|^2 - \|\chi_n\rho^2 v_n(t)\|^2\\
&\leq (1-c_H)\,\|\chi_n\rho v_n(t)\|^2\,.
\end{aligned} \tag{14.44}$$

Now, using the monotone convergence theorem and recalling the initial data to which we restrict in the hypotheses of the theorem, the last estimate implies that $u(t)$ belongs to $L^2(\Omega_\theta,\rho^{-2})$ and that it remains true after passing to the limit $n \to \infty$, *i.e.*,

$$\frac{1}{2}\frac{d}{dt}\|\rho^{-1}u(t)\|^2 \leq (1-c_H)\,\|u(t)\|^2\,. \tag{14.45}$$

At the same time, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 &= -\Big(\|\nabla u(t)\|^2 - E_1\|u(t)\|^2\Big)\\
&\leq -c_H\,\|\rho u(t)\|^2\\
&\leq -c_H\,\frac{\|u(t)\|^4}{\|\rho^{-1}u(t)\|^2}\,,
\end{aligned} \tag{14.46}$$

where the equality follows from (14.36), the first inequality follows from Theorem 14.2 and the last inequality is established by means of the Schwarz inequality.

Summing up, in view of (14.46) and (14.45), $a(t) := \|u(t)\|^2$ and $b(t) := \|\rho^{-1}u(t)\|^2$ satisfy the system of differential inequalities

$$\dot a \leq -2\,c_H\,\frac{a^2}{b}\,, \qquad \dot b \leq 2\,(1-c_H)\,a\,, \tag{14.47}$$

with the initial conditions $a(0) = \|u_0\|^2 =: a_0$ and $b(0) = \|\rho^{-1}u_0\|^2 =: b_0$. We distinguish two cases:

1. $\boxed{c_H \geq 1.}$ In this case, it follows from the second inequality of (14.47) that b is decreasing. Solving the first inequality of (14.47) with b being replaced by b_0, we then get

$$a(t) \leq a_0 \left[1 + 2\, c_H\, (a_0/b_0)\, t\right]^{-1}.$$

Dividing this inequality by b_0 and maximizing the resulting right hand side with respect to $a_0/b_0 \in (0,1)$, we finally get

$$\forall t \in [0,\infty)\,, \qquad \|u(t)\| \;\leq\; \|\rho^{-1}u_0\|\,(1 + 2\, c_H\, t)^{-1/2} \quad , \tag{14.48}$$

which in particular implies (14.41).

2. $\boxed{c_H < 1.}$ We "linearize" (14.47) by replacing one a of the square on the right hand side of first inequality by employing the second inequality of (14.47):

$$\frac{\dot a}{a} \leq -2\, c_H\, \frac{a}{b} \leq -\frac{c_H}{1 - c_H}\, \frac{\dot b}{b}\,.$$

This leads to

$$a/a_0 \leq (b/b_0)^{-\frac{c_H}{1-c_H}}\,.$$

Using this estimate in the original, non-linearized system (14.47), *i.e.* solving the system by eliminating b from the first and a from the second inequality of (14.47), we respectively obtain

$$a(t) \leq a_0 \left[1 + 2\,(a_0/b_0)\, t\right]^{-c_H}, \qquad b(t) \leq b_0 \left[1 + 2\,(a_0/b_0)\, t\right]^{1-c_H}.$$

Dividing the first inequality by b_0 and maximizing the resulting right hand side with respect to $a_0/b_0 \in (0,1)$, we finally get

$$\forall t \in [0,\infty)\,, \qquad \|u(t)\| \;\leq\; \|\rho^{-1}u_0\|\,(1 + 2\, t)^{-c_H/2} \quad , \tag{14.49}$$

which is equivalent to (14.41). □

Remark 14.2. We see that the power in the polynomial decay rate of Theorem 14.5 diminishes as $c_H \to 0$. Let us now argue that this cannot be improved by the present method of proof. Indeed, the first inequality of (14.46) is an application of the Hardy inequality of Theorem 14.2 and the second one is sharp. The Hardy inequality is also applied in the first inequality of (14.44). In the second inequality of (14.44), however, we have neglected a negative term. Applying the second inequality of (14.46) to it instead, we conclude with an improved system of differential inequalities

$$\dot a \leq -2\, c_H\, \frac{a^2}{b}\,, \qquad \dot b \leq 2\,(1 - c_H)\, a - 2\, \frac{a^2}{b}\,. \tag{14.50}$$

The corresponding system of differential equations has the explicit solution

$$\tilde a(t) = a_0 \left(\frac{\xi_0}{W\big[\xi_0 \exp(\xi_0 + 2t)\big]}\right)^{c_H}, \quad \tilde b(t) = a(t)\Big(1 + W\big[\xi_0 \exp(\xi_0 + 2t)\big]\Big),$$

where $\xi_0 := b_0/a_0 - 1 > 0$ and W denotes the Lambert W function (product log), *i.e.* the inverse function of $w \mapsto w \exp(w)$. Since

$$W\big[\xi_0 \exp(\xi_0 + 2t)\big] = 2\, t + o(t) \qquad \text{as} \qquad t \to \infty\,,$$

we see that the decay term $t^{-c_H/2}$ in (14.41) for $c_H < 1$ cannot be improved by replacing (14.47) with (14.50).

Remark 14.3. Note that the hypothesis (14.3) is not explicitly used in the proof of Theorem 14.5, it is just required that the inequality (14.26) holds with some positive constant c_H. For tubes satisfying (14.3), however, we know from Proposition 14.3 that the constant cannot exceed the value $1/2$. Consequently, irrespectively of the strength of twisting, Theorem 14.5 never represents an improvement upon Theorem 14.4. This is what we mean by the failure of a direct energy argument based on the Hardy inequality of Theorem 14.2.

14.5 The self-similarity transformation

Let us now turn to a completely different approach which leads to an improved decay rate regardless of the smallness of twisting.

14.5.1 Straightening of the tube

First of all, we reconsider the heat equation (14.36) in an untwisted tube Ω_0 by using the change of variables defined by the mapping $\mathcal{L}_\theta$. In view of the unitary transform (14.18), one can identify the Dirichlet Laplacian in $L^2(\Omega_\theta)$ with the operator (14.21) in $L^2(\Omega_0)$, and it is readily seen that (14.36) is equivalent to

$$u_t + H_\theta u - E_1 u = 0 \qquad \text{in} \qquad \Omega_0 \times (0,\infty)\,,$$

plus the Dirichlet boundary conditions on $\partial\Omega_0$ and an initial condition at $t = 0$. (We keep the same latter u for the solutions transformed to Ω_0.) More precisely, the weak formulation (14.37) is equivalent to

$$\big\langle v, u'(t)\big\rangle + Q_\theta\big(v, u(t)\big) - E_1\big(v, u(t)\big)_{L^2(\Omega_0)} = 0 \tag{14.51}$$

for each $v \in H^1_0(\Omega_0)$ and a.e. $t \in [0,\infty)$, with $u(0) = u_0 \in L^2(\Omega_0)$. Here $\langle\cdot,\cdot\rangle$ denotes the pairing of $H^1_0(\Omega_0)$ and $H^{-1}(\Omega_0)$. We know that the transformed solution u belongs to $C^0\big([0,\infty), L^2(\Omega_0)\big)$ by the semigroup theory.

14.5.2 Changing the time

The main idea is to adapt the method of self-similar solutions used in the case of the heat equation in the whole Euclidean space by Escobedo and Kavian [8] to the present problem. We perform the self-similarity transformation in the first (longitudinal) space variable only, while keeping the other (transverse) space variables unchanged.

More precisely, we consider a unitary transformation $\tilde U$ on $L^2(\Omega_0)$ which associates to every solution

$$u \in L^2_{\rm loc}\big((0,\infty), dt; L^2(\Omega_0, dx)\big)$$

of (14.51) a solution $\tilde u := \tilde U u$ in a new s-time weighted space

$$L^2_{\rm loc}\big((0,\infty), e^s ds; L^2(\Omega_0, dy)\big)$$

via (14.12). The inverse change of variables is given by

$$u(x_1,x_2,x_3,t) = (t+1)^{-1/4}\,\tilde u\big((t+1)^{-1/2}x_1, x_2, x_3, \log(t+1)\big)\,.$$

When evolution is posed in that context, $y = (y_1,y_2,y_3)$ plays the role of space variable and s is the new time. One can check that, in the new variables, the evolution is governed by (14.13).

More precisely, the weak formulation (14.51) transfers to

$$\big\langle \tilde v, \tilde u'(s) - \tfrac12\, y_1\, \partial_1 \tilde u(s)\big\rangle + \tilde Q_s\big(\tilde v, \tilde u(s)\big) - E_1\, e^s\, \big(\tilde v, \tilde u(s)\big)_{L^2(\Omega_0)} = 0 \tag{14.52}$$

for each $\tilde v \in H^1_0(\Omega_0)$ and a.e. $s \in [0,\infty)$, with $\tilde u(0) = \tilde u_0 := \tilde U u_0 = u_0$. Here $\tilde Q_s(\cdot,\cdot)$ denotes the sesquilinear form associated with

$$\begin{aligned} \tilde Q_s[\tilde u] &:= \|\partial_1 \tilde u - \sigma_s\, \partial_\tau \tilde u\|^2_{L^2(\Omega_0)} + e^s\, \|\nabla' \tilde u\|^2_{L^2(\Omega_0)} - \frac14\, \|\tilde u\|^2_{L^2(\Omega_0)}\,, \\ \tilde u \in \mathfrak{D}(\tilde Q_s) &:= H^1_0(\Omega_0)\,, \end{aligned}$$

where σ_s has been introduced in (14.14).

Note that the operator $\tilde H_s$ in $L^2(\Omega_0)$ associated with the form $\tilde Q_s$ has s-time-dependent coefficients, which makes the problem different from the whole-space case. In particular, the twisting represented by the function (14.14) becomes more and more "localized" in a neighbourhood of the origin $y_1 = 0$ for large time s.

14.5.3 The natural weighted space

Since $\tilde U$ acts as a unitary transformation on $L^2(\Omega_0)$, it preserves the space norm of solutions of (14.51) and (14.52), *i.e.*,

$$\|u(t)\|_{L^2(\Omega_0)} = \|\tilde u(s)\|_{L^2(\Omega_0)}\,. \tag{14.53}$$

This means that we can analyse the asymptotic time behaviour of the former by studying the latter.

However, the natural space to study the evolution (14.52) is not $L^2(\Omega_0)$ but rather the weighted space (14.8). For $k \in \mathbb{Z}$, we define

$$\mathcal{H}_k := L^2\big(\Omega_0, K^k(y_1)\, dy_1 dy_2 dy_3\big)\,.$$

Hereafter we abuse the notation a bit by denoting by K, initially introduced as a function on Ω_θ in (14.8), the analogous function on $\mathbb{R}$ too. Note that $K^{-1/2}$ is the first eigenfunction of the harmonic-oscillator Hamiltonian

$$h := -\frac{d^2}{dy_1^2} + \frac{1}{16}\, y_1^2 \qquad \text{in} \qquad L^2(\mathbb{R}) \tag{14.54}$$

(*i.e.* the Friedrichs extension of this operator initially defined on $C_0^\infty(\mathbb{R})$). The advantage of reformulating (14.52) in $\mathcal{H}_1$ instead of $\mathcal{H}_0 = L^2(\Omega_0)$ lies in the fact that then the governing elliptic operator has compact resolvent, as we shall see below (*cf* Proposition 14.8).

Let us also introduce the weighted Sobolev space

$$\mathcal{H}^1_k := H^1_0\big(\Omega_0, K^k(y_1)\, dy_1 dy_2 dy_3\big)\,,$$

defined as the closure of $C_0^\infty(\Omega_0)$ with respect to the norm $(\|\cdot\|^2_{\mathcal{H}_k} + \|\nabla\cdot\|^2_{\mathcal{H}_k})^{1/2}$. Finally, we denote by $\mathcal{H}^{-1}_k$ the dual space to $\mathcal{H}^1_k$.

14.5.4 The evolution in the weighted space

We want to reconsider (14.13) as a parabolic problem posed in the weighted space $\mathcal{H}_1$ instead of $\mathcal{H}_0$. We begin with a formal calculation. Choosing $\tilde v(y) := K(y_1)v(y)$ for the test function in (14.52), where $v \in C_0^\infty(\Omega_0)$ is arbitrary, we can formally cast (14.52) into the form

$$\big\langle v, \tilde u'(s)\big\rangle + a_s\big(v, \tilde u(s)\big) = 0\,. \tag{14.55}$$

Here $\langle\cdot,\cdot\rangle$ denotes the pairing of $\mathcal{H}^1_1$ and $\mathcal{H}^{-1}_1$, and

$$\begin{aligned} a_s(v,\tilde u) := {} & \big(\partial_1 v - \sigma_s\, \partial_\tau v, \partial_1 \tilde u - \sigma_s\, \partial_\tau \tilde u\big)_{\mathcal{H}_1} + e^s\, \big(\nabla' v, \nabla' \tilde u\big)_{\mathcal{H}_1} \\ & - E_1\, e^s\, \big(v, \tilde u\big)_{\mathcal{H}_1} - \frac{1}{2}\, \big(y_1\, v, \sigma_s\, \partial_\tau \tilde u\big)_{\mathcal{H}_1} - \frac{1}{4}\, \big(v, \tilde u\big)_{\mathcal{H}_1}\,. \end{aligned}$$

Note that a_s is not a symmetric form.

Of course, the formulae are meaningless in general, because the solution $\tilde u(s)$ and its derivative $\tilde u'(s)$ may not belong to $\mathcal{H}^1_1$ and $\mathcal{H}^{-1}_1$, respectively. We therefore proceed conversely by showing that (14.55) is actually well posed in $\mathcal{H}_1$ and that the solution solves (14.52) too. As for the former, we have:

Proposition 14.6. *For any $u_0 \in \mathcal{H}_1$, there exists a unique function $\tilde u$ such that*

$$\tilde u \in L^2_{\mathrm{loc}}\big((0,\infty); \mathcal{H}^1_1\big) \cap C^0\big([0,\infty); \mathcal{H}_1\big)\,, \qquad \tilde u' \in L^2_{\mathrm{loc}}\big((0,\infty); \mathcal{H}^{-1}_1\big)\,,$$

and it satisfies (14.55) *for each $v \in \mathcal{H}^1_1$ and a.e. $s \in [0,\infty)$, and $\tilde u(0) = u_0$.*

Proof. First of all, let us show that a_s is well-defined as a sesquilinear form with domain $\mathfrak{D}(a_s) := \mathcal{H}^1_1$ for any fixed $s \in [0,\infty)$. In view of the boundedness of σ_s (for every finite s) and the estimate (14.20), it only requires to check that $y_1 v \in \mathcal{H}_1$ provided $v \in \mathcal{H}^1_1$. Let $v \in C_0^\infty(\Omega_0)$. Then

$$\begin{aligned} \|y_1 v\|^2_{\mathcal{H}_1} &= 2\int_{\Omega_0} y_1\, |v(y)|^2\, \frac{dK(y_1)}{dy_1}\, dy \\ &= -2\int_{\Omega_0} \Big\{ |v(y)|^2 + 2\, y_1\, \Re\big[\bar v(y) \partial_1 v(y)\big] \Big\}\, K(y_1)\, dy \\ &\le 4\, \|y_1 v\|_{\mathcal{H}_1}\, \|\partial_1 v\|_{\mathcal{H}_1}\,. \end{aligned}$$

Consequently,

$$\|y_1 v\|_{\mathcal{H}_1} \le 4\, \|\partial_1 v\|_{\mathcal{H}_1} \le 4\, \|v\|_{\mathcal{H}^1_1}\,. \tag{14.56}$$

By density, this inequality extends to all $v \in \mathcal{H}^1_1$. Hence, $a_s(v,u)$ is well defined for all $s \ge 0$ and $v, u \in \mathcal{H}^1_1$ (we suppress the tilde over u in the rest of the proof). Then the Proposition follows by a theorem of J. L. Lions [1, Thm. X.9] about weak solutions of parabolic equations with time-dependent coefficients. We only need to verify its hypotheses:

1. Measurability. The function $s \mapsto a_s(v,u)$ is clearly measurable on $[0,\infty)$ for all $v, u \in \mathcal{H}^1_1$, since it is in fact continuous.

2. Boundedness. Let s_0 be an arbitrary positive number. Using the boundedness of $\dot\theta$, the estimates (14.20)

and (14.56), it is quite easy to show that there is a constant C, depending uniquely on s_0, $\|\dot\theta\|_{L^\infty(\mathbb{R})}$ and the geometry of ω (through a and E_1), such that

$$|a_s(v,u)| \le C\,\|v\|_{\mathcal{H}_1^1}\,\|u\|_{\mathcal{H}_1^1} \tag{14.57}$$

for all $s\in[0,s_0]$ and $v,u\in\mathcal{H}_1^1$.

3. Coercivity. Recall that a_s is not symmetric and that we consider complex functional spaces in this paper. However, since the real and imaginary parts of the solution $\tilde u$ of (14.55) evolve independently, one may restrict to real-valued functions v and $\tilde u$ there. Alternatively, it is enough to check the coercivity of the real part of a_s. We therefore need to show that there are positive constants ϵ and C such that the inequality

$$\Re\{a_s[v]\} \ge \epsilon\,\|v\|^2_{\mathcal{H}_1^1} - C\,\|v\|^2_{\mathcal{H}_1} \tag{14.58}$$

holds for all $v\in\mathcal{H}_1^1$ and $s\in[0,s_0]$, where $a_s[v]:=a_s(v,v)$. We have

$$\Re\{a_s[v]\} = \|\partial_1 v - \sigma_s\,\partial_\tau v\|^2_{\mathcal{H}_1} + e^s\,\|\nabla' v\|^2_{\mathcal{H}_1} - E_1\,e^s\,\|v\|^2_{\mathcal{H}_1} - \frac14\,\|v\|^2_{\mathcal{H}_1} - \frac12\,\Re\,(y_1\,v,\sigma_s\,\partial_\tau v)_{\mathcal{H}_1} \tag{14.59}$$

for all $v\in\mathcal{H}_1^1$. For every $v\in C_0^\infty(\Omega_0)$, an integration by parts shows that:

$$\Re\,(y_1\,v,\sigma_s\,\partial_\tau v)_{\mathcal{H}_1} = 0\,; \tag{14.60}$$

by density, this result extends to all $v\in\mathcal{H}_1^1$. Hence, the mixed term in (14.59) vanishes. We continue with estimating the first term on the right hand side of (14.59):

$$\begin{aligned}\|\partial_1 v - \sigma_s\,\partial_\tau v\|^2_{\mathcal{H}_1} &\ge \epsilon\,\|\partial_1 v\|^2_{\mathcal{H}_1} - \frac{\epsilon}{1-\epsilon}\,\|\sigma_s\,\partial_\tau v\|^2_{\mathcal{H}_1}\\ &\ge \epsilon\,\|\partial_1 v\|^2_{\mathcal{H}_1} - \frac{\epsilon}{1-\epsilon}\,e^s\,\|\dot\theta\|_{L^\infty(\mathbb{R})}\,a^2\,\|\nabla' v\|^2_{\mathcal{H}_1}\end{aligned}$$

valid for every $\epsilon\in(0,1)$ and $v\in\mathcal{H}_1^1$. Here the second inequality follows from the definition of σ_s in (14.14) and the estimate (14.20). Using (14.22) with help of Fubini's theorem, we therefore have

$$\|\partial_1 v - \sigma_s\,\partial_\tau v\|^2_{\mathcal{H}_1} + (1-\epsilon)\,e^s\,\|\nabla' v\|^2_{\mathcal{H}_1} \ge \epsilon\,\|\partial_1 v\|^2_{\mathcal{H}_1} + E_1\,e^s\left(1-\epsilon-\frac{\epsilon}{1-\epsilon}\,\|\dot\theta\|_{L^\infty(\mathbb{R})}\,a^2\right)\|v\|^2_{\mathcal{H}_1}$$

provided that ϵ is sufficiently small (so that the expression in the round brackets is positive). Putting this inequality into (14.59), recalling (14.60) and using the trivial bounds $1\le e^s\le e^{s_0}$ for $s\in[0,s_0]$, we conclude with

$$\Re\{a_s[v]\} \ge \epsilon\,\|\nabla v\|^2_{\mathcal{H}_1} - \left[E_1\,e^{s_0}\left(\epsilon+\frac{\epsilon}{1-\epsilon}\,\|\dot\theta\|_{L^\infty(\mathbb{R})}\,a^2\right)+\frac14\right]\|v\|^2_{\mathcal{H}_1}\,,$$

valid for all sufficiently small ϵ and all $v\in\mathcal{H}_1^1$. It is clear that the last inequality can be cast into the form (14.58), with a constant ϵ depending on a and $\|\dot\theta\|_{L^\infty(\mathbb{R})}$, and a constant C depending on s_0, $\|\dot\theta\|_{L^\infty(\mathbb{R})}$ and the geometry of ω (through a and E_1).

Now it follows from [1, Thm. X.9] that the unique solution $\tilde u$ of (14.55) satisfies

$$\tilde u\in L^2\big((0,s_0);\mathcal{H}_1^1\big)\cap C^0\big([0,s_0];\mathcal{H}_1\big)\,,\qquad \tilde u'\in L^2\big((0,s_0);\mathcal{H}_1^{-1}\big)\,.$$

Since s_0 is an arbitrary positive number here, we actually get a global continuous solution in the sense that $\tilde u\in C^0\big([0,\infty);\mathcal{H}_1\big)$. □

Remark 14.4. As a consequence of (14.57), (14.58) and the Lax-Milgram theorem, it follows that the form a_s is closed on its domain $\mathcal{H}_1^1$.

Now we are in a position to prove a partial equivalence of evolutions (14.52) and (14.55).

Proposition 14.7. *Let $u_0\in\mathcal{H}_1$. Let $\tilde u$ be the unique solution to* (14.55) *for each $v\in\mathcal{H}_1^1$ and a.e. $s\in[0,\infty)$, subject to the initial condition $\tilde u(0)=u_0$, that is specified in Proposition 14.6. Then $\tilde u$ is also the unique solution to* (14.52) *for each $\tilde v\in\mathcal{H}_0^1$ and a.e. $s\in[0,\infty)$, subject to the same initial condition.*

Proof. Choosing $v(y):=K(y_1)^{-1}\,\tilde v(y)$ for the test function in (14.55), where $\tilde v\in C_0^\infty(\Omega_0)$ is arbitrary, one easily checks that $\tilde u$ satisfies (14.52) for each $\tilde v\in C_0^\infty(\Omega_0)$ and a.e. $s\in[0,\infty)$. By density, this result extends to all $\tilde v\in\mathcal{H}_0^1$. □

14.5.5 Reduction to a spectral problem

As a consequence of the previous subsection, reducing the space of initial data, we can focus on the asymptotic time behaviour of the solutions to (14.55). Choosing $v := \tilde{u}(s)$ in (14.55) (and possibly combining with the conjugate version of the equation if we allow non-real initial data), we arrive at the identity

$$\frac{1}{2}\frac{d}{ds}\|\tilde{u}(s)\|^2_{\mathcal{H}_1} = -J^{(1)}_s[\tilde{u}(s)]\,, \tag{14.61}$$

where $J^{(1)}_s[\tilde{u}] := \Re\{a_s[\tilde{u}]\}$, $\tilde{u} \in \mathfrak{D}(J^{(1)}_s) := \mathfrak{D}(a_s) = \mathcal{H}^1_1$ (independent of s). Recalling (14.59) and (14.60), we have

$$J^{(1)}_s[\tilde{u}] = \|\partial_1\tilde{u} - \sigma_s\,\partial_\tau\tilde{u}\|^2_{\mathcal{H}_1} + e^s\,\|\nabla'\tilde{u}\|^2_{\mathcal{H}_1} - E_1\,e^s\,\|\tilde{u}\|^2_{\mathcal{H}_1} - \frac{1}{4}\,\|\tilde{u}\|^2_{\mathcal{H}_1}\,.$$

As a consequence of (14.57), (14.58) and the Lax-Milgram theorem, we know that $J^{(1)}_s$ is closed on its domain $\mathcal{H}^1_1$. It remains to analyse the coercivity of the form $J^{(1)}_s$.

More precisely, as usual for energy estimates, we replace the right hand side of (14.61) by the spectral bound, valid for each fixed $s \in [0,\infty)$,

$$\forall\tilde{u} \in \mathcal{H}^1_1\,, \qquad J^{(1)}_s[\tilde{u}] \geq \mu(s)\,\|\tilde{u}\|^2_{\mathcal{H}_1}\,, \tag{14.62}$$

where $\mu(s)$ denotes the lowest point in the spectrum of the self-adjoint operator $T^{(1)}_s$ in $\mathcal{H}_1$ associated with $J^{(1)}_s$. Then (14.61) together with (14.62) implies the exponential bound

$$\forall s \in [0,\infty)\,, \qquad \|\tilde{u}(s)\|_{\mathcal{H}_1} \leq \|\tilde{u}_0\|_{\mathcal{H}_1}\,e^{-\int_0^s \mu(r)dr}\,. \tag{14.63}$$

In this way, the problem is reduced to a spectral analysis of the family of operators $\{T^{(1)}_s\}_{s\geq 0}$.

14.5.6 Removing the weight

In order to investigate the operator $T^{(1)}_s$ in $\mathcal{H}_1$, we first map it into a unitarily equivalent operator $T^{(0)}_s$ in $\mathcal{H}_0$. This can be carried out via the unitary transform $\mathcal{U}_0 : \mathcal{H}_1 \to \mathcal{H}_0$ defined by

$$(\mathcal{U}_0\tilde{u})(y) := K^{1/2}(y_1)\,\tilde{u}(y)\,.$$

We define $T^{(0)}_s := \mathcal{U}_0 T^{(1)}_s \mathcal{U}_0^{-1}$, which is the self-adjoint operator associated with the quadratic form $J^{(0)}_s[v] := J^{(1)}_s[\mathcal{U}_0^{-1}v]$, $v \in \mathfrak{D}(J^{(0)}_s) := \mathcal{U}_0\,\mathfrak{D}(J^{(1)}_s)$. A straightforward calculation yields

$$J^{(0)}_s[v] = \|\partial_1 v - \sigma_s\,\partial_\tau v\|^2_{\mathcal{H}_0} + \frac{1}{16}\,\|y_1 v\|^2_{\mathcal{H}_0} + e^s\,\|\nabla' v\|^2_{\mathcal{H}_0} - E_1\,e^s\,\|v\|^2_{\mathcal{H}_0}\,. \tag{14.64}$$

It is easy to verify that the domain of $J^{(0)}_s$ coincides with the closure of $C^\infty_0(\Omega_0)$ with respect to the norm $(\|\cdot\|^2_{\mathcal{H}_0} + \|\nabla\cdot\|^2_{\mathcal{H}_0} + \|y_1\cdot\|^2_{\mathcal{H}_0})^{1/2}$. In particular, $\mathfrak{D}(J^{(0)}_s)$ is independent of s. Moreover, since this closure is compactly embedded in $\mathcal{H}_0$ (one can employ the well-known fact that (14.54) has purely discrete spectrum, which essentially uses the fact that the form domain of h is compactly embedded in $L^2(\mathbb{R})$), it follows that $T^{(0)}_s$ (and therefore $T^{(1)}_s$) is an operator with compact resolvent. In particular, we have:

Proposition 14.8. *$T^{(1)}_s \simeq T^{(0)}_s$ have purely discrete spectrum for all $s \in [0,\infty)$.*

Consequently, $\mu(s)$ is the lowest eigenvalue of $T^{(1)}_s$.

14.5.7 The asymptotic behaviour of the spectrum

In order to study the decay rate via (14.63), we need information about the limit of the eigenvalue $\mu(s)$ as the time s tends to infinity.

Since the function σ_s from (14.14) converges in the distributional sense to a multiple of the delta function supported at zero as $s \to \infty$, it is expectable (*cf* (14.64)) that the operator $T^{(0)}_s$ will converge, in a suitable sense, to the one-dimensional operator h from (14.54) with an extra Dirichlet boundary condition at zero. More precisely, the limiting operator, denoted by h_D, is introduced as the self-adjoint operator in $L^2(\mathbb{R})$ whose quadratic form acts in the same way as that of h but has a smaller domain

$$\mathfrak{D}(h^{1/2}_D) := \big\{\varphi \in \mathfrak{D}(h^{1/2}) \mid \varphi(0) = 0\big\}\,.$$

Alternatively, the form domain $\mathfrak{D}(h_D^{1/2})$ is the closure of $C_0^\infty(\mathbb{R}\setminus\{0\})$ with respect to the norm $(\|\cdot\|^2_{L^2(\mathbb{R})} + \|\nabla\cdot\|^2_{L^2(\mathbb{R})} + \|y_1\cdot\|^2_{L^2(\mathbb{R})})^{1/2}$.

To make this limit rigorous ($T_s^{(0)}$ and h_D act in different spaces), we follow [10] and decompose the Hilbert space $\mathcal{H}_0$ into an orthogonal sum

$$\mathcal{H}_0 = \mathfrak{H}_1 \oplus \mathfrak{H}_1^\perp\,,$$

where the subspace $\mathfrak{H}_1$ consists of functions of the form $\psi_1(y) = \varphi(y_1)\mathcal{J}_1(y')$. Recall that $\mathcal{J}_1$ denotes the positive eigenfunction of $-\Delta_D^\omega$ corresponding to E_1, normalized to 1 in $L^2(\omega)$. Given any $\psi\in\mathcal{H}_0$, we have the decomposition $\psi = \psi_1 + \phi$ with $\psi_1\in\mathfrak{H}_1$ as above and $\phi\in\mathfrak{H}_1^\perp$. The mapping $\pi:\varphi\mapsto\psi_1$ is an isomorphism of $L^2(\mathbb{R})$ onto $\mathfrak{H}_1$. Hence, with an abuse of notations, we may identify any operator h on $L^2(\mathbb{R})$ with the operator $\pi h\pi^{-1}$ acting on $\mathfrak{H}_1\subset\mathcal{H}_0$.

Proposition 14.9. *Let Ω_θ be twisted with $\theta\in C^1(\mathbb{R})$. Suppose that $\dot\theta$ has compact support. Then $T_s^{(0)}$ converges to $h_D\oplus 0^\perp$ in the strong-resolvent sense as $s\to\infty$,* i.e., *for every $F\in\mathcal{H}_0$,*

$$\lim_{s\to\infty}\left\|\left(T_s^{(0)}+1\right)^{-1}F - \left[\left(h_D+1\right)^{-1}\oplus 0^\perp\right]F\right\|_{\mathcal{H}_0} = 0\,.$$

Here $0^\perp$ denotes the zero operator on the subspace $\mathfrak{H}_1^\perp\subset\mathcal{H}_0$.

Proof. For any fixed $F\in\mathcal{H}_0$ and sufficiently large positive number z, let us set $\psi_s := (T_s^{(0)}+z)^{-1}F$. In other words, ψ_s satisfies the resolvent equation

$$\forall v\in\mathfrak{D}(J_s^{(0)})\,,\qquad J_s^{(0)}(v,\psi_s) + z\,(v,\psi_s)_{\mathcal{H}_0} = (v,F)_{\mathcal{H}_0}\,. \tag{14.65}$$

In particular, choosing ψ_s for the test function v in (14.65), we have

$$\begin{aligned}\|\partial_1\psi_s - \sigma_s\,\partial_\tau\psi_s\|^2_{\mathcal{H}_0} + \frac{1}{16}\,\|y_1\psi_s\|^2_{\mathcal{H}_0} + e^s\Big(\|\nabla'\psi_s\|^2_{\mathcal{H}_0} - E_1\|\psi_s\|^2_{\mathcal{H}_0}\Big) + z\,\|\psi_s\|^2_{\mathcal{H}_0}\\ = (\psi_s,F)_{\mathcal{H}_0} \le \frac14\,\|\psi_s\|^2_{\mathcal{H}_0} + \|F\|^2_{\mathcal{H}_0}\,.\end{aligned} \tag{14.66}$$

Henceforth we assume that $z>1/4$.

We employ the decomposition $\psi_s(y) = \varphi_s(y_1)\mathcal{J}_1(y') + \phi_s(y)$ where $\phi_s\in\mathfrak{H}_1^\perp$, *i.e.*,

$$\forall y_1\in\mathbb{R}\,,\qquad \big(\mathcal{J}_1,\phi_s(y_1,\cdot)\big)_{L^2(\omega)} = 0\,. \tag{14.67}$$

Then, for every $\epsilon\in(0,1)$,

$$\begin{aligned}\|\nabla'\psi_s\|^2_{\mathcal{H}_0} - E_1\|\psi_s\|^2_{\mathcal{H}_0} &= \epsilon\|\nabla'\phi_s\|^2_{\mathcal{H}_0} + (1-\epsilon)\|\nabla'\phi_s\|^2_{\mathcal{H}_0} - E_1\|\phi_s\|^2_{\mathcal{H}_0}\\ &\ge \epsilon\|\nabla'\phi_s\|^2_{\mathcal{H}_0} + \big[(1-\epsilon)E_2 - E_1\big]\|\phi_s\|^2_{\mathcal{H}_0}\,,\end{aligned}$$

where E_2 denotes the second eigenvalue of $-\Delta_D^\omega$. Since E_1 is (strictly) less then E_2, we can choose the ϵ so small that (14.66) implies

$$\|\phi_s\|^2_{\mathcal{H}_0}\le Ce^{-s}\qquad\text{and}\qquad \|\nabla'\phi_s\|^2_{\mathcal{H}_0}\le Ce^{-s}\,, \tag{14.68}$$

where C is a constant depending on ω and $\|F\|_{\mathcal{H}_0}$. At the same time, (14.66) yields

$$\|\varphi_s\|_{L^2(\mathbb{R})}\le C\,,\qquad \|y_1\varphi_s\|_{L^2(\mathbb{R})}\le C\,,\qquad\text{and}\qquad \|y_1\phi_s\|_{\mathcal{H}_0}\le C\,, \tag{14.69}$$

where C is a constant depending on $\|F\|_{\mathcal{H}_0}$.

To get an estimate on the longitudinal derivative of ψ_s, we handle the first three terms on left hand side of (14.66) as follows. Defining a new function $u_s\in\mathcal{H}_0$ by $\psi_s(y) = e^{s/4}u_s(e^{s/2}y_1,y')$ (*cf* the self-similarity transformation (14.12)) and making the change of variables $(x_1,x') = (e^{s/2}y_1,y')$, we have

$$\begin{aligned}J_s^{(0)}[\psi_s] &= e^s\|\partial_1u_s - \dot\theta\,\partial_\tau u_s\|^2_{\mathcal{H}_0} + \frac{e^{-s}}{16}\,\|x_1u_s\|^2_{\mathcal{H}_0} + e^s\Big(\|\nabla'u_s\|^2_{\mathcal{H}_0} - E_1\|u_s\|^2_{\mathcal{H}_0}\Big)\\ &\ge e^s\left\{\|\partial_1u_s - \dot\theta\,\partial_\tau u_s\|^2_{\mathcal{H}_0} + \|\nabla'u_s\|^2_{\mathcal{H}_0} - E_1\|u_s\|^2_{\mathcal{H}_0}\right\}\\ &\ge e^s\,c_H\,\|\rho\,u_s\|^2_{\mathcal{H}_0}\,,\\ &= e^s\,c_H\,\|\rho_s\psi_s\|^2_{\mathcal{H}_0}\,,\qquad\text{where}\quad \rho_s(y) := \rho(e^{s/2}y_1,y')\,.\end{aligned} \tag{14.70}$$

In the second inequality we have employed the Hardy inequality of Theorem 14.2; the constant c_H is positive by the hypothesis. Consequently, (14.66) yields

$$\|\rho_s\psi_s\|^2_{\mathcal{H}_0} \leq Ce^{-s}\,, \tag{14.71}$$

where C is a constant depending on $\dot\theta$, ω and $\|F\|_{\mathcal{H}_0}$. Now, proceeding as in the proof of (14.29), we get

$$\|\partial_1\psi_s - \sigma_s\,\partial_\tau\psi_s\|^2_{\mathcal{H}_0} + e^s\Big(\|\nabla'\psi_s\|^2_{\mathcal{H}_0} - E_1\|\psi_s\|^2_{\mathcal{H}_0}\Big) \\ \geq \epsilon\,\|\partial_1\psi_s\|^2_{\mathcal{H}_0} - \frac{\epsilon}{1-\epsilon}\,\|\dot\theta\|^2_{L^\infty(\mathbb{R})}\,a^2E_1\,e^s\,\|\psi_s\|^2_{L^2(I_s\times\omega)}$$

for every $\epsilon < \big(1 + a^2\|\dot\theta\|^2_{L^\infty(\mathbb{R})}\big)^{-1}$, where $I_s := e^{-s/2}I \equiv \{e^{-s/2}x_1\,|\,x_1\in I\}$ with $I := (\inf\operatorname{supp}\dot\theta, \sup\operatorname{supp}\dot\theta)$. Since

$$\|\psi_s\|_{L^2(I_s\times\omega)} \leq C\,\|\rho_s\psi_s\|_{\mathcal{H}_0}\,, \tag{14.72}$$

where C is a constant depending exclusively on I, (14.66) together with (14.71) implies $\|\partial_1\psi_s\|^2_{\mathcal{H}_0} \leq C$, where C is a constant depending on $\dot\theta$, ω and $\|F\|_{\mathcal{H}_0}$. Recalling (14.67), we therefore get the separate bounds

$$\|\partial_1\phi_s\|_{\mathcal{H}_0} \leq C \qquad \text{and} \qquad \|\dot\varphi_s\|_{L^2(\mathbb{R})} \leq C\,, \tag{14.73}$$

with the same constant C.

By (14.68), ϕ_s converges strongly to zero in $\mathcal{H}_0$ as $s\to\infty$. Moreover, it follows from (14.68), (14.69) and (14.73) that $\{\phi_s\}_{s\geq0}$ is a bounded family in $\mathfrak{D}(J_s^{(0)})$. Consequently, ϕ_s converges weakly to zero in $\mathfrak{D}(J_s^{(0)})$ as $s\to\infty$.

At the same time, it follows from (14.69) and (14.73) that $\{\varphi_s\}_{s\geq0}$ is a bounded family in $\mathfrak{D}(h^{1/2})$. Therefore it is precompact in the weak topology of $\mathfrak{D}(h^{1/2})$. Let φ_∞ be a weak limit point, *i.e.*, for an increasing sequence of positive numbers $\{s_n\}_{n\in\mathbb{N}}$ such that $s_n\to\infty$ as $n\to\infty$, $\{\varphi_{s_n}\}_{n\in\mathbb{N}}$ converges weakly to φ_∞ in $\mathfrak{D}(h^{1/2})$. Actually, we may assume that it converges strongly in $L^2(\mathbb{R})$ because $\mathfrak{D}(h^{1/2})$ is compactly embedded in $L^2(\mathbb{R})$.

Employing (14.67), (14.71) together with (14.72) gives

$$\|\varphi_s\|^2_{L^2(I_s)} \leq Ce^{-s}\,, \tag{14.74}$$

where C is a constant depending on $\dot\theta$, ω and $\|F\|_{\mathcal{H}_0}$. Multiplying this inequality by $e^{s/2}$ and taking the limit $s\to\infty$, we verify that

$$\varphi_\infty(0) = 0\,. \tag{14.75}$$

(We note that $\mathfrak{D}(h^{1/2}) \subset H^1(\mathbb{R})$ and that $H^1(J)$ is compactly embedded in $C^{0,\lambda}(J)$ for every $\lambda\in(0,1/2)$ and any bounded interval $J\subset\mathbb{R}$.)

Finally, let $\varphi\in C_0^\infty(\mathbb{R}\setminus\{0\})$ be arbitrary. Taking $v(x) := \varphi(x_1)\mathcal{J}_1(x')$ as the test function in (14.65), with s being replaced by s_n, and sending n to infinity, we easily check that

$$(\dot\varphi, \dot\varphi_\infty)_{L^2(\mathbb{R})} + \frac{1}{16}\,(y_1\varphi, y_1\varphi_\infty)_{L^2(\mathbb{R})} + z\,(\varphi,\varphi_\infty)_{L^2(\mathbb{R})} = (\varphi, f)_{L^2(\mathbb{R})}\,,$$

where $f(x_1) := (\mathcal{J}_1, F(x_1,\cdot))_{L^2(\omega)}$. That is, $\varphi_\infty = (h_D+z)^{-1}f$, for *any* weak limit point of $\{\varphi_s\}_{s\geq0}$.

Summing up, we have shown that ψ_s converges strongly to ψ_∞ in $\mathcal{H}_0$ as $s\to\infty$, where $\psi_\infty(y) := \varphi_\infty(y_1)\mathcal{J}_1(y') = \big[(h_D+z)^{-1}\oplus 0^\perp\big]F$. □

Remark 14.5. The crucial step in the proof is certainly the usage of the Hardy inequality in the second inequality of (14.70). Indeed, it enables one to control the mixed terms coming from the first term on the left hand side of (14.66). We would like to mention that instead of the Hardy inequality itself we could have used in (14.70) the corner-stone Lemma 14.1. This would leave to the lower bound $J_s^{(0)}[\psi_s] \geq e^s\,\lambda(\dot\theta, I)\,\|\psi_s\|^2_{L^2(I_s\times\omega)}$, which is sufficient to conclude the proof in the same way as above.

Corollary 14.3. *Let Ω_θ be twisted with $\theta\in C^1(\mathbb{R})$. Suppose that $\dot\theta$ has compact support. Then*

$$\lim_{s\to\infty}\mu(s) = 3/4\,.$$

Proof. In general, the strong-resolvent convergence of Proposition 14.9 is not enough to guarantee the convergence of spectra. However, in our case, since the spectra are purely discrete, the eigenprojections converge even in norm (*cf* [23]). In particular, $\mu(s)$ converges to the first eigenvalue of h_D. It remains to notice that the first eigenvalue of h_D coincides (in view of the symmetry) with the second eigenvalue of h which is 3/4. (For the spectrum of h, see any textbook dealing with quantum harmonic oscillator, *e.g.*, [13, Sec. 2.3].) □

14.5.8 The improved decay rate - Proof of Theorem 14.1

Now we have all the prerequisites to prove Theorem 14.1. Recall that the identity $\Gamma(\Omega_\theta) = 1/4$ for untwisted tubes is already established by Corollary 14.2. Throughout this subsection we therefore assume that Ω_θ is twisted with (14.3) and show that there is an extra decay rate.

We come back to (14.63). It follows from Corollary 14.3 that for arbitrarily small positive number ε there exists a (large) positive time s_ε such that for all $s \geq s_\varepsilon$, we have $\mu(s) \geq 3/4 - \varepsilon$. Hence, fixing $\varepsilon > 0$, for all $s \geq s_\varepsilon$, we have

$$-\int_0^s \mu(r)\,dr \leq -\int_0^{s_\varepsilon} \mu(r)\,dr - (3/4-\varepsilon)(s-s_\varepsilon) \leq (3/4-\varepsilon)s_\varepsilon - (3/4-\varepsilon)s\,,$$

where the second inequality is due to the fact that $\mu(s)$ is non-negative for all $s \geq 0$ (it is in fact greater than $1/4$, *cf* Proposition 14.10). At the same time, assuming $\varepsilon \leq 3/4$, we trivially have

$$-\int_0^s \mu(r)\,dr \leq 0 \leq (3/4-\varepsilon)s_\varepsilon - (3/4-\varepsilon)s$$

also for all $s \leq s_\varepsilon$. Summing up, (14.63) implies

$$\|\tilde{u}(s)\|_{\mathcal{H}_1} \leq C_\varepsilon\, e^{-(3/4-\varepsilon)s}\, \|\tilde{u}_0\|_{\mathcal{H}_1} \tag{14.76}$$

for every $s \in [0,\infty)$, where $C_\varepsilon := e^{s_\varepsilon} \geq e^{(3/4-\varepsilon)s_\varepsilon}$. Returning to the variables in the straightened tube via $u = \tilde{U}^{-1}\tilde{u}$, using (14.53) together with the point-wise estimate $1 \leq K$, and recalling that $\tilde{u}_0 = u_0$, it follows that

$$\|u(t)\|_{\mathcal{H}_0} = \|\tilde{u}(s)\|_{\mathcal{H}_0} \leq \|\tilde{u}(s)\|_{\mathcal{H}_1} \leq C_\varepsilon\,(1+t)^{-(3/4-\varepsilon)}\,\|u_0\|_{\mathcal{H}_1}$$

for every $t \in [0,\infty)$. Finally, we recall that the weight K in $\mathcal{H}_1$ depends on the longitudinal variable only, which is therefore left invariant by the mapping $\mathcal{L}_\theta$. Consequently, we apply the unitary transform (14.18) and conclude with

$$\|S(t)\|_{L^2(\Omega_\theta,K)\to L^2(\Omega_\theta)} = \sup_{u_0\in\mathcal{H}_1\setminus\{0\}} \frac{\|u(t)\|_{\mathcal{H}_0}}{\|u_0\|_{\mathcal{H}_1}} \leq C_\varepsilon\,(1+t)^{-(3/4-\varepsilon)}$$

for every $t \in [0,\infty)$. Since ε can be made arbitrarily small, this bound implies $\Gamma(\Omega_\theta) \geq 3/4$ and concludes thus the proof of Theorem 14.1.

14.5.9 The improved decay rate - an alternative statement

Theorem 14.1 provides quite precise information about the extra polynomial decay of solutions u of (14.2) in a twisted tube in the sense that the decay rate $\Gamma(\Omega_\theta)$ is at least three times better than in the untwisted case. On the other hand, we have no control over the constant C_Γ in (14.9) (in principle it may blow up as $\Gamma \to \Gamma(\Omega_\theta)$). As an alternative result, we therefore present also the following theorem, where we get rid of the constant C_Γ but the prize we pay is just a qualitative knowledge about the decay rate.

Theorem 14.6. *Let $\theta \in C^1(\mathbb{R})$ satisfy (14.3). We have*

$$\forall t \geq 0\,, \qquad \|S(t)\|_{L^2(\Omega_\theta,K)\to L^2(\Omega_\theta)} \;\leq\; (1+t)^{-(\gamma+1/4)}\;, \tag{14.77}$$

where γ is a non-negative constant depending on $\dot\theta$ and ω. Moreover, γ is positive if, and only if, Ω_θ is twisted.

In order to establish Theorem 14.6, the asymptotic result of Corollary 14.3 need to be supplied with information about values of $\mu(s)$ for finite times s.

Singling the dimensional decay rate out

It follows from Theorem 14.4 that there is at least a $1/4$ polynomial decay rate for the solutions of the heat equations. In the setting of self-similar solutions (recall (14.63) and the relation between the initial and self-similar times t and s given by (14.12)), this will be reflected in that we actually have $\mu(s) \geq 1/4$, regardless whether the tube is twisted or not. It is therefore natural to study rather the shifted operator $T_s^{(0)} - 1/4$. However, it is not obvious from (14.64) that such an operator is non-negative.

In order to introduce the shift explicitly into the structure of the operator, we therefore introduce another unitarily equivalent operator $T_s^{(-1)} := \mathcal{U}_{-1} T_s^{(0)} (\mathcal{U}_{-1})^{-1}$ in $\mathcal{H}_{-1}$, where the map $\mathcal{U}_{-1} : \mathcal{H}_0 \to \mathcal{H}_{-1}$ acts in the same way as $\mathcal{U}_0$:

$$(\mathcal{U}_{-1}v)(y) := K^{1/2}(y_1)\,v(y)\,.$$

$T_s^{(-1)}$ is the self-adjoint operator associated with the quadratic form

$$J_s^{(-1)}[w] := J_s^{(0)}[(\mathcal{U}_{-1})^{-1}w]\,, \qquad w \in \mathfrak{D}(J_s^{(-1)}) := \mathcal{U}_{-1}\,\mathfrak{D}(J_s^{(0)})\,.$$

Again, it is straightforward to check that

$$J_s^{(-1)}[w] = \|\partial_1 w - \sigma_s\,\partial_\tau w\|^2_{\mathcal{H}_{-1}} + e^s\,\|\nabla' w\|^2_{\mathcal{H}_{-1}} - E_1\,e^s\,\|w\|^2_{\mathcal{H}_{-1}} + \frac{1}{4}\,\|w\|^2_{\mathcal{H}_{-1}}\,.$$

Now it readily follows from the structure of the quadratic form that the shifted operator $T_s^{(-1)} - 1/4$ is non-negative. Moreover, it is positive if, and only if, the tube is twisted.

Proposition 14.10. *If Ω_θ is twisted with $\theta \in C^1(\mathbb{R})$, then we have*

$$\forall s \in [0,\infty), \qquad \mu(s) > 1/4\,.$$

Conversely, $\mu(s) = 1/4$ for all $s \in [0,\infty)$ if Ω_θ is untwisted.

Proof. Since $J_s^{(-1)}[w] - \frac{1}{4}\,\|w\|^2_{\mathcal{H}_{-1}} \geq 0$ for every $w \in \mathfrak{D}(J_s^{(-1)})$, we clearly have $\mu(s) \geq 1/4$, regardless whether the tube is twisted or not. By definition, if it is untwisted, then either $\sigma_s = 0$ identically in $\mathbb{R}$ for all $s \in [0,\infty)$ or $\partial_\tau \mathcal{J}_1 = 0$ identically in ω, where $\mathcal{J}_1$ is the positive eigenfunction corresponding to E_1 of the Dirichlet Laplacian in $L^2(\omega)$. Consequently, choosing $w(y) = \mathcal{J}_1(y')$ as a test function for $J_s^{(-1)}$, we also get the opposite bound $\mu(s) \leq 1/4$ in the untwisted case. To get the converse result, we can proceed exactly as in the proof of Lemma 14.1: Assuming $\mu(s) = 1/4$ in the twisted case, the variational definition of the eigenvalue $\mu(s)$ would imply

$$\|\sigma_s\|_{L^2(\mathbb{R},K^{-1})} = 0 \qquad \text{or} \qquad \|\partial_\tau \mathcal{J}_1\|_{L^2(\omega)} = 0\,,$$

a contradiction. □

Now we are in a position to prove Theorem 14.6.

Proof of Theorem 14.6

Assume (14.3). It follows from Proposition 14.10 and Corollary 14.3 that the number

$$\gamma := \inf_{s\in[0,\infty)} \mu(s) - 1/4 \tag{14.78}$$

is positive if, and only if, Ω_θ is twisted. In any case, (14.63) implies

$$\|\tilde{u}(s)\|_{\mathcal{H}_1} \leq \|\tilde{u}_0\|_{\mathcal{H}_1}\,e^{-(\gamma+1/4)s}$$

for every $s \in [0,\infty)$. Using this estimate instead of (14.76), but following the same type of arguments as in Section 14.5.8 below (14.76), we get

$$\|S(t)\|_{L^2(\Omega_\theta,K)\to L^2(\Omega_\theta)} \leq (1+t)^{-(\gamma+1/4)}$$

for every $t \in [0,\infty)$. This is equivalent to (14.77) and we know that γ is positive if Ω_θ is twisted. On the other hand, in view of Proposition 14.5, estimate (14.77) cannot hold with positive γ if the tube is untwisted. This concludes the proof of Theorem 14.6.

14.6 Conclusions

The classical interpretation of the heat equation (14.2) is that its solution u gives the evolution of the temperature distribution of a medium in the tube cooled down to zero on the boundary. It also represents the simplest version of the stochastic Fokker-Planck equation describing the Brownian motion in Ω_θ with killing boundary conditions. Then the results of the present paper can be interpreted as that the twisting implies a faster cool-down/death of the medium/Brownian particle in the tube. Many other diffusive processes in nature are governed by (14.2).

Our proof that there is an extra decay rate for solutions of (14.2) if the tube is twisted was far from being straightforward. This is a bit surprising because the result is quite expectable from the physical interpretation, if one notices that the twist (locally) enlarges the boundary of the tube, while it (locally) keeps the volume unchanged. (By "locally" we mean that it is the case for bounded tubes, otherwise both the quantities are infinite of course.) At the same time, the Hardy inequality (14.1) did not play a direct role in the proof of

Theorems 14.1 and 14.6 (although, combining any of the theorems with Theorem 14.2, we eventually know that the existence of the Hardy inequality is equivalent to the extra decay rate for the heat semigroup). It would be desirable to find a more direct proof of Theorem 14.1 based on (14.1).

We conjecture that the inequality of Theorem 14.1 can be replaced by equality, *i.e.*, $\Gamma(\Omega_\theta) = 3/4$ if the tube is twisted and (14.3) holds. The study of the quantitative dependence of the constant γ from Theorem 14.6 on properties of $\dot\theta$ and the geometry of ω also constitutes an interesting open problem. Note that the two quantities are related by $\gamma + 1/4 \le \Gamma(\Omega_\theta)$.

Throughout the paper we assumed (14.3). We expect that this hypothesis can be replaced by a mere vanishing of $\dot\theta$ at infinity to get Theorems 14.1 and 14.6 (and also Theorem 14.2). This less restrictive assumption is known to be enough to ensure (14.4) and there exist versions of (14.1) even if (14.3) is violated (*cf* [18]). However, it is quite possible that a slower decay of $\dot\theta$ at infinity will make the effect of twisting stronger. In particular, can $\Gamma(\Omega_\theta)$ be strictly greater than $3/4$ if the tube is twisted and $\dot\theta$ decays to zero very slowly at infinity?

Equally, it is not clear whether Proposition 14.3 holds if (14.3) is violated. There are some further open problems related to the Hardy inequality of Theorem 14.2. In particular, it is frustrating that the proof of the theorem does not extend to all $\dot\theta$ merely vanishing at infinity. In this context, it would be highly desirable to establish a more quantitative version of Lemma 14.1, *i.e.* to get a positive lower bound to $\lambda(\dot\theta, I)$ depending explicitly on $\dot\theta$, $|I|$ and ω.

On the other hand, a completely different situation will appear if one allows twisted tubes for which $\dot\theta$ does not vanish at infinity. Then the spectrum of $-\Delta_D^{\Omega_\theta}$ can actually start strictly above E_1 (*cf* [9] or [17, Corol. 6.6]) and an extra exponential decay rate for our semigroup $S(t)$ follows at once already in $L^2(\Omega_\theta)$. In such situations it is more natural to study the decay of the semigroup associated with $-\Delta_D^{\Omega_\theta}$ shifted by the lowest point in its spectrum. As a particularly interesting situation we mention the case of periodically twisted tubes, for which a systematic analysis based on the Floquet-Bloch decomposition could be developed in the spirit of [6, 20].

We expect that the extra decay rate will be induced also in other twisted models for which Hardy inequalities have been established recently [16, 15].

It would be also interesting to study the effect of twisting in other physical models. As one possible direction of this research, let us mention the question of the long time behaviour of the solutions to the dissipative wave equation [11, 12, 19].

Let us conclude the paper by a general conjecture. We expect that there is always an improvement of the decay rate for the heat semigroup if a Hardy inequality holds:

Conjecture 14.1. *Let Ω be an open connected subset of $\mathbb{R}^d$. Let H and H_+ be two self-adjoint operators in $L^2(\Omega)$ such that $\inf\sigma(H) = \inf\sigma(H_+) = 0$. Assume that there is a positive smooth function $\varrho : \Omega \to \mathbb{R}$ such that $H_+ \ge \varrho$, while $H - V$ is a negative operator for any non-negative non-trivial $V \in C_0^\infty(\Omega)$. Then there exists a positive function $K : \Omega \to \mathbb{R}$ such that*

$$\lim_{t\to\infty} \frac{\|e^{-H_+ t}\|_{L^2(\Omega,K)\to L^2(\Omega)}}{\|e^{-Ht}\|_{L^2(\Omega,K)\to L^2(\Omega)}} = 0\,.$$

A similar conjecture can be stated for the same type of operators in different Hilbert spaces. In this paper we proved the conjecture for the special situation where $H = H_0 - E_1$ and $H_+ = H_\theta - E_1$ (transformed Dirichlet Laplacians) in $L^2(\Omega)$, with $\Omega = \Omega_0$ (unbounded tube). In general, the proof seems to be a hardly accessible problem.

Acknowledgment

The work was initiated and continued in a series of workshops *Partial Differential Equations, Optimal Design and Numerics*, Benasque, Spain. The first author would like to thank the Basque Center for Applied Mathematics in Bilbao where part of this work was carried out too, for hospitality and support. The work was partially supported by the Czech Ministry of Education, Youth and Sports within the project LC06002, and by Grant MTM2008-03541 of the MICINN (Spain).

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Chapter 15

The heat equation in a twisted Dirichlet-Neumann waveguide

Published in: Journal of Differential Equations **250** (2011), 2334–2346

Joint work with: Enrique Zuazua

The asymptotic behaviour of the heat equation in a twisted Dirichlet-Neumann waveguide

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Abstract. We consider the heat equation in a straight strip, subject to a combination of Dirichlet and Neumann boundary conditions. We show that a switch of the respective boundary conditions leads to an improvement of the decay rate of the heat semigroup of the order of $t^{-1/2}$. The proof employs similarity variables that lead to a non-autonomous parabolic equation in a thin strip contracting to the real line, that can be analyzed on weighted Sobolev spaces in which the operators under consideration have discrete spectra. A careful analysis of its asymptotic behaviour shows that an added Dirichlet boundary condition emerges asymptotically at the switching point, breaking the real line in two half-lines, which leads asymptotically to the $1/2$ gain on the spectral lower bound, and the $t^{-1/2}$ gain on the decay rate in the original physical variables.

This result is an adaptation to the case of strips with twisted boundary conditions of previous results by the authors on geometrically twisted Dirichlet tubes.

15.1 Introduction

We consider the heat equation

$$u_t - \Delta u = 0 \tag{15.1}$$

in an infinite planar strip $\Omega := \mathbb{R} \times (-a, a)$ of half-width $a > 0$, subject to

$$\begin{cases} \text{Dirichlet boundary conditions on } \Gamma_\pi^D := (-\infty, 0) \times \{-a\} \cup (0, +\infty) \times \{a\}\,, \\ \text{Neumann boundary conditions on } \Gamma_\pi^N := (0, +\infty) \times \{-a\} \cup (-\infty, 0) \times \{a\}\,, \end{cases}$$

and to the initial condition

$$u(\cdot, 0) = u_0 \in L^2(\Omega)\,. \tag{15.2}$$

This model is considered as a 'twisted' counterpart of the explicitly solvable problem given by (see Figure 15.1):

$$\begin{cases} \text{Dirichlet boundary conditions on } \Gamma_0^D := (-\infty, +\infty) \times \{-a\}\,, \\ \text{Neumann boundary conditions on } \Gamma_0^N := (-\infty, +\infty) \times \{a\}\,. \end{cases}$$

Henceforth we shall use the common subscript

$$\theta \in \{0, \pi\}$$

when we want to deal with the two problems simultaneously (the value of θ suggests the rotation angle giving rise to twisting/untwisting).

Figure 15.1: Planar strips with untwisted (left) and twisted (right) boundary conditions; the thick and thin lines correspond to Dirichlet and Neumann boundary conditions, respectively.

The solution to (15.1)–(15.2) is given by $u(t) = e^{\Delta_\theta t} u_0$, where $e^{\Delta_\theta t}$ is the semigroup operator on $L^2(\Omega)$ associated with the Laplacian $-\Delta_\theta$ determined by the respective boundary conditions (depending on θ).

The operators $-\Delta_\pi$ and $-\Delta_0$ have the same spectrum

$$\sigma(-\Delta_\theta) = \sigma_{\mathrm{ess}}(-\Delta_\theta) = [E_1, \infty)\,, \qquad \text{where} \qquad E_1 := \left(\frac{\pi}{4a}\right)^2. \tag{15.3}$$

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Consequently, for all $t \geq 0$,

$$\left\| e^{\Delta_\theta t} \right\|_{L^2(\Omega)\to L^2(\Omega)} = e^{-E_1 t}\,, \tag{15.4}$$

irrespectively of the value of θ.

In this paper, we are interested in additional time decay properties of the heat semigroup, when the initial data are restricted to a subspace of the Hilbert space $L^2(\Omega)$. We restrict ourselves to the weighted space

$$L^2(\Omega, K) \qquad \text{with} \qquad K(x) := e^{x_1^2/4}\,, \tag{15.5}$$

which means that the initial data are required to be sufficiently rapidly decaying at the infinity of the strip. As a measure of the additional decay, we consider the (polynomial) *decay rate*

$$\gamma_\theta := \sup\left\{ \gamma \,\middle|\, \exists C_\gamma > 0,\, \forall t \geq 0,\ \left\| e^{(\Delta_\theta + E_1)t} \right\|_{L^2(\Omega,K)\to L^2(\Omega)} \leq C_\gamma\,(1+t)^{-\gamma} \right\}. \tag{15.6}$$

Our main result reads as follows:

Theorem 15.1. *We have $\gamma_0 = 1/4$, while $\gamma_\pi \geq 3/4$.*

Here the power $1/4$ corresponding to the untwisted case $\theta = 0$ reflects the quasi-one-dimensional nature of our model (recall that $d/4$ is the analogous decay rate for the heat semigroup in the L^2-space over the whole Euclidean space $\mathbb{R}^d$ for initial data in $L^1(\mathbb{R}^d) \supset L^2(\mathbb{R}^d, e^{|x|^2/4}dx)$). Indeed, the result for $\theta = 0$ follows easily by separation of variables (*cf* Section 15.3).

The essential content of Theorem 15.1 is that solutions to (15.1) when the strip is twisted (*i.e.* $\theta = \pi$) gain a further decay rate $1/2$. The proof of this statement is more involved and constitutes the main body of the paper (*cf* Section 15.4). It is based on the method of self-similar solutions developed in the whole Euclidean space by Escobedo and Kavian [5] and adapted to waveguide systems by the present authors in [10], where it was shown that the heat kernel decays faster in geometrically twisted tubes than in untwisted ones. An open problem is to show that the decay rate γ_π is precisely $3/4$ (*cf* Section 15.5).

The way how to understand the difference in the decay rates of Theorem 15.1 is due to a fine difference between the operators $-\Delta_0$ and $-\Delta_\pi$ in the spectral setting: Although the operators have the same spectrum (*cf* (15.3)), the shifted operator $-\Delta_0 - E_1$ is critical, while $-\Delta_\pi - E_1$ is subcritical. The latter is reflected in the existence of a Hardy-type inequality

$$-\Delta_\pi - E_1 \geq \rho \tag{15.7}$$

with a positive function ρ (while such an inequality cannot hold for $-\Delta_0 - E_1$).

Various Hardy inequalities for $-\Delta_\pi - E_1$ were established in [8]. A general conjecture on the influence of the subcriticality of an operator on the improvement of the decay of the associated semigroup was made in [10], where an analog of Theorem 15.1 was proved for the decay rate in three-dimensional Dirichlet tubes. We also refer to [6] where the conjecture (for not necessarily self-adjoint operators) is analysed from the point of view of heat kernels and its relationship with Davies' conjecture [2] is observed.

The organization of this paper is as follows. In the following Section 15.2 we give a precise definition of the Laplacians $-\Delta_\theta$ and the associated semigroups. The untwisted case is briefly treated in Section 15.3, obtaining, *inter alia*, the first statement of Theorem 15.1. The main body of the paper is represented by Section 15.4 where we develop the method of self-similar solutions to get the improved decay rate of Theorem 15.1 (and also to establish an alternative result, Theorem 15.2). The paper is concluded in Section 15.5 by referring to physical interpretations of the result and to some open problems.

15.2 Preliminaries

The Laplacians $-\Delta_\theta$ are introduced as the self-adjoint operators associated on $L^2(\Omega)$ with the quadratic form $\psi \mapsto \|\nabla\psi\|^2$ having the domains

$$\mathfrak{D}_\theta(\Omega) := \left\{ \psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma_\theta^D \right\}.$$

Here and in the sequel $\|\cdot\|$ denotes the norm in the Hilbert space $L^2(\Omega)$. It is possible to specify the operator domains (*cf* [3]), but we will not need them. We only mention the result from [3] that the set of restrictions of functions from $C_0^\infty(\mathbb{R}^2)$ to Ω that vanish on Γ_θ^D is dense in $\mathfrak{D}_\theta$ with respect to the $H^1(\Omega)$ norm (*cf* [3, App. B]).

In view of (15.3), both the operators $-\Delta_\theta$ satisfy the Poincaré-type inequality

$$-\Delta_\theta \geq E_1 \tag{15.8}$$

in the sense of forms on $L^2(\Omega)$. Here E_1 is the first eigenvalue of the one-dimensional operator $-\Delta_{DN}^{(-a,a)}$, *i.e.* the Laplacian in $L^2((-a,a))$ subject to the Dirichlet boundary condition at $-a$ and Neumann boundary condition at a. This inequality is sharp for $-\Delta_0$, while it follows from [8] that (15.8) can be improved to a Hardy-type inequality (15.7) if $\theta = \pi$.

Recalling (15.4) and that we are interested in additional decay properties of (15.1), it is natural to rather consider the shifted parabolic problem (obtained from the standard heat equation (15.1) by the replacement $u(x,t) \mapsto e^{-E_1 t}\, u(x,t)$):

$$u_t - \Delta u - E_1 u = 0\,, \tag{15.9}$$

subject to the initial condition (15.2) and the boundary conditions

$$u = 0 \quad \text{on} \quad \Gamma_\theta^D \times (0,\infty)\,, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_\theta^N \times (0,\infty)\,, \tag{15.10}$$

where n denotes the normal vector to the boundary $\partial\Omega$.

As usual, we consider the weak formulation of the problem and, with an abuse of notation, we denote by the same symbol u both the function on $\Omega \times (0,\infty)$ and the mapping $(0,\infty) \to L^2(\Omega)$. Standard semigroup theory implies that there exists a unique solution of (15.9)–(15.10), subject to the initial condition (15.2), that belongs to $C^0\big([0,\infty), L^2(\Omega)\big)$. More precisely, the solution is given by $u(t) = S_\theta(t) u_0$, where

$$S_\theta(t) := e^{(\Delta_\theta + E_1)t} \tag{15.11}$$

is the heat semigroup associated with the shifted Laplacian $-\Delta_\theta - E_1$.

It is easy to see that the real and imaginary parts of the solution u of (15.9) evolve separately. By writing $u = \Re(u) + i\,\Im(u)$ and solving (15.9) with initial data $\Re(u_0)$ and $\Im(u_0)$, we may therefore reduce the problem to the case of a real function u_0, without restriction. Consequently, all the functional spaces are considered to be real in the sequel.

15.3 The untwisted strip

If the strip is untwisted (*i.e.* $\theta = 0$), the heat equation (15.9) can be easily solved by separation of variables. Indeed, the Laplacian $-\Delta_0$ can be identified with the decomposed operator

$$(-\Delta^{\mathbb{R}}) \otimes 1 + 1 \otimes (-\Delta_{DN}^{(-a,a)}) \qquad \text{in} \qquad L^2(\mathbb{R}) \otimes L^2((-a,a))\,, \tag{15.12}$$

where $-\Delta^{\mathbb{R}}$ denotes the one-dimensional free Hamiltonian (*i.e.* the usual self-adjoint realization of the Laplacian in $L^2(\mathbb{R})$) and 1 stands for the identity operators in the appropriate spaces.

The eigenvalues and (normalized) eigenfunctions of $-\Delta_{DN}^{(-a,a)}$ are respectively given by ($n = 1, 2, \dots$)

$$E_n := (2n-1)^2 E_1\,, \qquad \mathcal{J}_n(y_2) := \sqrt{\frac{1}{a}}\, \sin\Big[\sqrt{E_n}\,(y_2 + a)\Big]\,, \tag{15.13}$$

while the spectral resolution of $-\Delta^{\mathbb{R}}$ is obtained by the Fourier transform. Then it is easy to see that the heat semigroup $S_0(t)$ is an integral operator with kernel

$$s_0(x,x',t) := \sum_{n=1}^{\infty} e^{-(E_n - E_1)t}\, \mathcal{J}_n(x_2)\, p(x_1,x_1',t)\, \mathcal{J}_n(x_2')\,, \tag{15.14}$$

where

$$p(x_1,x_1',t) := \frac{e^{-(x_1-x_1')^2/(4t)}}{\sqrt{4\pi t}}$$

is the well known heat kernel of $-\Delta^{\mathbb{R}}$.

Using the explicit form of the heat kernel, it is straightforward to establish the following bounds:

Proposition 15.1. *There exists a constant C such that for every $t \geq 1$,*

$$C^{-1}\, t^{-1/4} \leq \|S_0(t)\|_{L^2(\Omega,K)\to L^2(\Omega)} \leq C\, t^{-1/4}\,.$$

Proof. To get the lower bound, we may restrict to the class of initial data (15.2) of the form $u_0(x) = \varphi(x_1)\mathcal{J}_1(x_2)$ with $\varphi \in L^2(\mathbb{R}, K_1)$, $K_1(x_1) := e^{x_1^2/4}$. Then it is easy to see from (15.14) that

$$\|S_0(t)\|_{L^2(\Omega,K)\to L^2(\Omega)} \geq \|P(t)\|_{L^2(\mathbb{R},K_1)\to L^2(\mathbb{R})}\,,$$

where $P(t)$ is the heat semigroup of $-\Delta^{\mathbb{R}}$ for which the lower bound with $t^{-1/4}$ is well known (or can be easily established by taking $\varphi = K_1^{-\alpha}$ with any $\alpha > 1/2$ and evaluating the integrals with the kernel p explicitly). On the other hand, using the Schwarz inequality, we have

$$\begin{aligned}\|S_0(t)u_0\|^2 &\le \|u_0\|_K^2 \int_{\Omega\times\Omega} s_0(x,x',t)^2\, K(x')^{-1}\, dx\, dx' \\ &= \|u_0\|_K^2 \sum_{n=1}^{\infty} e^{-2(E_n-E_1)t} \int_{\mathbb{R}\times\mathbb{R}} p(x_1,x_1',t)^2\, K_1(x_1')^{-1}\, dx_1\, dx_1'\end{aligned}$$

for every $u_0 \in L^2(\Omega, K)$. Here the sum can be estimated by a constant independent of $t \ge 1$ and the integral (computable explicitly) is proportional to $t^{-1/2}$. □

Remark 15.1. It is clear from the proof that the bounds hold in less restrictive weighted spaces. Indeed, it is enough to have a corresponding result for the one-dimensional heat semigroup $P(t)$.

As a consequence of Proposition 15.1, we get:

Corollary 15.1. *We have* $\gamma_0 = 1/4$.

Proof. The lower bound of Proposition 15.1 implies $\gamma_0 \le 1/4$. The opposite inequality follows from the upper bound and (15.4). □

15.4 The self-similarity transformation

Our method to study the asymptotic behaviour of the heat equation (15.9) is to adapt the technique of self-similar solutions used in the case of the heat equation in the whole Euclidean space by Escobedo and Kavian [5] to the present problem. Following [10], devoted to the analysis of the heat kernel in twisted tubes, we perform the self-similarity transformation in the first (longitudinal) space variable only, while keeping the other (transverse) space variable unchanged.

15.4.1 An equivalent time-dependent problem

More precisely, we consider a unitary transformation U on $L^2(\Omega)$ which associates to every solution

$$u \in L^2_{\mathrm{loc}}\big((0,\infty), dt; L^2(\Omega, dx)\big)$$

of (15.9) a self-similar solution $\tilde u = Uu$ in a new s-time weighted space

$$L^2_{\mathrm{loc}}\big((0,\infty), e^s ds; L^2(\Omega, dy)\big)$$

via

$$\tilde u(y_1, y_2, s) = e^{s/4} u(e^{s/2} y_1, y_2, e^s - 1)\,.$$

The inverse change of variables is given by

$$u(x_1, x_2, t) = (t+1)^{-1/4}\, \tilde u\big((t+1)^{-1/2} x_1, x_2, \log(t+1)\big)\,.$$

When evolution is posed in that context, $y = (y_1, y_2)$ plays the role of space variable and s is the new time.

It is easy to check that, in the new variables, the evolution is governed by

$$\tilde u_s - \tfrac{1}{2}\, y_1\, \partial_1 \tilde u - \partial_1^2 \tilde u - e^s\, \partial_2^2 \tilde u - E_1\, e^s\, \tilde u - \tfrac{1}{4}\, \tilde u = 0 \tag{15.15}$$

subject to the same initial and boundary conditions as u in (15.2) and (15.10), respectively.

Remark 15.2. Note that (15.15) is a parabolic equation with s-time-dependent coefficients. The same occurs and has been previously analyzed in twisted three-dimensional tubes [10] and for a convection-diffusion equation in the whole space but with a variable diffusion coefficient [4]. A careful analysis of the behaviour of the underlying elliptic operators as s tends to infinity leads to a sharp decay rate for its solutions.

Since U acts as a unitary transformation on $L^2(\Omega)$, it preserves the space norm of solutions of (15.9) and (15.15), *i.e.*,

$$\|u(t)\| = \|\tilde u(s)\|\,. \tag{15.16}$$

This means that we can analyse the asymptotic time behaviour of the former by studying the latter.

However, the natural space to study the evolution (15.15) is not $L^2(\Omega)$ but rather the weighted space (15.5). Following the approach of [10] based on a theorem of J. L. Lions [1, Thm. X.9] about weak solutions of parabolic equations with time-dependent coefficients, it can be shown that (15.15) is well posed in the scale of Hilbert spaces

$$\mathfrak{D}_\theta(\Omega,K) \subset L^2(\Omega,K) \subset \mathfrak{D}_\theta(\Omega,K)^* \,,$$

with

$$\mathfrak{D}_\theta(\Omega,K) := \big\{ \tilde{u} \in H^1(\Omega,K) \mid \tilde{u} = 0 \text{ on } \Gamma_\theta^D \big\} \,,$$

where $H^1(\Omega,K)$ denotes the usual weighted Sobolev space.

15.4.2 Reduction to a spectral problem

Multiplying the equation (15.15) by $\tilde{u}K$ and integrating by parts (precisely this means that we use $\tilde{u}K$ as a test function in a weak formulation of (15.15)), we arrive at the identity

$$\frac{1}{2}\frac{d}{ds}\|\tilde{u}(s)\|_K^2 = -J_\theta^s[\tilde{u}(s)] \,. \tag{15.17}$$

Here $\|\cdot\|_K$ denotes the norm in (15.5) and

$$J_\theta^s[\tilde{u}] := \|\partial_1\tilde{u}\|_K^2 + e^s\,\|\partial_2\tilde{u}\|_K^2 - E_1\,e^s\,\|\tilde{u}\|_K^2 - \frac{1}{4}\,\|\tilde{u}\|_K^2$$

is a closed quadratic form with domain $\mathfrak{D}(J_\theta^s) := \mathfrak{D}_\theta(\Omega,K)$ (independent of s). It remains to analyse the coercivity of J_θ^s.

More precisely, as usual for energy estimates, we replace the right hand side of (15.17) by the spectral bound, valid for each fixed $s \in [0,\infty)$,

$$\forall \tilde{u} \in \mathfrak{D}(J_\theta^s)\,, \qquad J_\theta^s[\tilde{u}] \geq \mu_\theta(s)\,\|\tilde{u}\|_K^2 \,, \tag{15.18}$$

where $\mu_\theta(s)$ denotes the lowest point in the spectrum of the self-adjoint operator T_θ^s associated in $L^2(\Omega,K)$ with J_θ^s. Then (15.17) together with (15.18) implies the exponential bound

$$\forall s \in [0,\infty)\,, \qquad \|\tilde{u}(s)\|_K \leq \|\tilde{u}_0\|_K\, e^{-\int_0^s \mu_\theta(r)dr} \,. \tag{15.19}$$

In this way, the problem is reduced to a spectral analysis of the family of operators $\{T_\theta^s\}_{s\geq 0}$.

15.4.3 Study of the spectral problem

In order to investigate the operator T_θ^s in $L^2(\Omega,K)$, we first map it into a unitarily equivalent operator $\hat{T}_\theta^s := \mathcal{U}T_\theta^s\mathcal{U}^{-1}$ in $L^2(\Omega)$ via the unitary transform

$$\mathcal{U}\,\tilde{u} := K^{1/2}\,\tilde{u} \,.$$

By definition, $\hat{T}_\theta^s$ is the self-adjoint operator associated in $L^2(\Omega)$ with the quadratic form $\hat{J}_\theta^s[v] := J_\theta^s[\mathcal{U}^{-1}v]$, $v \in \mathfrak{D}(\hat{J}_\theta^s) := \mathcal{U}\,\mathfrak{D}(J_\theta^s)$. A straightforward calculation yields

$$\begin{aligned} \hat{J}_\theta^s[v] &= \|\partial_1 v\|^2 + \frac{1}{16}\,\|y_1 v\|^2 + e^s\,\|\partial_2 v\|^2 - E_1\,e^s\,\|v\|^2 \,, \\ v \in \mathfrak{D}(\hat{J}_\theta^s) &= \mathfrak{D}_\theta(\Omega) \cap L^2(\Omega, y_1^2\,dy) \,. \end{aligned} \tag{15.20}$$

In particular, $\mathfrak{D}(\hat{J}_\theta^s)$ is independent of s. In the distributional sense, we can write

$$\hat{T}_\theta^s = -\partial_1^2 + \frac{1}{16}\,y_1^2 - e^s\,\partial_2^2 - E_1\,e^s \,. \tag{15.21}$$

We observe that the 'longitudinal part' of $\hat{T}_\theta^s$ coincides with the quantum harmonic-oscillator Hamiltonian

$$H := -\frac{d^2}{dy_1^2} + \frac{1}{16}\,y_1^2 \qquad \text{in} \qquad L^2(\mathbb{R}) \tag{15.22}$$

(*i.e.* the Friedrichs extension of this operator initially defined on $C_0^\infty(\mathbb{R})$). We recall the well known fact that the form domain

$$\mathfrak{D}(H^{1/2}) = H^1(\mathbb{R}) \cap L^2(\mathbb{R}, y_1^2\,dy_1)$$

is compactly embedded in $L^2(\mathbb{R})$, so that the spectrum of H is purely discrete. In fact, the spectrum can be computed explicitly (see any textbook on quantum mechanics, *e.g.*, [7, Sec. 2.3]):

$$\sigma(H) = \left\{ \frac{1}{2}\left(n + \frac{1}{2}\right) \right\}_{n=0}^{\infty} . \tag{15.23}$$

Using now the discreteness of spectra of H and $-\Delta_{DN}^{(-a,a)}$ together with the minimax principle, one may easily conclude that also $\hat{T}_\theta^s$ (and therefore T_θ^s) is an operator with compact resolvent for all $s \in [0,\infty)$. In particular, $\mu_\theta(s)$ represents the lowest eigenvalue of T_θ^s.

15.4.4 The asymptotic behaviour of the spectrum

In order to study the decay rate via (15.19), we need information about the limit of the eigenvalue $\mu_\theta(s)$ as the time s tends to infinity. Notice that the scaling of the transverse variable in (15.21) corresponds to considering the operator $\hat{T}_\theta^1$ in the shrinking strip $\mathbb{R} \times (-e^{-s/2}a, e^{-s/2}a)$. This suggests that $\hat{T}_\theta^s$ will converge, in a suitable sense, to a one-dimensional operator of the type (15.22). We shall see that the difference between the twisted ($\theta = \pi$) and untwisted case ($\theta = 0$) consists in that the limit operator for the former is subject to an extra Dirichlet boundary condition at $y_1 = 0$.

Thus, simultaneously to H introduced in (15.22), let us therefore consider the self-adjoint operator H_D in $L^2(\mathbb{R})$ whose quadratic form acts in the same way as that of H but has a smaller domain

$$\mathfrak{D}(H_D^{1/2}) := \left\{ \varphi \in \mathfrak{D}(H^{1/2}) \mid \varphi(0) = 0 \right\} .$$

In fact, it is readily seen that $\hat{T}_0^s$ can be identified with the decomposed operator

$$H \otimes 1 + 1 \otimes (-e^s\, \Delta_{DN}^{(-a,a)} - E_1\, e^s) \qquad \text{in} \qquad L^2(\mathbb{R}) \otimes L^2((-a,a)) , \tag{15.24}$$

where 1 denotes the identity operators in the appropriate spaces. Using (15.23), it follows that $\mu_0(s) = 1/4$ for all $s \in [0,\infty)$. Consequently,

$$\mu_0(\infty) := \lim_{s\to\infty} \mu_0(s) = 1/4 . \tag{15.25}$$

Moreover, (15.24) can be used to show that $\hat{T}_0^s$ converges to H in the norm-resolvent sense as $s \to \infty$, if the latter is considered as an operator acting on the subspace of $L^2(\Omega)$ consisting of functions of the form $\varphi(y_1)\mathcal{J}_1(y_2)$, where $\mathcal{J}_1$ is introduced in (15.13).

It is more difficult (and more interesting) to establish the asymptotic behaviour of $\mu_\pi(s)$. A fine analysis of its behaviour leads to the key observation of the paper, ensuring a gain of $1/2$ in the decay rate in the twisted case.

We decompose the Hilbert space $L^2(\Omega)$ into an orthogonal sum

$$L^2(\Omega) = \mathfrak{H}_1 \oplus \mathfrak{H}_1^\perp , \tag{15.26}$$

where the subspace $\mathfrak{H}_1$ consists of functions of the form

$$\psi_1(y) = \varphi(y_1)\, \mathcal{J}_1\big(\operatorname{sgn}(-y_1)\, y_2\big) . \tag{15.27}$$

Notice that $y_2 \mapsto \mathcal{J}_1(-y_2)$ is an eigenfunction of $-\Delta_{ND}^{(-a,a)}$, *i.e.* the Laplacian in $L^2((-a,a))$ subject to the Neumann boundary condition at $-a$ and Dirichlet boundary condition at a (reversed boundary conditions with respect to $-\Delta_{DN}^{(-a,a)}$). Hence ψ_1 satisfies the boundary conditions of $-\Delta_\pi$. Given any $\psi \in L^2(\Omega)$, we have the decomposition $\psi = \psi_1 + \phi$ with $\psi_1 \in \mathfrak{H}_1$ as above and $\phi \in \mathfrak{H}_1^\perp$. The mapping $\iota : \varphi \mapsto \psi_1$ is an isomorphism of $L^2(\mathbb{R})$ onto $\mathfrak{H}_1$. Hence, with an abuse of notations, we may identify any operator h on $L^2(\mathbb{R})$ with the operator $\iota h \iota^{-1}$ acting on $\mathfrak{H}_1 \subset L^2(\Omega)$. Having this convention in mind, we state the following convergence result.

Proposition 15.2. *The operator $\hat{T}_\pi^s$ converges to $H_D \oplus 0^\perp$ in the strong-resolvent sense as $s \to \infty$,* i.e.,

$$\forall F \in L^2(\Omega) , \qquad \lim_{s\to\infty} \left\| \big(\hat{T}_\pi^s + 1\big)^{-1} F - \Big[\big(H_D + 1\big)^{-1} \oplus 0^\perp \Big]\, F \right\| = 0 .$$

Proof. We proceed as in the proof of [10, Prop. 5.4]. For any fixed $F \in L^2(\Omega)$, let us set $\psi_s := (\hat{T}_\pi^s + 1)^{-1} F$. In other words, ψ_s satisfies the resolvent equation

$$\forall v \in \mathfrak{D}(\hat{J}_\pi^s) , \qquad \hat{J}_\pi^s(v, \psi_s) + (v, \psi_s) = (v, F) , \tag{15.28}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ and $\hat{J}^s_\pi(\cdot,\cdot)$ is the sesquilinear form associated with (15.20). In particular, choosing ψ_s for the test function v in (15.28), we have

$$\|\partial_1\psi_s\|^2 + \frac{1}{16}\,\|y_1\psi_s\|^2 + e^s\Big(\|\partial_2\psi_s\|^2 - E_1\|\psi_s\|^2\Big) + \|\psi_s\|^2 = (\psi_s, F) \le \frac{1}{4}\,\|\psi_s\|^2 + \|F\|^2\,. \tag{15.29}$$

Notice that $\|\partial_2\psi_s\|^2 \ge E_1\|\psi_s\|^2$ by Fubini's theorem and the Poincaré inequality for $-\Delta_{DN}^{(-a,a)}$ and $-\Delta_{ND}^{(-a,a)}$. Consequently,

$$\|\partial_1\psi_s\|^2 \le C\,,\quad \|y_1\psi_s\|^2 \le C\,,\quad \|\psi_s\|^2 \le C\,,\quad \|\partial_2\psi_s\|^2 - E_1\|\psi_s\|^2 \le Ce^{-s}\,, \tag{15.30}$$

where C is a constant proportional to $\|F\|^2$.

Now we employ the decomposition

$$\psi_s(y) = \varphi_s(y_1)\,\mathcal{J}_1\big(\operatorname{sgn}(-y_1)\,y_2\big) + \phi_s(y)$$

where $\phi_s \in \mathfrak{H}_1^\perp$, *i.e.*,

$$\forall y_1 \in \mathbb{R}\,,\qquad \int_{-a}^{a} \mathcal{J}_1\big(\operatorname{sgn}(-y_1)\,y_2\big)\,\phi_s(y_1,y_2)\,dy_2 = 0\,. \tag{15.31}$$

That is, $y_2 \mapsto \phi_s(y_1,y_2)$ is orthogonal to the ground-state eigenfunction of $-\Delta_{DN}^{(-a,a)}$ (respectively of $-\Delta_{ND}^{(-a,a)}$) if $y_1 < 0$ (respectively $y_1 > 0$). Then

$$\begin{aligned}\|\partial_2\psi_s\|^2 - E_1\|\psi_s\|^2 &= \|\partial_2\phi_s\|^2 - E_1\|\phi_s\|^2\\ &= \tfrac{1}{2}\,\|\partial_2\phi_s\|^2 + \tfrac{1}{2}\,\|\partial_2\phi_s\|^2 - E_1\|\phi_s\|^2\\ &\ge \tfrac{1}{2}\,\|\partial_2\phi_s\|^2 + \big(\tfrac{1}{2}\,E_2 - E_1\big)\,\|\phi_s\|^2\,,\end{aligned}$$

where $E_2 = 9E_1$ denotes the second eigenvalue of $-\Delta_{DN}^{(-a,a)}$ (which coincides with that of $-\Delta_{ND}^{(-a,a)}$). Thus it follows from the last inequality of (15.30) that

$$\|\phi_s\|^2 \le Ce^{-s} \qquad \text{and} \qquad \|\partial_2\phi_s\|^2 \le Ce^{-s}\,, \tag{15.32}$$

where C is a constant proportional to $\|F\|^2$.

It follows from (15.30) that $\{\psi_s\}_{s>0}$ is a bounded family in $\mathfrak{D} := H^1(\Omega) \cap L^2(\Omega, y_1^2 dy_1)$ (equipped with the intersection topology). Therefore it is precompact in the weak topology of $\mathfrak{D}$. Let ψ_∞ be a weak limit point, *i.e.*, for an increasing sequence of positive numbers $\{s_n\}_{n\in\mathbb{N}}$ such that $s_n \to \infty$ as $n \to \infty$, $\{\psi_{s_n}\}_{n\in\mathbb{N}}$ converges weakly to ψ_∞ in $\mathfrak{D}$. Actually, we may assume that it converges strongly in $L^2(\Omega)$ because $\mathfrak{D}$ is compactly embedded in $L^2(\Omega)$. Since $\{\phi_{s_n}\}_{n\in\mathbb{N}}$ converges strongly to zero in $L^2(\Omega)$ due to (15.32), we know that $\psi_\infty \in \mathfrak{H}_1$, *i.e.*,

$$\psi_\infty(y) = \varphi_\infty(y_1)\,\mathcal{J}_1\big(\operatorname{sgn}(-y_1)\,y_2\big)$$

with some $\varphi_\infty \in L^2(\mathbb{R})$. Since the weak derivative $\partial_1\psi_\infty \in L^2(\Omega)$ exists, we necessarily have $\varphi_\infty \in H^1(\mathbb{R})$ and

$$\varphi_\infty(0) = 0\,.$$

Finally, let $\varphi \in C_0^\infty(\mathbb{R}\setminus\{0\})$ be arbitrary. Taking

$$v(y) := \varphi(y_1)\,\mathcal{J}_1\big(\operatorname{sgn}(-y_1)\,y_2\big)$$

as the test function in (15.28), with s being replaced by s_n, and sending n to infinity, we easily check that

$$(\dot\varphi, \dot\varphi_\infty)_{L^2(\mathbb{R})} + \frac{1}{16}\,(y_1\varphi, y_1\varphi_\infty)_{L^2(\mathbb{R})} + (\varphi,\varphi_\infty)_{L^2(\mathbb{R})} = (\varphi, f)_{L^2(\mathbb{R})}\,,$$

where

$$f(y_1) := \int_{-a}^{a} \mathcal{J}_1\big(\operatorname{sgn}(-y_1)\,y_2\big)\,F(y_1,y_2)\,dy_2\,.$$

That is, $\varphi_\infty = (H_D+1)^{-1}f$, for *any* weak limit point of $\{\varphi_s\}_{s\ge0}$. Summing up, we have shown that ψ_s converges strongly to ψ_∞ in $L^2(\Omega)$ as $s \to \infty$, where $\psi_\infty(y) = \big[(H_D+z)^{-1} \oplus 0^\perp\big]F$. □

Corollary 15.2. *One has*

$$\mu_\pi(\infty) := \lim_{s\to\infty} \mu(s) = 3/4\,.$$

Proof. In general, the strong-resolvent convergence of Proposition 15.2 is not sufficient to guarantee the convergence of spectra. However, in our case, since the spectra are purely discrete, the eigenprojections converge even in norm (*cf* [12]). In particular, $\mu_\pi(s)$ converges to the first eigenvalue of H_D as $s \to \infty$. It remains to notice that the first eigenvalue of H_D coincides (in view of the symmetry) with the second eigenvalue of H which is $3/4$ due to (15.23). □

15.4.5 A lower bound to the decay rate

We come back to (15.19). Recalling (15.25) and Corollary 15.2, we know that for arbitrarily small positive number ε there exists a (large) positive time s_ε such that for all $s \geq s_\varepsilon$, we have $\mu_\theta(s) \geq \mu_\theta(\infty) - \varepsilon$. Hence, fixing $\varepsilon > 0$, for all $s \geq s_\varepsilon$, we have

$$\begin{aligned} -\int_0^s \mu_\theta(r)\,dr &\leq -\int_0^{s_\varepsilon} \mu_\pi(r)\,dr - [\mu_\theta(\infty) - \varepsilon](s - s_\varepsilon) \\ &\leq [\mu_\theta(\infty) - \varepsilon]s_\varepsilon - [\mu_\theta(\infty) - \varepsilon]s\,, \end{aligned}$$

where the second inequality is due to the fact that $\mu_\theta(s)$ is non-negative for all $s \geq 0$ (it is in fact greater than or equal to $1/4$, *cf* Proposition 15.3 below). At the same time, assuming $\varepsilon \leq 1/4$, we trivially have

$$-\int_0^s \mu_\theta(r)\,dr \leq 0 \leq [\mu_\theta(\infty) - \varepsilon]s_\varepsilon - [\mu_\theta(\infty) - \varepsilon]s$$

also for all $s \leq s_\varepsilon$. Summing up, for every $s \in [0, \infty)$, we have

$$\|\tilde{u}(s)\|_K \leq C_\varepsilon\, e^{-[\mu_\theta(\infty)-\varepsilon]s}\, \|\tilde{u}_0\|_K\,, \tag{15.33}$$

where $C_\varepsilon := e^{s_\varepsilon} \geq e^{[\mu_\theta(\infty)-\varepsilon]s_\varepsilon}$.

Now we return to the original variables $(x_1, x_2, t) = (e^{s/2}y_1, y_2, e^s - 1)$. Using (15.16) together with the point-wise estimate $1 \leq K$, and recalling that $\tilde{u}_0 = u_0$, it follows from (15.33) that

$$\|u(t)\| = \|\tilde{u}(s)\| \leq \|\tilde{u}(s)\|_K \leq C_\varepsilon\, (1+t)^{-[\mu_\theta(\infty)-\varepsilon]}\, \|u_0\|_K$$

for every $t \in [0, \infty)$. Consequently, we conclude with

$$\|S_\theta(t)\|_{L^2(\Omega,K)\to L^2(\Omega)} = \sup_{u_0 \in L^2(\Omega,K)\setminus\{0\}} \frac{\|u(t)\|}{\|u_0\|_K} \leq C_\varepsilon\, (1+t)^{-[\mu_\theta(\infty)-\varepsilon]}$$

for every $t \in [0, \infty)$. Since ε can be made arbitrarily small, this bound implies

$$\gamma_\theta \geq \mu_\theta(\infty)\,.$$

This together with Corollary 15.1 proves Theorem 15.1.

15.4.6 A global upper bound to the heat semigroup

Theorem 15.1 provides quite precise information about the extra polynomial decay of solutions u of (15.9) in a twisted tube in the sense that the decay rate γ_π is better by a factor $1/2$ than in the untwisted case. On the other hand, we have no control over the constant C_γ in (15.6) (in principle it may blow up as $\gamma \to \gamma_\theta$). We therefore conclude this section by establishing a global (in time) upper bound to the heat semigroup (*i.e.* we get rid of the constant C_γ) but the prize we pay is just a qualitative knowledge about the decay rate. It is a consequence of (15.19) and the following result:

Proposition 15.3. $\forall s \geq 0, \qquad \mu_0(s) = 1/4\,, \qquad \mu_\pi(s) > 1/4\,.$

Proof. The identity for μ_0 is readily seen from the decomposition (15.24) and (15.23). Using Fubini's theorem and the minimax principle, it is also easy to deduce from (15.20) that $\mu_\pi(s) \geq 1/4$ for all $s \geq 0$. To show that the inequality is strict, let us assume by contradiction that $\mu_\pi(s) = 1/4$ for some $s \geq 0$. Let v denote the corresponding eigenfunction of $\hat{T}^s_\pi$. Then the identity $\hat{J}^s_\pi[v] = \mu_\pi(s)\|v\|^2$ yields

$$\|\partial_1 v\|^2 + \frac{1}{16}\,\|y_1 v\|^2 = \frac{1}{4}\,\|v\|^2 \qquad \text{and} \qquad \|\partial_2 v\|^2 - E_1\,\|v\|^2 = 0\,. \tag{15.34}$$

Using the direct-sum decomposition (15.26), the second identity implies that v is of the form (15.27). The continuity of the eigenfunction v inside Ω in turn requires that $\varphi(0) = 0$. However, this contradicts the first identity in (15.34) which says, in view of (15.23), that φ is the first (therefore nowhere vanishing) eigenfunction of H. □

Combining this result with Corollary 15.2, we see that the number

$$c_\theta := \inf_{s\in[0,\infty)} \mu_\theta(s) - 1/4 \tag{15.35}$$

is positive if $\theta = \pi$ and zero if $\theta = 0$. In any case, (15.19) implies

$$\|\tilde{u}(s)\|_K \leq \|\tilde{u}_0\|_K \, e^{-(c_\theta + 1/4)s}$$

for every $s \in [0, \infty)$. Using this estimate instead of (15.33), but following the same type of arguments as in Section 15.4.5 below (15.33), we thus conclude with:

Theorem 15.2. *We have*

$$\forall t \in [0, \infty), \qquad \|S_\theta(t)\|_{L^2(\Omega,K)\to L^2(\Omega)} \leq (1+t)^{-(c_\theta+1/4)} \,,$$

where $c_\pi > 0$ *(and* $c_0 = 0$*).*

15.5 Conclusions

The classical interpretation of the heat equation (15.1) is that its solution u gives the evolution of the temperature distribution of a medium in the strip Ω surrounded by a perfect insulator on the Neumann boundary Γ_θ^N and by a substance of a high-heat capacity and of zero temperature on the Dirichlet boundary Γ_θ^D. It also represents the simplest version of the stochastic Fokker-Planck equation describing the Brownian motion in Ω which is normally reflected on Γ_θ^N and killed on Γ_θ^D (*cf* [11] for a probabilistic setting in an analogous higher-dimensional model). Then the results of the present paper can be interpreted as that the twisting of boundary conditions (*i.e.* $\theta = \pi$) implies a faster cool-down/death of the medium/Brownian particle in the strip. Many other diffusive processes in nature are governed by (15.1).

Our proof that there is an extra decay rate for solutions of (15.1) if the boundary conditions are twisted was far from being straightforward. This is a bit surprising because the result is quite expectable from the physical interpretation, if one notices that the twist makes it more difficult for the Brownian particle to pass through the channel at $\{x_1 = 0\}$, because of the proximity of killing boundary conditions. At the same time, the Hardy inequality (15.7) did not play any role in the proof of Theorem 15.1 (although, combining the theorem with the results of [8], we eventually know that the existence of the Hardy inequality is equivalent to the extra decay rate for the heat semigroup). It would be desirable to find a more direct proof of Theorem 15.1 based on (15.7).

We conjecture that the inequality of Theorem 15.1 can be replaced by equality, *i.e.*, $\gamma_\pi = 3/4$ for twisted strip. The question of optimal value of the constant c_π (and its quantitative dependence on the half-width a) from Theorem 15.2 also constitutes an interesting open problem. Note that the two quantities are related by $c_\pi + 1/4 \leq \gamma_\theta$.

The present paper can be viewed as a continuation of the research initiated by our work [10], where we investigated the large-time behaviour of the heat semigroup in geometrically twisted Dirichlet tubes. It confirms that the effect of twisting (leading to the subcriticality of the Laplacian and implying an improvement of the decay rate of the associated heat semigroup) is more general, namely it holds true also in waveguide systems twisted via boundary conditions. We expect that the extra decay rate will be induced also in the systems twisted via embedding of the strip into a negatively curved manifold, for which the existence of Hardy inequalities is already known [9].

More generally, recall that we expect that there is always an improvement of the decay rate for the heat semigroup of a subcritical operator (*cf* [10, Conjecture in Sec. 6] and [6, Conjecture 1]).

Acknowledgment

The work was partially supported by the Czech Ministry of Education, Youth and Sports within the project LC06002, and by Grant MTM2008-03541 the MICINN (Spain), project PI2010-04 of the Basque Government and the ERC Advanced Grant FP7-246775 NUMERIWAVES.

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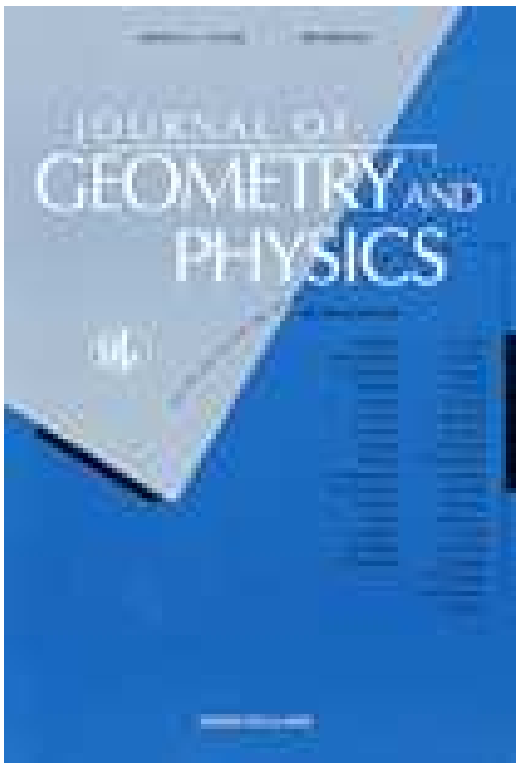
Part III

Quantum traveller on manifolds

Chapter 16

Quantum strips on surfaces



Published in: Journal of Geometry and Physics **45** (2003), 203–217

Quantum strips on surfaces

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Abstract. Motivated by the theory of quantum waveguides, we investigate the spectrum of the Laplacian, subject to Dirichlet boundary conditions, in a curved strip of constant width that is defined as a tubular neighbourhood of an infinite curve in a two-dimensional Riemannian manifold. Under the assumption that the strip is asymptotically straight in a suitable sense, we localise the essential spectrum and find sufficient conditions which guarantee the existence of geometrically induced bound states. In particular, the discrete spectrum exists for strips in non-negatively curved manifolds which are studied in detail. The general results are used to recover and revisit the known facts about quantum strips in the plane. As an example of strips in non-positively curved manifolds, we consider strips on ruled surfaces.

16.1 Introduction

The theory of quantum waveguides constitutes a beautiful domain of mathematical physics in which one meets an interesting interaction of analysis and geometry. Recall that the configuration space Ω of a waveguide is usually modelled by tubular neighbourhoods of infinite curves in $\mathbb{R}^d$, $d = 2, 3$ (quantum strips, tubes), or surfaces in $\mathbb{R}^3$ (quantum layers), while the dynamics is governed by the Laplace operator with Dirichlet boundary conditions. It is due to an admirable progress of mesoscopis physics that such models do really represent actual nanostructures which are produced in the laboratory nowadays. We refer to [6, 22] for the physical background and references.

A common, particularly interesting property of these systems is that the curvature of the reference curve or surface may produce bound states of the Laplacian below the essential spectrum. This phenomenon was demonstrated first in a rigourous way by P. Exner and P. Šeba for curved strips in the plane, [10]. Numerous subsequent studies improved their result and generalised it to space tubes. For more information and other spectral and scattering properties, see the review paper [6] and references therein. The evidently more complicated case of quantum layers was investigated quite recently in [7, 8, 9].

Up to this time, the ambient manifold of the quantum waveguide has been usually identified with a flat Euclidean space $\mathbb{R}^d$, $d = 2, 3$. This restriction is obviously due to the physical reasons, however, at least from the mathematical point of view, one may be interested equally in the situations when it is a general Riemannian manifold $\mathcal{A}$ of dimension $d \geq 2$. The principal interest of the present work is to initiate this study by considering the simplest non-trivial case, $d = 2$, when the configuration space Ω is a tubular neighbourhood of constant radius $a > 0$ about an infinite curve Σ on a surface $\mathcal{A}$.

Let us describe the contents of the paper. The strip configuration space Ω itself is properly defined in Section 16.2.1. Through all the paper, we suppose that the strip is globally parameterised by a system of geodesic coordinates based on the reference curve Σ. In accordance with [15], we call them *Fermi coordinates*, [11], although they had already been considered by C. F. Gauss. A comprehensive discussion of such a coordinate system has been given by F. Fiala, [12], in order to prove some isoperimetric inequalities; see also [17]. A modern definition of Fermi coordinates of tubes about a submanifold of a general Riemannian manifold can be found in [15]. We introduce them for our purposes in Section 16.2.2.

In Section 16.2.3, the Hamiltonian H of our system is identified with the Friedrichs extension of the Laplacian, $-\Delta$ on $L^2(\Omega)$, which is expressed in Fermi coordinates and defined initially on $C_0^\infty(\Omega)$. The construction is based on the quadratic-form approach of [4, Chap. 6]. Two trivial classes of quantum strips are then mentioned in Section 16.2.4. If the curvature of the ambient space is identically equal to zero on Ω, the strip is called *flat* and the spectrum of H coincides with the spectrum of strips in the plane, [6]. A generalisation of straight strips in the plane is represented by *geodesic* strips, for which the reference curve is in addition a geodesic. In that case, we find that the spectrum is the interval $[\kappa_1^2, \infty)$, where $\kappa_1 := \pi/(2a)$.

Section 16.3 is devoted to a heuristic analysis of the Hamiltonian H. Using a unitary transformation, it can be identified with a Schrödinger-like operator with a potential expressed by means of the metric of Ω. The latter operator acquires a very instructive form in the formal limit when the width of the strip tends to zero. In particular, we reveal an effective potential which is given by a combination of curvatures of Σ and $\mathcal{A}$. The

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result is compared with the case of strips in the plane and used as a motivation for the spectral analysis of H in the subsequent sections.

In Section 16.4, we localise the essential spectrum under the assumption that the strip is *asymptotically geodesic* in a suitable sense. Using a Neumann bracketing argument together with the minimax principle, we find in Theorem 16.1 that the threshold of $\sigma_{\rm ess}(H)$ is then bounded from below by κ_1^2.

Section 16.5 is devoted to the analysis of the spectrum below the energy κ_1^2. Using a variational technique standard in the theory of quantum waveguides, we find two sufficient conditions which guarantee that this part of spectrum is not empty, *cf* Theorems 16.2 and 16.3. These conditions require that the strip is non-negatively curved in an integral sense; see (16.21) for the precise meaning of the statement. Combining these results with Theorem 16.1, we arrive at Corollary 16.1 which contains the main result of this paper concerning the existence of a non-trivial discrete spectrum in quantum strips.

Since the condition (16.21) is clearly satisfied for non-negatively curved strips, this situation is investigated in detail in Section 16.6. We simplify some assumptions, we have put on the geometry of Ω, and sum up the spectral results in Theorem 16.4. Apart from a significant generalisation, it recovers and revisits the known results for the quantum strips in the plane.

To the best of our knowledge, it is for the first time when the spectrum of a curved strip embedded in a non-trivial manifold has been investigated. An exception is the paper [2], where I. J. Clark and A. J. Bracken deal with a special class of quantum strips in $\mathbb{R}^3$, which are made up from segments perpendicular to an infinite space curve Σ. They introduce the Hamiltonian in a formal way, derive the effective potential mentioned above and make some conjectures on the influence of the torsion of Σ on the spectrum, however, do not perform any spectral analysis itself. Actually, their paper is a preliminary to [1], where bound states in space quantum waveguides with torsion are investigated. The strip of [2] is a part of a *ruled* surface $\mathcal{A}$ based on Σ; we examine this situation briefly in Section 16.7.

We conclude the paper by Section 16.8, where some open problems and directions of a future research are mentioned. A particularly interesting question concerns possible applications to physics.

16.2 Preliminaries

16.2.1 Definitions

Let $\mathcal{A}$ be a non-compact two-dimensional Riemannian manifold of class C^2 and let K denote its Gauss curvature. We require that K is a continuous function on $\mathcal{A}$, which holds if $\mathcal{A}$ is of class C^3 or if it is embedded in $\mathbb{R}^3$. Even if it is not necessary for our construction, we shall assume that $\mathcal{A}$ is geodesically complete.

Let Σ be a simple, infinite curve of class C^2 embedded in $\mathcal{A}$ and let k denote its curvature. (We do not require that $\mathcal{A}$ is embedded in $\mathbb{R}^3$, however, if it is that case, k means the geodesic curvature of Σ.) We may assume that Σ is given by the image of the mapping $p : \mathbb{R} \to \mathcal{A}$ such that $|p'| = 1$. It represents the C^2-parameterisation of the curve by its arc length. We note that k is a continuous function on Σ.

Let $a > 0$ and $I := (-a, a)$. The strip Ω of width $2a$ is defined as the a-tubular neighbourhood of Σ in $\mathcal{A}$:

$$\Omega := \{x \in \mathcal{A} \mid \mathrm{dist}(x, \Sigma) < a\} . \tag{16.1}$$

As usual, the distance $\mathrm{dist}(x, \Sigma)$ means here the length of the minimal geodesic joining x with Σ. We want to introduce the Laplacian in Ω and investigate its spectrum. Our strategy is to map the curved strip (16.1) onto the straight one, $\Omega_0 := \mathbb{R} \times I$, by the use of Fermi coordinates which are defined in the following subsection.

16.2.2 Fermi Coordinates

We denote by $T_x\mathcal{A}$ the tangent space to $\mathcal{A}$ at $x \in \mathcal{A}$ and recall that the exponential map, $\exp_x : T_x\mathcal{A} \to \mathcal{A}$, is the identification $t \mapsto \gamma_t(1)$, where γ_t is the unique geodesic (parameterised by arc length) in $\mathcal{A}$ with $\gamma_t(0) = x$ and $\gamma_t'(0) = t$. We define

$$\mathcal{L} : \mathbb{R}^2 \to \mathcal{A} : \Big\{(s, u) \mapsto \exp_{p(s)} (u\, n(s)) \mid n \in N_p\Sigma\Big\} , \tag{16.2}$$

where $N_p\Sigma$ denote the orthogonal complement of $T_p\Sigma$ in $T_p\mathcal{A}$, and always assume that

⟨H1⟩ $\quad \mathcal{L} : \Omega_0 \to \Omega$ is a diffeomorphism for some $a > 0$.

Then the inverse of $\mathcal{L}$ determines the system of Fermi "coordinates" (s, u) and one has

$$\Omega = \mathcal{L}(\Omega_0) . \tag{16.3}$$

Remark 16.1. Hereafter we shall use the standard component notation of tensor analysis with the range of indices being $1, 2$ and associate them with Fermi coordinates via the identification, $(1, 2) \leftrightarrow (s, u)$. The partial derivatives will be denoted by commas. From now on the curvature K shall be considered as a function of Fermi coordinates (s, u); k is a function of s.

The metric tensor of Ω in Fermi coordinates is given by $G_{ij} := \langle \mathcal{L}_{,i}, \mathcal{L}_{,j} \rangle$, where "$\langle \cdot, \cdot \rangle$" denotes the inner product induced by the Riemannian metric on $\mathcal{A}$. Note that $s \mapsto \mathcal{L}(s, u)$ traces the curves parallel to Σ at a fixed distance $|u|$ and that the curve $u \mapsto \mathcal{L}(s, u)$ is a unit-speed geodesic orthogonal to Σ for any fixed s. The generalised Gauss Lemma, [15, Sec. 2.4], implies that these curves meet orthogonally and one arrives at the diagonal form of the metric tensor

$$(G_{ij}) = \begin{pmatrix} f(s,u)^2 & 0 \\ 0 & 1 \end{pmatrix} . \tag{16.4}$$

According to [16, 18], the function f is continuous and has continuous partial derivatives $f_{,u}$, $f_{,uu}$ satisfying the Jacobi equation

$$f_{,uu} + K f = 0 \qquad \text{with} \qquad \begin{array}{c} f(\cdot, 0) = 1 \,, \\ f_{,u}(\cdot, 0) = k \,. \end{array} \tag{16.5}$$

The determinant of the metric tensor, $G := \det(G_{ij}) = f^2$, defines through $d\Omega := G(s, u)^{\frac{1}{2}} ds du$ the surface element of the strip.

Remark 16.2. If Σ was a compact curve, then the condition $\langle \mathsf{H1} \rangle$ could always be achieved for sufficiently small a. Recall also that the same holds true for infinite strips in the plane if one assumes in addition that Ω does not overlap, [6]. In our case, the situation is analogous. The inverse function theorem implies that $\mathcal{L} : \Omega_0 \to \Omega$ is a local diffeomorphism provided f is uniformly strictly positive and bounded, *cf* posterior assumption $\langle \mathsf{H2} \rangle$. This can be achieved for a small enough because $f(\cdot, 0) = 1$. The condition $\langle \mathsf{H1} \rangle$ will be then fulfilled if we do not allow in addition an overlapping of the strip.

We needed the ambient manifold $\mathcal{A}$ just in order to define the strip by means of (16.1) or (16.3). Once the construction is over, we may forget about the rest of $\mathcal{A}$ and consider its part Ω only. It will be our configuration space. Note that the closure, $\bar{\Omega}$, is a manifold with boundary.

16.2.3 Hamiltonian

After geometric preliminaries, let us define the Hamiltonian of our system. We consider a non-relativistic quantum particle within the two-dimensional region Ω of impenetrable boundary. As usual, we put $\hbar^2/(2m) = 1$, where $\hbar$ denotes Planck's constant and m the mass of the particle. Then the Hamiltonian could be identified with the Laplace operator, $-\Delta$ on $L^2(\Omega)$, with an appropriate domain of functions which vanish on $\partial\Omega$. However, we proceed differently and always understand this Laplacian in the generalised (form) sense.

In detail, using Fermi coordinates, we shall identify the Hilbert space $L^2(\Omega)$ with $\mathcal{H} := L^2(\Omega_0, d\Omega)$. Let us consider the quadratic form on $\mathcal{H}$ given by

$$Q(\psi, \phi) := \big(\psi_{,i}, G^{ij} \phi_{,j}\big)_{\mathcal{H}} \,, \qquad \mathrm{Dom}\, Q := W_0^{1,2}(\Omega_0, d\Omega) \,, \tag{16.6}$$

where (G^{ij}) is the inverse of (G_{ij}). Assuming that the metric is uniformly elliptic in the sense that the condition

$\langle \mathsf{H2} \rangle \qquad \exists c_\pm > 0 \quad \forall (s, u) \in \Omega_0 \ : \quad c_- \leq f(s, u) \leq c_+$

is valid, it follows that the form Q is densely defined, non-negative, symmetric and closed on its domain. Consequently, there exists a non-negative self-adjoint operator H associated to Q which satisfies $\mathrm{Dom}\, H \subset \mathrm{Dom}\, Q$. It will be our Hamiltonian. We refer to [4, Chap. 6] for more details and proofs concerning the above construction.

Remark 16.3. Although H is formally equal to the operator $-G^{-\frac{1}{2}} \partial_i G^{\frac{1}{2}} G^{ij} \partial_j$, *i.e.* the Laplacian, $-\Delta$, expressed in Fermi coordinates, we shall be particularly concerned not to assume that the metric is differentiable. If, however, the metric is sufficiently smooth then the operator H is indeed given by this expression with Dirichlet boundary conditions in the classical sense. We stress that under our assumptions, f is known to be differentiable w.r.t. u only.

16.2.4 Flat and Geodesic Strips

Assume that the strip is *flat* in the sense that the curvature K is equal to zero everywhere on Ω, *i.e.* $K \equiv 0$. Then the Jacobi equation (16.5) has the exact solution

$$f(s,u) = 1 + u\,k(s)\,. \tag{16.7}$$

This is a well-known result for the strips in the plane, however, we note that the same holds as well for the strips on cylinders, on surfaces of the shape of corrugated iron, *etc.* Since the Hamiltonian is expressed via the metric which depends on f only, we may immediately adapt to the flat strips all the results which has been previously derived for quantum strips in the plane, [6]. In particular, the discrete spectrum will always exist as soon as the strip is non-trivially curved, $k \not\equiv 0$, and asymptotically straight, $k \overset{\infty}{\longrightarrow} 0$.

On the contrary, if (in addition to $K \equiv 0$) the reference curve is a geodesic, *i.e.* $k \equiv 0$, then the function f equals 1 identically, and therefore

$$H = H_0 := -\Delta_D^{\Omega_0} \qquad \text{on} \qquad L^2(\Omega_0)\,.$$

Consequently, the discrete spectrum is empty and

$$\sigma(H_0) = \sigma_{\mathrm{ess}}(H_0) = \left[\kappa_1^2, \infty\right)\,, \tag{16.8}$$

where κ_1^2 denotes the first eigenvalue of the Dirichlet Laplacian on the transverse section, $-\Delta_D^I$. These systems generalises the straight strips in the plane and will be called here *geodesic*. We will use them as a comparative class of quantum strips whose spectrum is known explicitly.

The operator $-\Delta_D^I$ occurs often in the present work. We note that it is nothing else than the quantum Hamiltonian of the one-dimensional infinite square well of width $2a$. In what follows we shall use its family of eigenfunctions $\{\chi_n\}_{n=1}^{\infty}$ which is given by

$$\chi_n(u) := \begin{cases} \sqrt{\frac{1}{a}}\,\cos\kappa_n u & \text{if } \ n \text{ is odd,} \\ \sqrt{\frac{1}{a}}\,\sin\kappa_n u & \text{if } \ n \text{ is even,} \end{cases} \tag{16.9}$$

where $\kappa_n^2 := (\kappa_1 n)^2$ with $\kappa_1 := \pi/(2a)$ are the corresponding eigenvalues. The ground-state χ_1 will be very important for us because it represents a generalised eigenvector of the geodesic strip corresponding to the threshold of the essential spectrum (16.8).

16.3 Motivation

This part is devoted to heuristic considerations in order to motivate the spectral analysis of the Hamiltonian in the following sections. It is possible, but beyond the scope of this paper, to examine the conditions under which the thin-width limit process below is justified. Since we use it just for motivation purposes, we shall do the limit in a formal way only. To this end (but only through this section!), we shall assume that f is an analytic function.

Let us recall first the observation which initiated the attempts to prove the existence of bound states in quantum strips in the plane, *cf* [19, 21, 3, 26]. The Hamiltonian of such a strip is unitarily equivalent to a Schrödinger-like operator with a potential expressed by means of the curvature k of the reference curve and the transverse coordinate $u \in I$. Making formally the thin-width limit, $a \to 0$, in the expression for the transformed Hamiltonian, the potential becomes equal to $-k^2/4$. The latter always represents an *attractive* interaction as soon as the strip is non-trivially curved, $k \not\equiv 0$, and asymptotically straight, $k \overset{\infty}{\longrightarrow} 0$. Consequently, the limit operator possesses bound states below its essential spectrum. As we have mentioned in the introduction, one proves that these bound states "survive" also in the actual quantum strips of non-zero widths.

In order to find the effective potential in our situation, we introduce the unitary transformation $U : \mathcal{H} \to L^2(\Omega_0)$ given by $\psi \mapsto G^{\frac{1}{4}}\psi$, which leads to

$$\tilde{H} := UHU^{-1} = -G^{-\frac{1}{4}}\partial_i G^{\frac{1}{2}} G^{ij}\partial_j G^{-\frac{1}{4}}\,. \tag{16.10}$$

Commuting $G^{-\frac{1}{4}}$ with the gradient components, we cast this operator into a form which has a simpler kinetic part but contains a potential,

$$\tilde{H} = -\partial_i G^{ij}\partial_j + V \qquad \text{with} \qquad V := (G^{ij}J_{,j})_{,i} + J_{,i}G^{ij}J_{,j}\,, \tag{16.11}$$

where $J := \ln G^{\frac{1}{4}}$. This expression is valid for any smooth metric G_{ij}. Employing the particular form (16.4) of our metric tensor together with the Jacobi equation (16.5), we get

$$V = \frac{1}{f^2}\left[\frac{1}{2}\,\frac{f_{,ss}}{f} - \frac{5}{4}\left(\frac{f_{,s}}{f}\right)^2\right] - \frac{1}{2}\,K - \frac{1}{4}\left(\frac{f_{,u}}{f}\right)^2 . \tag{16.12}$$

To make the limit when the width of the strip, $2a$, tends to zero, we note first that the function f admits, as a solution of (16.5), the following asymptotic expansion w.r.t. u:

$$f(s,u) = 1 + u\,k(s) - \tfrac{1}{2}\,u^2\,K(s,0) + r(s,u)\,, \tag{16.13}$$

where the remainder r is $\mathcal{O}(u^3)$ for any fixed s. Putting this expansion into (16.12) and (16.11), and making the limit $u \to 0$ in the expression for V and G^{ij}, we see that, up to higher order terms in $u \in I$, the operator H decouples formally into the direct sum of the operators

$$-\Delta^{\mathbb{R}} + V_{\text{eff}} \quad \text{on} \quad L^2(\mathbb{R}) \qquad \text{and} \qquad -\Delta_D^I \quad \text{on} \quad L^2(I)\,, \tag{16.14}$$

where

$$V_{\text{eff}}(s) := -\tfrac{1}{4}\,k(s)^2 - \tfrac{1}{2}\,K(s,0)\,. \tag{16.15}$$

The first term in V_{eff} is identical with the effective potential for the thin strips in the plane, while the second one reflects the fact that our strip is in addition embedded in a curved manifold now.

Assume that the curvatures k and K vanish at the infinity of the strip. In distinction to the planar case, the potential (16.15) may not represent an attractive interaction for any non-trivially curved strip. (For, it suffices to consider the strip constructed over a geodesic, $k \equiv 0$, on a surface of negative curvature, $K < 0$.) Nevertheless, if the curvature K is, say, non-negative (and $k \not\equiv 0$ provided $K \equiv 0$), then the potential V_{eff} always represents an attractive interaction. Consequently, the direct sum of the limit operators of (16.14) possesses bound states below its essential spectrum. The aim of this paper is to state an analogous sufficient condition which guarantees the existence of a non-trivial discrete spectrum for the actual Hamiltonian H of the strips of non-zero widths.

To conclude this section, we stress again that the thin-width-limit procedure we have used to derive the operators (16.14) and the effective potential (16.15) of thin strips is formal only. (One reason is that the transverse operator $-\Delta_D^I$ gives rise to infinite normal oscillations as $a \to 0$.) Nevertheless, we note that a similar thin-neighbourhood limit was performed rigourously by R. Froese and I. Herbst, [13], in efforts to treat the time evolution around a compact n-dimensional submanifold of $\mathbb{R}^{n+m}$, $m \geq 1$. There the confinement was realised by a harmonic potential transverse to the manifold and the limit was carried out by means of a dilation procedure followed by averaging in the normal direction. The situation when $\mathbb{R}^{n+m}$ is replaced by a Riemannian manifold of the same dimension was treated formally in [23]; there one can also recover the effective potential (16.15).

16.4 Essential Spectrum

In Section 16.2.4, we have seen that the essential spectrum of a geodesic strip ($k, K \equiv 0$) starts by the first eigenvalue κ_1^2 of the transverse operator $-\Delta_D^I$. Since the metric tensor is the identity matrix ($f \equiv 1$) in this case and the essential spectrum is determined by the behaviour of the metric at infinity only, we expect that the same will hold true if a curved quantum strip behaves like a geodesic strip *asymptotically* in the sense

$\langle$H3$\rangle$ $\qquad f \xrightarrow{\infty} 1\,.$

By the symbol "$\xrightarrow{\infty}$" we mean precisely the uniform limit w.r.t. u as $|s|$ tends to $+\infty$, *i.e.*

$$\lim_{s_0 \to +\infty} \sup_{(s,u)\in\Omega_0, |s|>s_0} |f(s,u) - 1| = 0\,.$$

Remark 16.4. Note that the assumption $\langle$H3$\rangle$ together with $\langle$H1$\rangle$ implies the condition $\langle$H2$\rangle$ for any half-width less than a. In detail, since f is continuous it is bounded locally, and cannot be equal to 0 on Ω_0 because of $\langle$H1$\rangle$. The asymptotic assumption $\langle$H3$\rangle$ then controls the uniform behaviour of f at infinity.

Theorem 16.1. *Assume* $\langle$H1$\rangle$, $\langle$H2$\rangle$, *and suppose that the strip is asymptotically geodesic,* i.e. $\langle$H3$\rangle$. *Then*

$$\inf \sigma_{\text{ess}}(H) \geq \kappa_1^2\,.$$

Proof. The idea is inspired by the proof of Theorem 4.1 in [8]. For any $s_0 > 0$, let us define $\Omega_{0,\mathrm{int}} := (-s_0, s_0)\times I$ and $\Omega_{0,\mathrm{ext}} := (\Omega_0 \setminus \Omega_{0,\mathrm{int}})^\circ$. The images of $\Omega_{0,\mathrm{int}}$ and $\Omega_{0,\mathrm{ext}}$ by the mapping $\mathcal{L}$ divide the strip Ω into an interior and exterior part, respectively. Imposing the Neumann boundary condition at the two curves separating these sets, we arrive at the decoupled Hamiltonian $H^N = H^N_{\mathrm{int}} \oplus H^N_{\mathrm{ext}}$. More precisely, it is obtained as the operator associated with the quadratic form Q^N acting as (16.6), however, with the domain $\mathrm{Dom}\, Q^N := \mathrm{Dom}\, Q^N_{\mathrm{int}} \oplus \mathrm{Dom}\, Q^N_{\mathrm{ext}}$, where

$$\mathrm{Dom}\, Q^N_\omega := \left\{\psi \in W^{1,2}(\Omega_{0,\omega}, d\Omega) \mid \psi(\cdot, \pm a) = 0\right\}, \qquad \omega \in \{\mathrm{int}, \mathrm{ext}\}.$$

The corresponding quadratic forms Q^N_ω act like Q, however, on appropriately restricted Hilbert spaces $\mathcal{H}_\omega := L^2(\Omega_{0,\omega}, d\Omega)$. Since $H \geq H^N$ and the spectrum of H^N_{int} is purely discrete, [4, Chap. 7], the minimax principle, [24, Sec. XIII. 1], gives the estimate

$$\inf \sigma_{\mathrm{ess}}(H) \geq \inf \sigma_{\mathrm{ess}}(H^N_{\mathrm{ext}}) \geq \inf \sigma(H^N_{\mathrm{ext}}).$$

Hence it is sufficient to find a lower bound on H^N_{ext}. However, by virtue of (16.6) and (16.4), we have for all $\psi \in \mathrm{Dom}\, Q^N_{\mathrm{ext}}$:

$$\begin{aligned} Q^N_{\mathrm{ext}}[\psi] \;&\geq\; \|\psi_{,u}\|^2_{\mathcal{H}_{\mathrm{ext}}} \;\geq\; \big(\inf_{\Omega_{0,\mathrm{ext}}} G^{\frac{1}{2}}\big)\, \|\psi_{,u}\|^2_{L^2(\Omega_{0,\mathrm{ext}})} \\ &\geq\; \big(\inf_{\Omega_{0,\mathrm{ext}}} G^{\frac{1}{2}}\big)\; \kappa_1^2\, \|\psi\|^2_{L^2(\Omega_{0,\mathrm{ext}})} \;\geq\; \big(\inf_{\Omega_{0,\mathrm{ext}}} G^{\frac{1}{2}}\big)\big(\sup_{\Omega_{0,\mathrm{ext}}} G^{\frac{1}{2}}\big)^{-1} \kappa_1^2\, \|\psi\|^2_{\mathcal{H}_{\mathrm{ext}}}. \end{aligned}$$

In the third inequality, we have used the bound $-\Delta^I_D \geq \kappa_1^2$. The obtained estimate on Q^N_{ext} is valid for any metric of the block form (16.4) even if the function f is replaced by a matrix. However, here we have $G^{\frac{1}{2}} = f$ and the infimum and supremum tend to 1 as $s_0 \to \infty$ by the assumption ⟨H3⟩. The claim then easily follows by the fact that s_0 can be chosen arbitrarily large. □

Remark 16.5. This threshold estimate is sufficient for the subsequent investigation of the discrete spectrum which is our goal in this paper. In order to prove the opposite estimate, one may employ a Dirichlet bracketing argument instead of the Neumann one we have used. Next, to show that all energies above κ_1^2 belong to the spectrum, one has to construct an appropriate Weyl sequence. This can be done under an assumption stronger than ⟨H3⟩ which involves derivatives of f as well.

16.5 Discrete Spectrum

The aim of this section is to prove two conditions sufficient for the Hamiltonian to have a non-empty spectrum below κ_1^2. Since we have shown that the essential spectrum does not start below this value for the asymptotically geodesic strips, the conditions yields immediately the existence of curvature-induced bound states. The proofs here are based on the variational strategy of finding a trial function ψ from the form domain of H such that

$$\tilde{Q}[\psi] := Q[\psi] - \kappa_1^2\, \|\psi\|^2_{\mathcal{H}} < 0. \tag{16.16}$$

The idea which goes back to J. Goldstone and R. L. Jaffe, [14], is to construct a trial function by deforming χ_1 of (16.9), which represents a generalised eigenfunction of energy κ_1^2 for the geodesic strip. In particular, if the strip is geodesic, then $\tilde{Q}[\chi_1] = 0$. The latter has to be understood in a generalised sense because χ_1 is not integrable w.r.t. s and as such it does not belong to $\mathrm{Dom}\, Q$. Let us use this function in the curved case. We start with a formal calculation:

$$\begin{aligned} \tilde{Q}[\chi_1] \;&=\; \big(\chi_{1,s}, f^{-1}\chi_{1,s}\big) + (\chi_{1,u}, f\chi_{1,u}) - \kappa_1^2\, (\chi_1, f\chi_1) \\ &=\; -\,(\chi_1, f_{,u}\,\chi_{1,u}) = \tfrac{1}{2}\,(\chi_1, f_{,uu}\,\chi_1) = -\tfrac{1}{2}\,(\chi_1, Kf\chi_1), \end{aligned} \tag{16.17}$$

where the inner product is in the Hilbert space $L^2(\Omega_0)$. The first equality is the definition of Q and $\|\cdot\|_{\mathcal{H}}$, in the second one we have used the fact that χ_1 does not depend on s and integrated by parts w.r.t. u, in the third one we have integrated by parts once more, and the last equality follows by (16.5). The resulting integral will be well defined if we assume

⟨H4⟩ $\qquad K \in L^1(\Omega_0, d\Omega)$.

Hence we obtain immediately

Theorem 16.2. *Assume* ⟨H1⟩, ⟨H2⟩, ⟨H4⟩, *and suppose that*

$$(\chi_1, K\chi_1)_{\mathcal{H}} > 0\,. \tag{16.18}$$

Then

$$\inf\sigma(H) < \kappa_1^2\,.$$

Proof. It remains to regularise χ_1 in such a way that the formal result (16.17) would be justified in a limit. For any $n \in \mathbb{N}\setminus\{0\}$, we define $\psi_n := \varphi_n\chi_1$, where, for example,

$$\varphi_n(s) := \begin{cases} 1 & \text{if} \quad |s| \in [0,n), \\ (2n-|s|)/n & \text{if} \quad |s| \in [n,2n), \\ 0 & \text{if} \quad |s| \in [2n,\infty). \end{cases}$$

Although ψ_n is not smooth, it is a continuous function of compact support in Ω_0 satisfying a Lipschitz condition and as such it as an admissible trial function from $\mathrm{Dom}\,Q$; *cf* [4, Thm. 6.1.5]. Since the variables (s,u) are separated in ψ_n, we arrive easily at

$$\tilde{Q}[\psi_n] = \big(\psi_{n,s}, f^{-1}\psi_{n,s}\big) - \tfrac{1}{2}\,(\psi_n, Kf\psi_n)\,, \tag{16.19}$$

where the first term vanishes as $n\to\infty$ because

$$0 < \big(\psi_{n,s}, f^{-1}\psi_{n,s}\big) \le c_-^{-1}\|\varphi_n'\|^2_{L^2(\mathbb{R})} = 2c_-^{-1}n^{-1}\,.$$

We have employed here ⟨H2⟩ and the normalisation of χ_1. Since $\varphi_n\to 1$ point-wise and from below as $n\to\infty$ and K is supposed to be integrable, the second term in (16.19) converges to the negative integral (16.17) by the dominated convergence theorem. Consequently, there exists a fixed n_0 such that $\tilde{Q}[\psi_{n_0}]$ is negative and the proof is finished. □

It may not be easy to verify the sufficient condition (16.18) for a given ambient surface $\mathcal{A}$ and reference curve Σ. Nevertheless, it is clear that it holds true for any strip of positive curvature, $K>0$. On the other hand, the condition is not satisfied for the strips in the plane where, however, it is well known that any non-trivial curvature of Σ pushes the spectrum of H below the energy κ_1^2. The following result shows that the same holds true for a more general class of quantum strips, including the flat case too.

Theorem 16.3. *Assume* ⟨H1⟩, ⟨H2⟩, ⟨H4⟩, *and suppose that*

$$(\chi_1, K\chi_1)_{\mathcal{H}} = 0\,. \tag{16.20}$$

If $K\equiv 0$, we require in addition that $k\not\equiv 0$. Then

$$\inf\sigma(H) < \kappa_1^2\,.$$

Proof. Let us start with formal considerations. By virtue of (16.17), the condition (16.20) implies that $\tilde{Q}[\chi_1] = 0$. It is the result which one obtains for the strips in the plane. There the usual strategy is to deform slightly the function χ_1 on a curved part of the strip in order to obtain a negative value of the functional $\tilde{Q}$. In particular, let $\varepsilon\in\mathbb{R}$ and there exist a real function ϕ of compact support in Ω_0 such that it belongs to $\mathrm{Dom}\,Q$ and $\tilde{Q}(\phi,\chi_1)$ is not equal to zero. Writing

$$\tilde{Q}[\chi_1+\varepsilon\phi] = \tilde{Q}[\chi_1] + 2\varepsilon\,\tilde{Q}(\phi,\chi_1) + \varepsilon^2\tilde{Q}[\phi]\,,$$

and since the first term at the r.h.s. equals zero, we can choose ε sufficiently small and of a suitable sign so that the sum of the last two terms is negative.

The result is then justified by using the mollifier φ_n from the proof of the previous theorem in order to regularise χ_1. Since the function φ_n equals one on an interval growing as $n\to\infty$ and ϕ has a compact support, we can take n sufficiently large so that $\tilde{Q}(\phi,\varphi_n\chi_1)$ does not depend on n. Hence it suffices to find an appropriate function ϕ which verifies the above properties.

We take $\phi(s,u) := j(s,u)^2\chi_1'(u)$, where j is a non-zero infinitely smooth real function with a compact support on a region in Ω_0 where $f_{,u}$ does not change sign and it is not identically zero. Such a region surely exists because $f_{,u}$ is a continuous function satisfying (16.5). Then an explicit calculation yields

$$\tilde{Q}(\phi,\chi_1) = -\,(j\chi_1', f_{,u}\,j\chi_1') \neq 0\,.$$

This establishes the proof by virtue of the above considerations. □

Remark 16.6. If $K \equiv 0$, we have already mentioned that the idea of the proof belongs to [14]. Nevertheless, the deformation is not given explicitly there. An explicit deformation function can be found in [25], however, it gives a satisfactory result only if $K \equiv 0$. Our ϕ is inspired by the explicit expression for the deformation function used in [6, Thm. 2.1].

An immediate consequence of Theorems 16.1, 16.2 and 16.3 is the following

Corollary 16.1. *Assume* $\langle \mathsf{H1} \rangle$, $\langle \mathsf{H2} \rangle$, $\langle \mathsf{H3} \rangle$, $\langle \mathsf{H4} \rangle$, *and suppose that*

$$(\chi_1, K\chi_1)_{\mathcal{H}} \geq 0\,. \tag{16.21}$$

If $K \equiv 0$, *we require in addition that* $k \not\equiv 0$. *Then*

$$\sigma_{\mathrm{disc}}(H) \neq \emptyset\,,$$

i.e., *there exists at least one isolated eigenvalue of finite multiplicity situated below* κ_1^2.

16.6 Non-Negative Curvature

Since the condition (16.21) is clearly satisfied for non-negatively curved strips, we shall suppose that $K \geq 0$ through all this section and investigate this situation in detail. Since the integral $(\chi_1, K\chi_1)_{\mathcal{H}}$ is always well defined, we may not assume the assumption $\langle \mathsf{H4} \rangle$. This includes to use the monotone convergence theorem instead of the dominated one we have used in the proofs of Theorems 16.2 and 16.3.

An integration of the Jacobi equation (16.5) yields the following identity

$$\forall (s,u) \in \Omega_0 : \qquad f_{,u}(s,u) = k(s) - \int_0^u K(s,\xi)\, f(s,\xi)\, d\xi\,. \tag{16.22}$$

Since K is non-negative and f positive, we have immediately

$$f(s,u) \leq 1 + u\, k(s)\,. \tag{16.23}$$

Let $a\|k\|_\infty < 1$. Putting the inequality (16.23) into (16.22), we get an opposite bound

$$f(s,u) \geq 1 + u\, k(s) - \tfrac{1}{2}\, u^2 \left(1 + \tfrac{1}{3}\, u\, k(s)\right) \sup_{\xi \in I} K(s,\xi)\,. \tag{16.24}$$

It follows from (16.23) and (16.24) that the condition $\langle \mathsf{H2} \rangle$ can always be achieved for bounded curvatures and a small enough. More specifically, a condition on the half-width is expressed by means of the following inequality

$$\frac{1}{6}\, a^2\, \|K\|_\infty + \frac{2}{3 - a\|k\|_\infty} \;<\; 1\,. \tag{16.25}$$

(The supremum norm of K is taken over the strip only.) We note that the condition $a\|k\|_\infty < 1$ is a usual assumption in the theory of quantum strips in the plane, while the presence of K in (16.25) is due to the curved ambient space $\mathcal{A}$.

Furthermore, it is clear from (16.23) and (16.24) that the asymptotic condition $\langle \mathsf{H3} \rangle$ is satisfied if we assume

$\langle \mathsf{H3}' \rangle \qquad k \xrightarrow{\infty} 0 \qquad$ and $\qquad K \xrightarrow{\infty} 0\,.$

We refer to the beginning of Section 16.4 for the exact definition of "$\xrightarrow{\infty}$". The first limit is the usual assumption on the asymptotic straightness of the strips in the plane, while the second requires that the surface Ω is asymptotically flat. The latter restricts the asymptotic behaviour of the ambient space $\mathcal{A}$.

Finally, we remind that also the basic assumption $\langle \mathsf{H1} \rangle$ can always be achieved for sufficiently small a if one assumes in addition that the strip does not overlap, *cf* Remark 16.2. Summing up the above considerations together with the results of the precedent sections, we conclude by

Theorem 16.4. *Let* Ω *be a strip of non-negative curvature,* i.e. $K \geq 0$, *which does not overlap and satisfies the condition* (16.25) *together with* $a\|k\|_\infty < 1$. *If it is not a geodesic strip,* i.e. $k \not\equiv 0$ *or* $K \not\equiv 0$, *then* $\inf \sigma(H) < \kappa_1^2$. *If it is in addition an asymptotically geodesic strip,* i.e. $\langle \mathsf{H3}' \rangle$, *then the essential spectrum starts above* κ_1^2 *and* H *has at least one isolated eigenvalue of finite multiplicity.*

This theorem generalises the known results for strips in the plane, [6], which are a particular case of the flat strips, $K \equiv 0$. Moreover, the condition which enables us to localise the essential spectrum is weaker in the sense that it does not contain derivatives of the curvature k of the reference curve. However, the most important generalisation concerns the quantum strips on non-trivially curved manifolds with a positive curvature. An instructive example in $\mathbb{R}^3$ is given by the infinite strips on the paraboloid of revolution.

16.7 Ruled Strips

In Section 16.2.4, we have found an explicit form of the metric (16.4) in Fermi coordinates for the flat strips which represent a trivial situation ($K \equiv 0$). In general, however, it is not at all an easy problem to find f because it requires to determine the geodesics orthogonal to the reference curve Σ and integrate the Jacobi equation (16.5) over these geodesics. Nevertheless, there is a non-trivial class of strips in $\mathbb{R}^3$ where the metric is easy to calculate. For, consider the strip Ω constructed by segments orthogonal to a space curve Σ. Such a strip is a part of a *ruled* surface $\mathcal{A}$ based on Σ, [20, Def. 3.7.4]. As we have mentioned in the introduction, the Hamiltonian H of a quantum particle in the ruled strips had already been investigated in [2]. The aim of the present paper is just to derive another expression for f, which suits better to our approach, and discuss some properties of H. A more detailed spectral analysis of the ruled strips will be discussed elsewhere.

Let Σ be a simple, infinite curve of class C^3 embedded in $\mathbb{R}^3$ and $p : \mathbb{R} \to \mathbb{R}^3$ be its parameterisation by the arc length s. We assume that the set $\{p', n, b\}$, where n and b are the unit normal and binormal vectors, respectively, is well defined and forms a right-handed Frenet triad frame. We use the symbols κ and τ for the curvature and torsion of Σ, respectively. One general class of ruled surfaces $\mathcal{A}$ is defined via $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^3$,

$$\mathcal{L}(s,u) := p(s) + u\,[n(s)\cos\theta(s) - b(s)\sin\theta(s)]\,, \tag{16.26}$$

where $\theta : \mathbb{R} \to \mathbb{R}$ is a function of class C^1. The ruled strip Ω is then given by (16.3) so that $\langle \mathsf{H1} \rangle$ and $\langle \mathsf{H2} \rangle$ hold true. The mapping (16.26) does really represent the Fermi-coordinate chart (16.2) with the metric of the form (16.4). Employing the Frenet-Serret formulae, an explicit calculation yields

$$f(\cdot,u)^2 \;=\; \big(1 - u\,\kappa\cos\theta\big)^2 + u^2(\tau - \theta')^2 \tag{16.27}$$

$$K(\cdot,u) \;=\; -\frac{(\tau-\theta')^2}{f(\cdot,u)^2}\,, \qquad k \;=\; -\kappa\cos\theta\,. \tag{16.28}$$

It is clear that any ruled strip has always a non-positive curvature. Consequently, the sufficient condition (16.21) is achieved only in the limit case, $K \equiv 0$, which corresponds to $\theta' = \tau$. In that case, Ω is a flat strip which may not be necessary a part of plane, however.

Combining (16.27) with (16.28), we get

$$f(s,u) = \frac{1 + u\,k(s)}{\sqrt{1 + u^2 K(s,u)}}\,, \tag{16.29}$$

which is an expression of a more transparent structure from the intrinsic point of view of this paper. At the same time, it is clear from (16.29) that the condition $\langle \mathsf{H2} \rangle$ holds true provided

$$a\|k\|_\infty < 1 \qquad \text{and} \qquad a^2\|K\|_\infty < 1\,, \tag{16.30}$$

and the assumption $\langle \mathsf{H1} \rangle$ then follows by the additional requirement that Ω does not overlap. Next, the ruled strip is asymptotically geodesic under the assumption $\langle \mathsf{H3'} \rangle$, which implies that $\inf \sigma_{\mathrm{ess}}(H) \geq \kappa_1^2$ by Theorem 16.1. However, an open question is whether there exist bound states below the threshold of the essential spectrum provided $K \not\equiv 0$.

16.8 Concluding Remarks

The main interest of this paper was to investigate spectral properties of the Laplacian $-\Delta$, subject to Dirichlet boundary conditions, in the strip region Ω defined as the tubular neighbourhood of an infinite curve Σ in a two-dimensional Riemannian manifold $\mathcal{A}$. The strategy was to express the operator $-\Delta$ under suitable assumptions, $\langle \mathsf{H1} \rangle$, $\langle \mathsf{H2} \rangle$, in geodesic coordinates based on Σ. We were mainly interested in the existence of the discrete spectrum. In particular, using some variational techniques, we proved that there are bound states below the essential spectrum provided the strip is not geodesic, $K \not\equiv 0$ or $k \not\equiv 0$, but asymptotically geodesic, $\langle \mathsf{H3} \rangle$, and positively curved "in the mean" in the sense of (16.21). The latter sufficient conditions hold particularly true for the strips of a non-negative curvature which were investigated in detail. The obtained results represent a generalisation of quantum strips in the plane, [6].

An interesting problem is to decide whether the discrete spectrum exists for some negatively curved quantum strips as well. The simplest model is probably given by the ruled strips introduced in the previous section. It is also desirable to investigate the spectrum of quantum strips on surfaces more precisely using some perturbation and numerical methods. Another direction of a future research consists in quantum strips which are not asymptotically geodesic; this may include periodically or randomly curved strips too. Following [5], we also

expect that interesting new features may be brought by a switch of the boundary condition. Apart from the spectral analysis, the scattering problem represents another challenge facing the theory of quantum strips.

The present paper has been motivated by the theory of quantum waveguides. If one deals with a curved quantum waveguide in the plane, a reasonable model is given by the two-dimensional Laplacian in an infinite strip in $\mathbb{R}^2$, [6]. However, we stress here that the two-dimensional Laplacian in the strip on a curved surface does not represent the actual Hamiltonian of a space quantum waveguide. For, a quantum particle in a strip-like waveguide is forced to move close to Ω by means of a constraining potential (representing a high chemical potential between different semiconductor materials) but, due to tunnelling effect, it can be found, even if not too far, outside the strip in the space $\mathbb{R}^3$ too. Even if this effect is not important for the waveguide in the plane because the motion of the particle in the direction transverse to the plane can be separated, it is not negligible for waveguides on a curved surface. In this paper, we dealt with a more general situation when the ambient space $\mathcal{A}$ of the waveguide may not be embedded in $\mathbb{R}^3$. Our results are interesting from the mathematical point of view, however, it is worth to know whether they could be interpreted physically as well.

Acknowledgments

The author wishes to express his gratitude to Professors Pierre Duclos, Pavel Exner, and Jean Nourrigat for useful discussions. The author would also like to thank the referee for helpful suggestions. The work has been partially supported by GA AS CR grant IAA 1048101.

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Errata

There is a mistake in the power of the denominator in the first expression of (16.28). The correct formula (which can be also found in the following Chapters 17 and 18) reads

$$K(\cdot,u) = -\frac{(\tau-\theta')^2}{f(\cdot,u)^4}\,. \tag{16.31}$$

Consequently, the conclusions (16.29) and (16.30) are incorrect. The correct formula for the metric epressed in terms of k and K reads

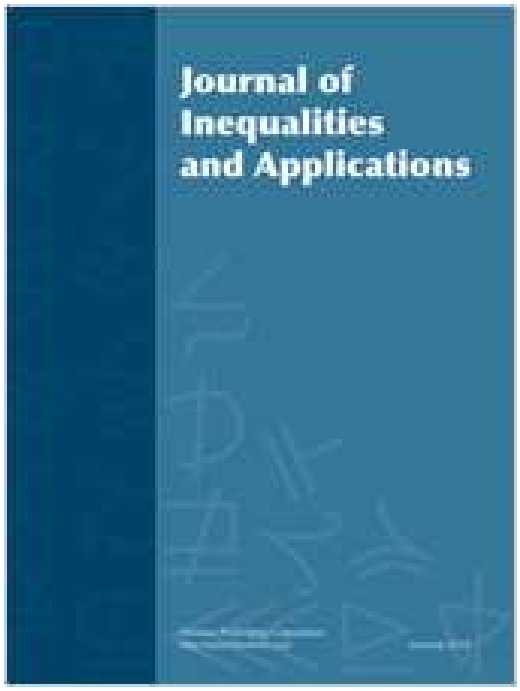
$$f(\cdot,u)^2 = \frac{-1+\sqrt{1+4\,u^2\,K\,(1+u\,k)^2}}{2\,u^2\,K}\,, \tag{16.32}$$

which is well defined provided

$$a\|k\|_\infty < 1 \qquad \text{and} \qquad a^2\|K\|_\infty < \frac{1}{8}\,. \tag{16.33}$$

Chapter 17

Hardy inequalities in strips on ruled surfaces



Published in: Journal of Inequalities and Applications **2006** (2006), Article ID 46409, 10 pages.

Hardy inequalities in strips on ruled surfaces

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Abstract. We consider the Dirichlet Laplacian in infinite two-dimensional strips defined as uniform tubular neighbourhoods of curves on ruled surfaces. We show that the negative Gauss curvature of the ambient surface gives rise to a Hardy inequality and use this to prove certain stability of spectrum in the case of asymptotically straight strips about mildly perturbed geodesics.

17.1 Introduction

Problems linking the geometry of two-dimensional manifolds and the spectrum of associated Laplacians have been considered for more than a century. While classical motivations come from theories of elasticity and electromagnetism, the same rather simple models can be also remarkably successful in describing even rather complicated phenomena in quantum heterostructures. Here an enormous amount of recent research has been undertaken on both the theoretical and experimental aspects of binding in curved strip-like waveguide systems.

More specifically, as a result of theoretical studies it is well known now that the Dirichlet Laplacian in an infinite planar strip of uniform width always possesses eigenvalues below its essential spectrum whenever the strip is curved and asymptotically straight. We refer to [13, 15] for initial proofs and to [8, 21, 19] for reviews with many references on the topic. The existence of the curvature-induced bound states is interesting from several respects. First of all, one deals with a purely quantum effect of geometrical origin, with negative consequences for the electronic transport in nanostructures. From the mathematical point of view, the strips represent a class of non-compact non-complete manifolds for which the spectral results of this type are non-trivial, too.

At the same time, a couple of results showing that the attractive interaction due to bending can be eliminated by appropriate additional perturbations have been established quite recently. Dittrich and Kříž [7] demonstrated that the discrete spectrum of the Laplacian in any asymptotically straight planar strip is empty provided the curvature of the boundary curves does not change sign and the Dirichlet condition on the locally shorter boundary is replaced by the Neumann one. A different proof of this result and extension to Robin boundary conditions were performed in [14]. Ekholm and Kovařík [10] obtained the same conclusion for the purely Dirichlet Laplacian in a mildly curved strip by introducing a local magnetic field perpendicular to the strip. The purpose of the present paper is to show that the same types of repulsive interaction can be created if the ambient space of the strip is a negatively curved manifold instead of the Euclidean plane.

A spectral analysis of the Dirichlet Laplacian in infinite strips embedded in curved two-dimensional manifolds was performed for the first time by the present author in [18]. He derived a sufficient condition which guarantees the existence of discrete eigenvalues in asymptotically straight strips; in particular, the bound states exist in strips on positively curved surfaces and in curved strips on flat surfaces. He also performed heuristic considerations suggesting that the discrete spectrum might be empty for certain strips on negatively curved surfaces. Similar conjectures were also made previously for strips on ruled surfaces in [5]. However, a rigorous treatment of the problem remained open.

In the present paper, we derive several Hardy inequalities for mildly curved strips on ruled surfaces, which proves the conjecture for this class of strips. A ruled surface is generated by straight lines translating along a curve in the Euclidean space; hence its Gauss curvature is always non-positive. The reason why we restrict to ruled surfaces in this paper is due to the fact that the Jacobi equation determining the metric in geodesic coordinates is explicitly solvable, so that rather simple formulae are available. Nevertheless, it should be possible to extend the present ideas to other classes of non-positively curved surfaces for which more precise information about geodesics are available.

Hardy inequalities represent a powerful technical tool in more advanced theoretical studies of elliptic operators. We refer to the book [22] for an exhaustive study and generalizations of the original inequality due to Hardy. Interesting Hardy inequalities on non-compact Riemannian manifolds were established in [2]. In the quantum-waveguide context, various types of Hardy inequality were derived in [10, 1, 11] in order to prove certain stability of spectrum of the Laplacian in tubular domains.

Here the last reference is the closest to the issue of the present paper. Indeed, the authors of [11] considered a three-dimensional tube constructed by translating a non-circular two-dimensional cross-section along an infinite curve and obtained that the twisting due to an appropriate construction eliminates the curvature-induced discrete spectrum in the regime of mild curvature. Formally, the strips of the present paper can be

viewed as a singular case of [11] when the cross-section is replaced by a segment and the effect of twisting is hidden in the curvature of the ambient space. While [11] and the present paper exhibit these similarity features, and also the technical handling of the problems is similar, they differ in some respects. On the one hand, the present situation is simpler, since it happens that the negative curvature of the ambient space gives rise to an explicit repulsive potential (*cf* (17.10) below) which leads to a Hardy inequality in a more direct way than in [11]. On the other hand, we do not perform the unitary transformation of [11] in order to replace the Laplacian on the Hilbert space of a curved strip by a Schrödinger-type operator on a "straighten" Hilbert space, but work directly with "curved" Hilbert spaces. This technically more complicated approach has an advantage that we need to impose no conditions whatsoever on the derivatives of curvatures.

Although we are not aware of a direct physical interpretation of the Laplacian in infinite strips if the ambient space has a non-trivial curvature, there exists an indirect motivation coming from the theory of quantum layers studied in [9, 3, 20]. In these references, the Dirichlet Laplacian in tubular neighbourhoods of a surface in the Euclidean space is used for the quantum Hamiltonian (*cf* [12] for a similar model). Taking our strip as the reference surface, the layer model of course differs from the present one, but a detailed study of the latter is important to understand certain spectral properties of the former. Similar layer problems are also considered in other areas of physics away from quantum theories, *cf* [16]. Finally, the present problem is a mathematically interesting one in the context of spectral geometry.

The organization of the paper is as follows. The ambient ruled surface, the strip and the corresponding Dirichlet Laplacian are properly defined in the preliminary Section 17.2. In Section 17.3, we consider the special situation of the strip being straight in a generalized sense. If the Gauss curvature of such a strip does not vanish identically and the strip is thin enough, we derive a central Hardy inequality of the present paper, *cf* Theorem 17.1. In fact, the latter is established by means of a "local" Hardy inequality, *cf* (17.11), which might be also interesting for applications. In Section 17.4, we apply Theorem 17.1 to mildly curved strips and prove certain stability of spectrum, *cf* Theorem 17.2. As an intermediate result, we obtain a general Hardy inequality for mildly curved strips on ruled surfaces, *cf* (17.15).

17.2 Preliminaries

Given two bounded continuous functions κ and τ defined on $\mathbb{R}$ with κ being positive, let $\Gamma : \mathbb{R} \to \mathbb{R}^3$ be the unit-speed curve whose curvature and torsion are κ and τ, respectively. Γ is determined uniquely up to congruent transformations and possesses a distinguished C^1-smooth Frenet frame $\{\dot\Gamma, N, B\}$ consisting of tangent, normal and binormal vector fields, respectively (*cf* [17, Chap. 1]). It is also convenient to include the case of κ and τ being equal to zero identically, which corresponds to Γ being a straight line with a constant Frenet frame.

Given a bounded C^1-smooth function θ defined on $\mathbb{R}$, let us introduce the mapping $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}^3$ via

$$\mathcal{L}(s,t) := \Gamma(s) + t\,\big[N(s)\cos\theta(s) - B(s)\sin\theta(s)\big]\,. \tag{17.1}$$

$\mathcal{L}$ represents a ruled surface (*cf* [17, Def. 3.7.4]) provided it is an immersion. The latter is ensured by requiring that the metric tensor $G \equiv (G_{ij})$ induced by $\mathcal{L}$, *i.e.*

$$G_{ij} := (\partial_i\mathcal{L})\cdot(\partial_j\mathcal{L})\,, \qquad i,j\in\{1,2\}\,,$$

where the dot denotes the scalar product in $\mathbb{R}^3$, is positive definite. Employing the Serret-Frenet formulae (*cf* [17, Sec. 1.3]), we find

$$G = \begin{pmatrix} h^2 & 0 \\ 0 & 1 \end{pmatrix}\,, \qquad h(s,t) := \sqrt{\big[1-t\,\kappa(s)\cos\theta(s)\big]^2 + t^2\,\big[\tau(s)-\dot\theta(s)\big]^2}\,. \tag{17.2}$$

Hence, it is enough to assume that t is sufficiently small so that the first term in the square root defining h never vanishes.

More restrictively, given a positive number a, we always assume

$$a\,\|\kappa\cos\theta\|_\infty < 1\,, \tag{17.3}$$

so that also h^{-1} is bounded, and define a *ruled strip* of width $2a$ to be the Riemannian manifold

$$\Omega := \big(\mathbb{R}\times(-a,a), G\big)\,. \tag{17.4}$$

That is, Ω is a non-compact and non-complete surface which is fully characterized by the functions κ, τ, θ and the number a. It is easy to verify that the Gauss curvature K of Ω is non-positive, namely,

$$K = -(\tau-\dot\theta)^2\,h^{-4}\,. \tag{17.5}$$

Moreover, if the mapping $\mathcal{L}$ is injective, then the image $\mathcal{L}\big(\mathbb{R}\times(-a,a)\big)$ has indeed the geometrical meaning of a non-self-intersecting strip and Ω represents its parameterization in geodesic coordinates.

Remark 17.1. In (17.2), let us write k instead of $\kappa\cos\theta$ and σ instead of $\tau-\dot\theta$, and assume that k and σ are given bounded continuous functions on $\mathbb{R}$. Then, abandoning the geometrical interpretation in terms of ruled surfaces based on Γ, (17.4) can be considered as an abstract Riemannian manifold, with $a\|k\|_\infty<1$ being the only restriction. The spectral results of this paper extend automatically to this more general situation by applying the above identification.

Our object of interest is the Dirichlet Laplacian in Ω, *i.e.*, the unique self-adjoint operator $-\Delta_D^\Omega$ associated with the closure of the quadratic form Q defined in the Hilbert space

$$\mathcal{H}:=L^2(\Omega)\equiv L^2\big(\mathbb{R}\times(-a,a),h(s,t)\,ds\,dt\big) \tag{17.6}$$

by the prescription

$$Q[\psi]:=\big(\partial_i\psi,G^{ij}\partial_j\psi\big)_{\mathcal{H}}\,,\qquad \psi\in\mathrm{Dom}(Q):=C_0^\infty\big(\mathbb{R}\times(-a,a)\big)\,, \tag{17.7}$$

where $(G^{ij}):=G^{-1}$ and the summation is assumed over the indices $i,j\in\{1,2\}$. Given $\psi\in\mathrm{Dom}(Q)$, we have

$$Q[\psi]=\|h^{-1}\partial_1\psi\|^2_{\mathcal{H}}+\|\partial_2\psi\|^2_{\mathcal{H}}\,.$$

Under the stated assumptions, it is clear that the form domain of $-\Delta_D^\Omega$ is just the Sobolev space $W^{1,2}\big(\mathbb{R}\times(-a,a)\big)$. If $\mathcal{L}$ is injective, then $-\Delta_D^\Omega$ is nothing else than the Dirichlet Laplacian defined in the open subset $\mathcal{L}\big(\mathbb{R}\times(-a,a)\big)$ of the ruled surface (17.1) and expressed in the "coordinates" (s,t).

17.3 Geodesic strips

The ruled strip Ω is called a *geodesic strip* and denoted by Ω_0 if the reference curve Γ is a geodesic on $\mathcal{L}$. Since $\kappa\cos\theta$ is the geodesic curvature of Γ (when the latter is considered as a curve on $\mathcal{L}$), it is clear that Ω is a geodesic strip provided Γ is either a straight line (*i.e.* geodesic in $\mathbb{R}^3$) or the straight lines $t\mapsto\mathcal{L}(s,t)-\Gamma(s)$ generating the ruled surface (17.1) are tangential to the binormal vector field for each fixed s. The metric (17.2) corresponding to Ω_0 acquires the form

$$G_0:=\begin{pmatrix}h_0^2&0\\0&1\end{pmatrix}\,,\qquad h_0(s,t):=\sqrt{1+t^2\big[\tau(s)-\dot\theta(s)\big]^2}\,, \tag{17.8}$$

and we denote by $\mathcal{H}_0$, Q_0 and $-\Delta_D^{\Omega_0}$, respectively, the corresponding Hilbert space defined in analogy to (17.6), the corresponding quadratic form defined in analogy to (17.7) and the associated Dirichlet Laplacian in Ω_0.

If $\tau-\dot\theta$ is equal to zero identically, *i.e.* Ω_0 is a flat surface due to (17.5), it is easy to see that the spectrum of $-\Delta_D^{\Omega_0}$ coincides with the interval $[E_1,\infty)$, where

$$E_1:=\pi^2/(2a)^2$$

is the lowest eigenvalue of the Dirichlet Laplacian in $(-a,a)$. In this section, we prove that the presence of Gauss curvature leads to a Hardy inequality for the difference $-\Delta_D^{\Omega_0}-E_1$, which has important consequences for the stability of spectrum.

Theorem 17.1. *Given a positive number a and bounded continuous functions τ and $\dot\theta$, let Ω_0 be the Riemannian manifold $\big(\mathbb{R}\times(-a,a),G_0\big)$ with the metric given by (17.8). Assume that $\tau-\dot\theta$ is not identically zero and that $a\,\|\tau-\dot\theta\|_\infty<\sqrt{2}$. Then, for all $\psi\in W^{1,2}\big(\mathbb{R}\times(-a,a)\big)$ and any s_0 such that $(\tau-\dot\theta)(s_0)\neq0$, we have*

$$Q_0[\psi]-E_1\,\|\psi\|^2_{\mathcal{H}_0}\;\geq\;c\,\|\rho^{-1}\psi\|^2_{\mathcal{H}_0}\qquad\text{with}\qquad\rho(s,t):=\sqrt{1+(s-s_0)^2}\,.$$

Here c is a positive constant which depends on s_0, a and $\tau-\dot\theta$.

It is possible to find an explicit lower bound for the constant c; we give an estimate in (17.13) below.

Theorem 17.1 implies that the presence of Gauss curvature represents a repulsive interaction in the sense that there is no spectrum below E_1 for all small potential-type perturbations having $\mathcal{O}(s^{-2})$ decay at infinity. Moreover, in the following Section 17.4, we show that this is also the case for appropriate perturbations of the metric (17.8).

In order to prove Theorem 17.1, we introduce the function $\lambda:\mathbb{R}\to\mathbb{R}$ by:

$$\lambda(s):=\inf_{\varphi\in C_0^\infty((-a,a))\setminus\{0\}}\frac{\int_{-a}^a|\dot\varphi(t)|^2\,h_0(s,t)\,dt}{\int_{-a}^a|\varphi(t)|^2\,h_0(s,t)\,dt}\;-\;E_1 \tag{17.9}$$

and keep the same notation for the function $\lambda\otimes1$ on $\mathbb{R}\times(-a,a)$. We have

Lemma 17.1. *Under the hypotheses of Theorem 17.1, λ is a continuous non-negative function which is not identically equal to zero.*

Proof. For any fix $s \in \mathbb{R}$, we make the change of test function $\phi := \sqrt{h_0(s,\cdot)}\,\varphi$, integrate by parts and arrive at

$$\lambda(s) = \inf_{\phi \in C_0^\infty((-a,a))\setminus\{0\}} \frac{\int_{-a}^{a} \left(|\dot\phi(t)|^2 - E_1\,|\phi(t)|^2 + V(s,t)\,|\phi(t)|^2 \right) dt}{\int_{-a}^{a} |\phi(t)|^2\,dt}$$

with

$$V(s,t) := \frac{[\tau(s) - \dot\theta(s)]^2 \left(2 - t^2\,[\tau(s) - \dot\theta(s)]^2 \right)}{4\,h_0(s,t)^4}\,. \tag{17.10}$$

Under the hypotheses of Theorem 17.1, the function V is clearly continuous, non-negative and not identically zero. These facts together with the Poincaré inequality $\int_{-a}^{a} |\dot\phi|^2 \geq E_1 \int_{-a}^{a} |\phi|^2$ valid for any $\phi \in C_0^\infty((-a,a))$ yield the claims of the Lemma. □

Assuming that the conclusion of Lemma 17.1 holds and using the definition (17.9), we get the estimate

$$Q_0[\psi] - E_1\,\|\psi\|_{\mathcal{H}_0}^2 \;\geq\; \left\| h_0^{-1} \partial_1 \psi \right\|_{\mathcal{H}_0}^2 + \left\| \lambda^{1/2} \psi \right\|_{\mathcal{H}_0}^2 \tag{17.11}$$

valid for any $\psi \in C_0^\infty\big(\mathbb{R} \times (-a,a)\big)$. Neglecting the first term on the right hand side of (17.11), the inequality is already a Hardy inequality. However, for applications it is more convenient to replace the Hardy weight λ in (17.11) by the positive function $c\,\rho^{-2}$ of Theorem 17.1. This is possible by employing the contribution of the first term based on

Lemma 17.2. *For any $\psi \in C_0^\infty\big(\mathbb{R} \times (-a,a)\big)$,*

$$\big(1 + a^2\,\|\tau - \dot\theta\|_\infty^2\big)^{-1/2} \left\| \rho^{-1} \psi \right\|_{\mathcal{H}_0}^2 \leq\; 16\,\big(1 + a^2\,\|\tau - \dot\theta\|_\infty^2\big)^{1/2} \left\| h_0^{-1} \partial_1 \psi \right\|_{\mathcal{H}_0}^2 + \big(2 + 64/|I|^2\big) \left\| \chi_I\,\psi \right\|_{\mathcal{H}_0}^2,$$

where I is any bounded subinterval of $\mathbb{R}$, χ_I denotes the characteristic function of the set $I \times (-a,a)$ and ρ is the function of Theorem 17.1 with s_0 being the centre of I.

Proof. The Lemma is based on the following version of the one-dimensional Hardy inequality:

$$\int_{\mathbb{R}} \frac{|u(x)|^2}{x^2}\,dx \;\leq\; 4 \int_{\mathbb{R}} |\dot u(x)|^2\,dx \tag{17.12}$$

valid for all $u \in W_0^{1,2}(\mathbb{R})$ with $u(0) = 0$. Put $b := |I|/2$. We define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(s) := \begin{cases} 1 & \text{for } \; |s - s_0| \geq b\,, \\ |s - s_0|/b & \text{for } \; |s - s_0| < b\,, \end{cases}$$

and keep the same notation for the function $f \otimes 1$ on $\mathbb{R} \times (-a,a)$. For any $\psi \in C_0^\infty\big(\mathbb{R} \times (-a,a)\big)$, let us write $\psi = f\psi + (1-f)\psi$. Applying (17.12) to the function $s \mapsto (f\psi)(s,t)$ with t fixed, we arrive at

$$\begin{aligned} \int \frac{|\psi|^2}{\rho^2} \;&\leq\; 2 \int \frac{|f\psi|^2}{\rho^2 - 1} + 2 \int \chi_I\,|(1-f)\psi|^2 \\ &\leq\; 16 \int |\partial_1 f|^2 |\psi|^2 + 16 \int |f|^2 |\partial_1 \psi|^2 + 2 \int \chi_I\,|(1-f)\psi|^2 \\ &\leq\; 16 \int |\partial_1 \psi|^2 + \big(2 + 16/b^2\big) \int \chi_I\,|\psi|^2\,, \end{aligned}$$

where the integration sign indicates the integration over $\mathbb{R} \times (-a,a)$. Recalling the definition of $\mathcal{H}_0$ and using the estimates

$$1 \;\leq\; h_0^2 \;\leq\; 1 + a^2\,\|\tau - \dot\theta\|_\infty^2\,,$$

the Lemma follows at once. □

Now we are in a position to prove Theorem 17.1.

Proof of Theorem 17.1. It suffices to prove the Theorem for functions ψ from the dense subspace $C_0^\infty\big(\mathbb{R}\times(-a,a)\big)$. Assume the hypotheses of Theorem 17.1 so that the conclusion of Lemma 17.1 holds. Let I be any closed interval on which λ is positive. Writing

$$\big\|\lambda^{1/2}\psi\big\|^2_{\mathcal{H}_0}=\epsilon\,\big\|\lambda^{1/2}\psi\big\|^2_{\mathcal{H}_0}+(1-\epsilon)\,\big\|\lambda^{1/2}\psi\big\|^2_{\mathcal{H}_0}\qquad\text{with}\qquad\epsilon\in(0,1]\,,$$

neglecting the second term of this decomposition, estimating the first one by an integral over $I\times(-a,a)$ and applying Lemma 17.2, the inequality (17.11) yields

$$\begin{aligned}Q_0[\psi]-E_1\,\|\psi\|^2_{\mathcal{H}_0}&\\ \geq\;&\Big[1-16\,\epsilon\min_I\lambda\,\big(2+64/|I|^2\big)^{-1}\big(1+a^2\,\|\tau-\dot\theta\|^2_\infty\big)^{1/2}\Big]\,\big\|h_0^{-1}\partial_1\psi\big\|^2_{\mathcal{H}_0}\\ &+\epsilon\min_I\lambda\,\big(2+64/|I|^2\big)^{-1}\big(1+a^2\,\|\tau-\dot\theta\|^2_\infty\big)^{-1/2}\,\big\|\rho^{-1}\psi\big\|^2_{\mathcal{H}_0}\,.\end{aligned}$$

Choosing ϵ as the minimum between 1 and the value such that the first term on the right hand side of the last estimate vanishes, we get the claim of Theorem 17.1 with

$$c\;\geq\;\min\left\{\frac{\min_I\lambda}{\big(2+64/|I|^2\big)\,\big(1+a^2\,\|\tau-\dot\theta\|^2_\infty\big)^{1/2}}\,,\frac{1}{16\,\big(1+a^2\,\|\tau-\dot\theta\|^2_\infty\big)}\right\}.\tag{17.13}$$

□

17.4 Mildly curved strips

Recall that the spectrum of $-\Delta_D^{\Omega_0}$ coincides with the interval $[E_1,\infty)$ provided the Gauss curvature (17.5) vanishes everywhere in the geodesic strip Ω_0. On the other hand, it was proved in [18] that $-\Delta_D^\Omega$ always possesses a spectrum below E_1 provided the Gauss curvature (17.5) vanishes everywhere but Γ is not a geodesic on $\mathcal{L}$. In this section, we use the Hardy inequality of Theorem 17.1 to show that the presence of Gauss curvature prevents the spectrum to descend even if Γ is mildly curved.

Theorem 17.2. *Given a positive number a and bounded continuous functions κ, τ and $\dot\theta$, let Ω be the Riemannian manifold (17.4) with the metric given by (17.2). Assume that $\tau-\dot\theta$ is not identically zero and that $a\,\|\tau-\dot\theta\|_\infty<\sqrt{2}$. Assume also that for all $s\in\mathbb{R}$,*

$$|\kappa(s)\cos\theta(s)|\;\leq\;\varepsilon(s):=\frac{\varepsilon_0}{1+s^2}\qquad\textit{with}\qquad\varepsilon_0\in[0,a^{-1})\,.$$

Then there exists a positive number C such that $\varepsilon_0\leq C$ implies

$$-\Delta_D^\Omega\;\geq\;E_1\,.\tag{17.14}$$

Here C depends on a and on the constants c and s_0 of Theorem 17.1.

As usual, the inequality (17.14) is to be considered in the sense of forms. Actually, a stronger, Hardy-type inequality holds true, *cf* (17.15) below.

An explicit lower bound for the constant C is given by the estimates made in the proof of Theorem 17.2 below.

As a direct consequence of Theorem 17.2, we get that the spectrum $[E_1,\infty)$ is stable as a set provided the difference $\tau-\dot\theta$ vanishes at infinity:

Corollary 17.1. *In addition to hypotheses of Theorem 17.2, assume that $\tau(s)-\dot\theta(s)$ tends to zero as $|s|\to\infty$. Then*

$$\operatorname{spec}(-\Delta_D^\Omega)=[E_1,\infty)\,.$$

Proof. Following the proof of [7, Sec. 3.1] or [19, Sec. 5] based on a general characterization of essential spectrum adopted from [6], it is possible to show that the essential spectrum $-\Delta_D^\Omega$ coincides with the interval $[E_1,\infty)$, while Theorem 17.2 ensures that there is no spectrum below E_1. □

Proof of Theorem 17.2. Let ψ belong to $C_0^\infty\big(\mathbb{R}\times(-a,a)\big)$. The proof is based on an algebraic comparison of $Q[\psi]-E_1\,\|\psi\|^2_{\mathcal{H}}$ with $Q_0[\psi]-E_1\,\|\psi\|^2_{\mathcal{H}_0}$ and the usage of Theorem 17.1. For every $(s,t)\in\mathbb{R}\times(-a,a)$, we have

$$f_-(s):=\sqrt{1-\frac{a\varepsilon(s)[2+a\varepsilon(s)]}{1+a^2\|\tau-\dot\theta\|^2_\infty}}\leq\frac{h(s,t)}{h_0(s,t)}\leq\sqrt{1+a\varepsilon(s)[2+a\varepsilon(s)]}=:f_+(s)\,.$$

Here the lower bound is well defined and positive provided $\varepsilon_0 \leq (3a)^{-1}$, and both the bounds behave as $1 + \mathcal{O}(\varepsilon(s))$ as $\varepsilon_0 \to 0$; we keep the same notation $f_\pm$ for the functions $f_\pm \otimes 1$ on $\mathbb{R} \times (-a, a)$. Consequently,

$$\begin{aligned} Q[\psi] - E_1 \|\psi\|^2_{\mathcal{H}} \;\geq\; & \int_{\mathbb{R}\times(-a,a)} f_+^{-1}\, h_0^{-1}\, |\partial_1 \psi|^2 \\ & + \int_{\mathbb{R}} ds\, f_-(s) \int_{-a}^{a} dt\, h_0(s,t) \left(|\partial_2 \psi(s,t)|^2 - E_1\, |\psi(s,t)|^2 \right) \\ & - E_1 \int_{\mathbb{R}\times(-a,a)} (f_+ - f_-)\, h_0\, |\psi|^2 \,. \end{aligned}$$

Since the term in the second line is non-negative due to (17.9) and Lemma 17.2, we can further estimate as follows:

$$\begin{aligned} Q[\psi] - E_1 \|\psi\|^2_{\mathcal{H}} \;\geq\; & \min\left\{ f_+(0)^{-1}, f_-(0) \right\} \left(Q_0[\psi] - E_1 \|\psi\|^2_{\mathcal{H}_0} \right) \\ & - E_1 \int_{\mathbb{R}\times(-a,a)} (f_+ - f_-)\, h_0\, |\psi|^2 \,. \end{aligned}$$

Using Theorem 17.1, we finally obtain

$$Q[\psi] - E_1 \|\psi\|^2_{\mathcal{H}} \;\geq\; \left\| w^{1/2} \psi \right\|^2_{\mathcal{H}_0} , \tag{17.15}$$

where

$$w(s,t) := \frac{c \, \min\left\{ f_+(0)^{-1}, f_-(0) \right\}}{1 + (s - s_0)^2} - E_1 \left[f_+(s) - f_-(s) \right]$$

is positive for all sufficiently small ε_0. □

Acknowledgement

The work has been supported by the Czech Academy of Sciences and its Grant Agency within the projects IRP AV0Z10480505 and A100480501.

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Chapter 18

The Brownian traveller on manifolds

Preprint on: arXiv:1108.3191 [math.AP] (2011)

Joint work with: Martin Kolb

The Brownian traveller on manifolds

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Abstract. We study the influence of the intrinsic curvature on the large time behaviour of the heat equation in a tubular neighbourhood of an unbounded geodesic in a two-dimensional Riemannian manifold. Since we consider killing boundary conditions, there is always an exponential-type decay for the heat semigroup. We show that this exponential-type decay is slower for positively curved manifolds comparing to the flat case. As the main result, we establish a sharp extra polynomial-type decay for the heat semigroup on negatively curved manifolds comparing to the flat case. The proof employs the existence of Hardy-type inequalities for the Dirichlet Laplacian in the tubular neighbourhoods on negatively curved manifolds and the method of self-similar variables and weighted Sobolev spaces for the heat equation.

18.1 Introduction

The intimate intertwining between properties of Brownian motion (or alternatively the heat flow) on a Riemannian manifold and the curvature properties of the manifold are a classical research question that has been investigated extensively (see, *e.g.*, [19, 20, 12, 13, 31, 17]) and has led to deep results and new methods, which turned out to be also of importance in other fields of mathematics. One of the main themes here is to characterize probabilistic properties via geometric ones and vice versa. Thinking of the Brownian particle as a 'traveller' in a curved space we continue this line of research and investigate the influence of the curvature on its large time behaviour.

However, in contrast to previous works, we restrict the motion of the Brownian particle to a tubular neighbourhood of a curve in the Riemannian manifold and kill it when it leaves this quasi-one-dimensional subset. This line of research seems to have its origin in the mathematical physics literature, where one aims to describe the dynamics of quantum particles in very thin almost one-dimensional waveguides. The constraint on the Brownian motion to the quasi-one-dimensional subsets leads to additional effects not present in the case of an unrestricted stochastic conservative motion. It particular it will turn out that the behaviour of the Brownian particle in the tube-like set is sensitive to local perturbations of the geometry.

A more precise description of our setting is the following. Let the ambient space of the Brownian traveller be a complete non-compact *two-dimensional* Riemannian manifold $\mathcal{A}$ (not necessarily embedded in the Euclidean space $\mathbb{R}^3$) with Gauss curvature K. We restrict to the case of *locally perturbed traveller* by assuming that K is compactly supported.

We further assume that the motion is *quasi-one-dimensional* in the sense that the Brownian traveller is forced to move along an infinite curve Γ on the surface $\mathcal{A}$. To focus on the effects induced by the intrinsic curvature K itself, we suppress side effects induced by the curvature of the curve by assuming that Γ is a geodesic.

The constraint to move along the geodesic curve is introduced by imposing *killing boundary conditions* on the boundary of the tubular neighbourhood

$$\Omega := \{q \in \mathcal{A} \mid \operatorname{dist}(q, \Gamma) < a\} \,, \tag{18.1}$$

where a is a positive (not necessarily small) number. That is, the Brownian traveller 'dies' whenever it hits the boundary $\partial\Omega$ of the strip Ω.

The problem is mathematically described by the diffusion equation

$$\begin{cases} \partial_t u - \Delta_q u = 0 & \text{in} \quad \Omega \times (0,\infty)\,, \\ u = 0 & \text{on} \quad \partial\Omega \times (0,\infty)\,, \\ u = u_0 & \text{on} \quad \Omega \times \{0\}\,, \end{cases} \tag{18.2}$$

in the space time variables $(q,t) \in \Omega \times (0,\infty)$, where u_0 is an initial datum. More specifically, for the Dirac distribution $u_0(q) = \delta(q - q_0)$, the solution $u(q,t)$ is related to the density of the transition probability of the Brownian motion starting at $q_0 \in \Omega$ as follows. Let us denote by $\mathbb{E}_q$ (respectively, $\mathbb{P}_q$) the expectation

(respectively, probability) of Brownian motion $(X_t)_{t\geq 0}$ on the manifold $\mathcal{A}$ started at $q \in \mathcal{A}$ and let $\tau_\Omega := \inf\{t > 0 \mid X_t \in \partial\Omega\}$ denote the first exit time. Then

$$u(q,t) = \mathbb{E}_q\big[u_0(X_t), \tau_\Omega > t\big] \tag{18.3}$$

solves equation (18.2). If $u_0 = \chi_B$ for some measurable set $B \subset \Omega$, we get

$$u(q,t) = \mathbb{P}_q\big(X_t \in B, \tau_\Omega > t\big)\,, \tag{18.4}$$

which is the probability that the Brownian particle survived up to time t and is in B at time t.

Now imagine a Brownian traveller in Ω and we imagine that he/she reached his/her goal when hitting the boundary. The ultimate question we would like to address in this paper is to decide *which geometry is better to travel*. By the 'good geometry' we understand that which enables the Brownian traveller to reach his/her goal as soon as possible or 'to escape from his/her starting point as far as possible'. More precisely, we are interested in quantifying the large time of (18.4) for bounded sets $B \subset \Omega_0$.

In any case, the question is related to the large time decay of the solutions of (18.2) as regards the curvature K. We mainly study a Hilbert-space version of the problem by analysing the asymptotic behaviour of the heat semigroup on $L^2(\Omega)$ associated with (18.2). Nevertheless, we establish some pointwise results about the large time behaviour of $u(q,t)$ as well.

Our results are informally summarized in Table 18.1.

curvature	positive	zero	negative
transport	*bad*	*critical*	*good*
probability decay	$e^{\gamma t}\, e^{-E_1 t}$	$t^{-1/2}\, e^{-E_1 t}$	$t^{-3/2}\, e^{-E_1 t}$

Table 18.1: An informal summary of our results.

There $E_1 := \pi^2/(2a)^2$ denotes the lowest Dirichlet eigenvalue of the strip cross-section $(-a,a)$ and γ is a positive number. As explained above, the vague statements about transport in Table 18.1 should be understood in the spirit of the large time decay of the solutions to (18.2) stated there. It turns out that the solutions of (18.2) has worse (respectively, better) decay properties if K is non-negative (respectively, non-positive) as a consequence of the existence of stationary solutions (respectively, Hardy-type inequalities). More general results, involving surfaces with sign changing curvatures, are established in this paper.

The effect of curvature on the transience/recurrence of a Brownian particle have been extensively studied (see [12] for a nice review). It turned out that on manifolds with 'large' negative curvature Brownian motion leaves compact subsets faster than on manifolds with non-negative curvature. But local changes of the Riemannian metric cannot change transience to recurrence or vice versa. Observe that for the results presented in Table 18.1 this is non longer true. In probabilistic literature this corresponds to the R-recurrence/R-transience dichotomy (see [46, 45]) or in analytic literature to the critical/subcritical dichotomy (see, *e.g.*, [36], or [34, 35] for a brief overview). Indeed, in our setting the Brownian motion in the negatively curved tube with *compactly supported* curvature is E_1-transient in contrast to the case of no curvature.

The organization of this paper is as follows. In the forthcoming Sections 18.2 and 18.3 we properly define the configuration space of the Brownian traveller and the associated heat equation (18.2), respectively. The case of zero curvature is briefly mentioned in Section 18.4. In Section 18.5 we consider direct consequences in a more general situation when the curvature vanishes at infinity. The influence of positive curvature on the Brownian traveller is studied in Section 18.6. The main part of the paper consists of Section 18.7, where we establish the existence of Hardy-type inequalities in negatively curved manifolds and develop the method of self-similar variables for the heat equation to reveal the subtle effect of negative curvature. The paper is concluded by Section 18.8 where we summarize our results and refer to some open problems.

18.2 Geometric preliminaries

We start by imposing some natural hypotheses to give an instructive geometrical interpretation of the configuration space Ω of the Brownian traveller. The conditions will be considerably weakened later when we reconsider the problem in an abstract setting.

18.2.1 The configuration space

Let us assume that the Riemannian manifold $\mathcal{A}$ is of class C^2 and that its Gauss curvature K is continuous. The latter holds under the additional assumption that $\mathcal{A}$ is either of class C^3 (by Gauss's Theorema Egregium) or that it is embedded in $\mathbb{R}^3$ (by computing principal curvatures).

Any geodesic curve Γ on $\mathcal{A}$ is C^2-smooth and, without loss of generality, we may assume that it is parameterized by arc-length. To enable the traveller to propagate to infinity, we consider unbounded geodesics Γ only. For a moment, we make the strong hypothesis that $\Gamma : \mathbb{R} \to \mathcal{A}$ is an embedding.

Since Γ is parameterized by arc-length, the derivative $T := \dot{\Gamma}$ defines the unit tangent vector field along Γ. Let N be the unit normal vector field along Γ which is uniquely determined as the C^1-smooth mapping from $\mathbb{R}$ to the tangent bundle of $\mathcal{A}$ by requiring that $N(s)$ is orthogonal to $T(s)$ and that $\{T(s), N(s)\}$ is positively oriented for all $s \in \mathbb{R}$ (*cf* [41, Sec. 7.B]).

The feature of our model is that the Brownian traveller is assumed to be confined to the strip-like a-tubular neighbourhood (18.1). By definition, Ω is the set of points q in $\mathcal{A}$ for which there exists a geodesic of length less than a from q meeting Γ orthogonally. More precisely, we introduce a mapping $\mathcal{L}$ from the flat strip

$$\Omega_0 := \mathbb{R} \times (-a, a) \tag{18.5}$$

(considered as a subset of the tangent bundle of $\mathcal{A}$) to the manifold $\mathcal{A}$ by setting

$$\mathcal{L}(x_1, x_2) := \exp_{\Gamma(x_1)}(N(x_1)\, x_2)\,, \tag{18.6}$$

where $\exp_q$ is the exponential map of $\mathcal{A}$ at $q \in \mathcal{A}$. Then we have

$$\Omega = \mathcal{L}(\Omega_0)\,. \tag{18.7}$$

Note that $x_1 \mapsto \mathcal{L}(x_1, x_2)$ traces the curves parallel to Γ at a fixed distance $|x_2|$, while the curve $x_2 \mapsto \mathcal{L}(x_1, x_2)$ is a geodesic orthogonal to Γ for any fixed x_1. See Figure 2.9.

18.2.2 The Fermi coordinates

Making the hypothesis that

$$\mathcal{L} : \Omega_0 \to \Omega \text{ is a diffeomorphism}\,, \tag{18.8}$$

we get a convenient parametrization of Ω via the (Fermi or geodesic parallel) 'coordinates' $x = (x_1, x_2)$ determined by (18.6), *cf* Figure 2.9. We refer to [10, Sec. 2] and [15] for the notion and properties of Fermi coordinates. In particular, it follows by the generalized Gauss lemma that the metric G induced by (18.6) acquires the diagonal form:

$$G = \begin{pmatrix} f^2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{18.9}$$

where f is continuous and has continuous partial derivatives $\partial_2 f$, $\partial_2^2 f$ satisfying the Jacobi equation

$$\partial_2^2 f + K f = 0 \qquad \text{with} \qquad \begin{cases} f(\cdot, 0) = 1\,, \\ \partial_2 f(\cdot, 0) = 0\,. \end{cases} \tag{18.10}$$

Here K is considered as a function of the Fermi coordinates (x_1, x_2).

By the inverse function theorem, a sufficient condition to ensure (18.8) is that $\mathcal{L}$ is injective and f positive. The latter can always be achieved for sufficiently small a as the following lemma shows.

Lemma 18.1. *Let $K \in L^\infty(\Omega_0)$ and $\|K\|_\infty a^2 < 1$. For every $x \in \Omega_0$, we have*

$$1 - \frac{\bar{K}(x_1)\, a^2}{1 - \bar{K}(x_1)\, a^2} \;\le\; f(x) \;\le\; 1 + \frac{\bar{K}(x_1)\, a^2}{1 - \bar{K}(x_1)\, a^2}\,, \tag{18.11}$$

where $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\Omega_0)}$ and

$$\bar{K}(x_1) := \operatorname*{ess\,sup}_{x_2 \in (-a,a)} |K(x_1, x_2)|\,.$$

Proof. Integrating (18.10), we arrive at the identity

$$\forall x \in \Omega_0\,, \qquad \partial_2 f(x) = -\int_0^{x_2} (Kf)(x_1, \xi)\, d\xi\,.$$

Consequently,

$$|\partial_2 f(x)| \le a\, \bar{K}(x_1)\, \bar{f}(x_1)\,, \qquad \text{with} \qquad \bar{f}(x_1) := \sup_{\xi \in (-a,a)} |f(x_1, \xi)|\,, \tag{18.12}$$

for all $x \in \Omega_0$. By the mean value theorem, we deduce the bounds

$$\forall x \in \Omega_0\,, \qquad 1 - a^2\, \bar{K}(x_1)\, \bar{f}(x_1) \le f(x) \le 1 + a^2\, \bar{K}(x_1)\, \bar{f}(x_1)\,. \tag{18.13}$$

Taking the supremum over $x_2 \in (-a, a)$, the upper bound leads to the upper bound of (18.11). Finally, using the upper bound of (18.11) to estimate $\bar{f}$ in the lower bound of (18.13), we conclude with the lower bound of (18.11). □

18.2.3 The abstract setting

It follows from the preceding subsection that, under the hypothesis (18.8), we can identify $\Omega \subset \mathcal{A}$ with the Riemannian manifold (Ω_0, G). However, the assumption (18.8) is not really essential provided that one is ready to abandon the geometrical interpretation of Ω as a tubular neighbourhood embedded in $\mathcal{A}$.

Indeed, (Ω_0, G), with the metric G determined by (18.9) and (18.10), can be considered as an abstract Riemannian manifold for which the boundedness of K and a restriction of a are the only important hypotheses. More specifically, we assume

$$K \in L^\infty(\Omega_0) \qquad \text{and} \qquad \|K\|_\infty a^2 < \frac{1}{2}\,. \tag{18.14}$$

Then the Jacobi equation (18.10) admits a solution $f(x_1, \cdot) \in H^2((-a, a))$ for every $x_1 \in \mathbb{R}$ and it follows from Lemma 18.1 that f is bounded and uniformly positive on Ω_0.

In the sequel, we therefore allow for self-intersections and low regularity of Ω by considering (Ω_0, G) as an abstract configuration space of the Brownian traveller. The mere boundedness of the metric G is sufficient to establish the desired results.

18.3 Analytic and probabilistic preliminaries

In this section, we give a precise meaning to the evolution problem (18.2).

18.3.1 The generator of motion

The meaning of $-\Delta_q u$ in (18.2) should be understood as an action of the Laplace-Beltrami operator $-\Delta$ in the Riemannian manifold Ω. In the Fermi coordinates, considering $-\Delta$ as a differential expression in Ω_0, we have

$$-\Delta = -|G|^{-1/2}\partial_i |G|^{1/2} G^{ij} \partial_j = -f^{-1}\partial_1 f^{-1}\partial_1 - f^{-1}\partial_2 f \partial_2\,. \tag{18.15}$$

Here the first identity is a general formula for the Laplace-Beltrami operator in a manifold equipped with the metric G, with the usual notation for the determinant $|G| := \det(G)$ and the coefficients G^{ij} of the inverse metric G^{-1}, and using the Einstein summation convention. The second identity employs the special form of the metric (18.9) in the Fermi coordinates.

The objective of this subsection is to associate to the differential expression (18.15) a self-adjoint operator H_K in the Hilbert space

$$L^2_f(\Omega_0) := L^2\big(\Omega_0, f(x)\,dx\big)\,, \tag{18.16}$$

a space isomorphic to $L^2(\Omega)$ via the Fermi coordinates. In order to implement the Dirichlet boundary conditions of (18.2), we introduce H_K as the Friedrichs extension of (18.15) initially defined on smooth functions of compact support in Ω_0 (*cf* [2, Sec. 6]). That is, H_K is the unique self-adjoint operator associated on (18.16) with the quadratic form

$$h_K[\psi] := \big(\partial_i\psi, G^{ij}\partial_j\psi\big)_f\,, \qquad \psi \in \mathfrak{D}(h_K) := H^1_0(\Omega_0, G)\,. \tag{18.17}$$

Here $(\cdot,\cdot)_f$ denotes the inner product in (18.16) and $H^1_0(\Omega_0, G)$ denotes the completion of $C^\infty_0(\Omega_0)$ with respect to the norm $\|\cdot\|_{\mathfrak{D}(h_K)} := (h_K[\cdot] + \|\cdot\|^2_f)^{1/2}$, with $\|\cdot\|_f$ denoting the norm in (18.16). The dependence of H_K on the curvature K is understood through the dependence of f on K, *cf* (18.10).

Under our hypothesis (18.14), it follows from Lemma 18.1 that $\|\cdot\|_f$ is equivalent to the usual norm $\|\cdot\| := \|\cdot\|_1$ in $L^2(\Omega_0) = L^2_1(\Omega_0)$ (*i.e.* $f = 1$) and, moreover, the $\mathfrak{D}(h_K)$-norm is equivalent to the usual norm in the Sobolev space $H^1(\Omega_0)$. Consequently,

$$\mathfrak{D}(h_K) = H^1_0(\Omega_0)\,.$$

However, it is important to keep in mind that, although $H^1(\Omega_0, G)$ and $H^1_0(\Omega_0)$ coincide as vector spaces, their topologies are different.

Remark 18.1. Under extra regularity assumptions involving derivatives of f, it is possible to show that H_K acts as (18.15) on the domain $H^1_0(\Omega_0) \cap H^2(\Omega_0)$. However, we shall not need these facts, always considering H_K in the form sense described above.

18.3.2 The dynamics

As usual, we consider the weak formulation of the parabolic problem (18.2). We say a Hilbert space-valued function $u \in L^2_{\mathrm{loc}}\big((0,\infty); H^1_0(\Omega_0, G)\big)$, with the weak derivative $u' \in L^2_{\mathrm{loc}}\big((0,\infty); [H^1_0(\Omega_0, G)]^*\big)$, is a (global) solution of (18.2) provided that

$$\big\langle v, u'(t)\big\rangle_f + h_K\big(v, u(t)\big) = 0 \tag{18.18}$$

for each $v \in H^1_0(\Omega_0, G)$ and a.e. $t \in [0,\infty)$, and $u(0) = u_0$. Here $h_K(\cdot,\cdot)$ denotes the sesquilinear form associated with (18.17) and $\langle\cdot,\cdot\rangle_f$ stands for the pairing of $H^1_0(\Omega_0, G)$ and its dual $[H^1_0(\Omega_0, G)]^*$. With an abuse of notation, we denote by the same symbol u both the function on $\Omega_0 \times (0,\infty)$ and the mapping $(0,\infty) \to H^1_0(\Omega_0, G)$.

Standard semigroup theory implies that there indeed exists a unique solution of (18.18) that belongs to $C^0\big([0,\infty); L^2_f(\Omega_0)\big)$. More precisely, the solution is given by $u(t) = e^{-tH_K} u_0$, where e^{-tH_K} is the semigroup associated with H_K.

It is easy to see that the real and imaginary parts of the solution u of (18.2) evolve separately. By writing $u = \Re(u) + i\,\Im(u)$ and solving (18.2) with initial data $\Re(u_0)$ and $\Im(u_0)$, we may therefore reduce the problem to the case of a real function u_0, without restriction. This reflects the fact that e^{-tH_K} is positivity preserving. Consequently, the functional spaces can be considered to be real when investigating the heat equation (18.2).

Indeed, the quadratic form h_K is a Dirichlet form, to which we can associate a strong Markov process with continuous paths (Brownian motion on (Ω_0, G)). In order to do so let us first extend f to $\mathbb{R}^2$ by setting it equal to 1 outside Ω_0. Moreover, let us define the Dirichlet form $\tilde{h}_K$ in $L^2(\mathbb{R}^2, f(x)\,dx)$ by

$$\tilde{h}_K[\psi] := \int_{\mathbb{R}^2} \overline{\partial_i \psi(x)}\, G^{ij}(x)\, \partial_j \psi(x)\, f(x)\, dx\,, \qquad \psi \in \mathfrak{D}(\tilde{h}_K) := H^1(\mathbb{R}^2)\,.$$

Then there exists a strong Markov process $(X_t)_{t\geq 0}$ with continuous paths, which is associated to $\tilde{h}_K$. According to Theorem 4 in [42] the process is conservative. We use $\mathbb{E}_x$ (respectively, $\mathbb{P}_x$) to denote the expectation (respectively, probability) conditional on $X_0 = x$. Since Dirichlet boundary conditions correspond to killing in the probabilistic picture, we have the following probabilistic representation

$$e^{-tH_K} u_0(x) = \mathbb{E}_x\big[u_0(X_t), \tau_{\Omega_0} > t\big] \tag{18.19}$$

for almost every $x \in \Omega_0$.

18.3.3 Basic properties

In our first proposition we collect some fundamental properties of the stochastic process $(X_t)_{t\geq 0}$.

Proposition 18.1. *Assume* (18.14).

- *The stochastic process $(X_t)_{t\geq 0}$ has the strong Feller property and is therefore well-defined for every $x \in \Omega_0$. In particular, the right hand side of* (18.19) *is continuous for every $u_0 \in L^\infty(\Omega_0)$.*
- *The stochastic process $(X_t)_{t\geq 0}$ has a continuous transition function $k_t(\cdot,\cdot)$ with respect to $f(x)\,dx$, which satisfy a Gaussian bound,* i.e. *for some constants $C_1 > 0$, $C_2 > 0$, one has*

$$\forall x, y \in \Omega_0, \qquad k_t(x,y) \leq \frac{C_1}{t}\, e^{\frac{|x-y|^2}{C_2 t}}\,.$$

Proof. The first assertion follows immediately from the second one by a standard use of Lebesgue's dominated convergence theorem.

In order to prove the second assertion, let us denote by $\tilde{H}_K$ the unique self-adjoint operator associated to $\tilde{h}_K$. Observe that according to [32, Thm. 1.1] the semigroup $e^{-t\tilde{H}_K}$ has an integral kernel, satisfying a Gaussian upper bound. As e^{tH_K} is dominated by $e^{-t\tilde{H}_K}$ (using either [33] or the probabilistic representation), this bound for $\tilde{H}_K$ carries over to H_K. In order to prove the regularity assertion concerning the transition kernel, observe that the Dirichlet form $\tilde{h}_K$ corresponds to a uniformly elliptic operator (in the sense of [37, Sec. 4]) on the subset Ω_0 of the Riemannian manifold $\mathbb{R}^2$ with Euclidean metric. Thus, according to the remark below Theorem 6.3 in [37] (compare also [43]), it therefore follows that the transition kernel is locally Hölder continuous. □

In this work we are mainly interested in the large time behaviour of the stochastic process $(X_t)_{\tau_{\Omega_0} > t \geq 0}$, which is well-known to be connected to spectral properties of its generator H_K. The spectral mapping theorem yields

$$\big\|e^{-tH_K}\big\|_{L^2_f(\Omega_0)\to L^2_f(\Omega_0)} = e^{-\lambda_K t} \tag{18.20}$$

for each time $t \geq 0$, where λ_K denotes the lowest point in the spectrum of H_K, *i.e.*, $\lambda_K := \inf \sigma(H_K)$. Hence, it is important to understand the low-energy properties of H_K in order to study the large time behaviour of the solutions of (18.2).

From equation (18.20) and Proposition 18.1 we deduce the following result showing that the exponential rate of decay of $\mathbb{P}_x\big(X_t \in B, \tau_{\Omega_0} > t\big)$ is given by the lowest point in the spectrum.

Proposition 18.2. *Assume* (18.14). *For any measurable subset* $B \subset \Omega_0$ *and every* $x \in \Omega_0$,

$$-\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x\big(X_t \in B, \tau_{\Omega_0} > t\big) = \lambda_K.$$

Proof. We apply arguments from [39] and [38] used there in the context of Schrödinger operators. First observe that the positive Sub-Markov operators e^{-tH_K} act as bounded operators on the space $L^\infty(\Omega_0, f(x)\,dx)$ and by duality also on $L^1(\Omega_0, f(x)\,dx)$. Let us set

$$\alpha_p := \lim_{t\to\infty} \frac{1}{t} \log \big\| e^{-tH_K} \big\|_{L^p_f(\Omega_0)\to L^p_f(\Omega_0)},$$

with the notation $L^p_f(\Omega_0) := L^p(\Omega_0, f(x)\,dx)$. Then we have $\alpha_p = \alpha_{p'}$ $(p^{-1}+p'^{-1} = 1)$ and $\alpha_p \leq \alpha_q$ $(2 \leq p \leq q)$. On the other hand, using the Gaussian bound in Proposition 18.1, we get for $\psi \in L^2_f(\Omega_0)$, $t > 2$ and some constant $C > 0$,

$$\big\| e^{-tH_K}\psi \big\|_{L^\infty_f(\Omega_0)} \leq C \big\| e^{-tH_K}\psi \big\|_{L^2_f(\Omega_0)} \leq C\, e^{-\lambda_K(t-1)} \|\psi\|_{L^2_f(\Omega_0)}\,.$$

Let ψ denote the indicator function of the set $\Omega_0 \cap B(0,r)$, where $B(0,r)$ denotes the ball with radius r centered at 0. Then we get for $x \in \Omega_0$

$$\mathbb{P}_x\big(X_t \in \Omega_0 \cap B(0,r), \tau_{\Omega_0} > t\big) = e^{-tH_K}\psi(x) \leq C\, e^{\lambda_K} \sqrt{r}\, e^{-\lambda_K t}.$$

On the other hand we have (see also [37, p. 429]) for some $C_1' > 0$

$$\mathbb{P}_x\big(X_t \in \Omega_0 \cap B(0,r), \tau_{\Omega_0} > t\big) = \int_{\Omega_0\cap B(0,r)} k_t(x,y) f(y)\, dy \leq e^{r^2/C_1' t}.$$

Choosing $r = \rho t$ with sufficiently large ρ, this finishes the proof of the upper bound

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x\big(X_t \in B, \tau_{\Omega_0} > t\big) \leq \lambda_K.$$

In order to proof the assertion of the Lemma we follow the proof of Theorem A.1.2. in [39]. It is sufficient to prove that for every $\varepsilon > 0$ there exists a constant $c > 0$ such that for sufficiently large $t > 0$

$$\mathbb{P}_x\big(X_t \in B, \tau_{\Omega_0} > t\big) = e^{-\lambda_K t} \chi_B(x) \geq c e^{-(\lambda_K+\varepsilon)t}.$$

We set $H_K' := H_K - \lambda_K$. There exists a smooth compactly supported $\psi \in H^1_0(\Omega_0)$ with $\|\psi\|_f = 1$ such that $h_K[\psi] - \lambda_K < \varepsilon/2$. Let W be $-\varepsilon$ on some bounded ball containing the support of ψ and 0 otherwise. Then the operators $H_K' - W$ and H_K' have the same essential spectrum. ¿From the inequality $(\psi, (H_K' - W)\psi)_{L^2_f(\Omega_0)} < -\varepsilon/2$, we conclude that the bottom of the spectrum $\lambda_{K,W}$ of $H_K' - W$ is a negative isolated eigenvalue and the associated ground state ψ_0 can be chosen to be non-negative. Since $e^{-t(H_K'-W)} \leq e^{\varepsilon t} e^{-tH_K'}$, we then arrive at $(t > 1)$

$$\begin{aligned} e^{\lambda_{K,W} t} e^{-t_0(H_K-W)} \chi_B(x) &= e^{\lambda_{K,W} t} \big(e^{-(H_K-W)1}(x,\cdot), e^{-(t-1)H_K'} \chi_B \big)_f \\ &\xrightarrow[t\to\infty]{} \psi_0(x) \big(\psi_0, \chi_B\big)_f\,, \end{aligned}$$

and therefore at $Ce^{-\varepsilon t} \leq e^{-tH_K'}\chi_B(x)$ for some constant $c > 0$. □

A better understanding of low-energy properties of H_K leads to much more precise estimates.

18.4 Flat manifolds

We say that (a submanifold of) $\mathcal{A}$ is *flat* if its Gauss curvature K is identically equal to zero (on the submanifold). The Brownian motion in a flat ambient space is easy to understand because Ω coincides with the straight Euclidean strip Ω_0, *i.e.* G is identity, for which the heat equation (18.2) can be solved by separation of variables.

18.4.1 Separation of variables

By the 'separation of variables' mentioned above we mean precisely that the Dirichlet Laplacian $H_0 = -\Delta_D^{\Omega_0}$ on $L^2(\Omega_0)$ can be identified with the decomposed operator

$$(-\Delta^{\mathbb{R}}) \otimes 1 + 1 \otimes (-\Delta_D^{(-a,a)}) \qquad \text{in} \qquad L^2(\mathbb{R}) \otimes L^2((-a,a))\,. \tag{18.21}$$

Here we denote by $-\Delta_D^U$ the Dirichlet Laplacian on $L^2(U)$ for any open Euclidean set U, suppress the subscript D if the boundary of U is empty, and 1 stands for the identity operators in the appropriate spaces. In a probabilistic language, (18.21) is essentially a reformulation of the fact that the horizontal and the vertical component of $(X_t)_{t\geq 0}$ are independent.

The eigenvalues and (normalized) eigenfunctions of $-\Delta_D^{(-a,a)}$ are respectively given by $(n = 1, 2, \dots)$

$$E_n := \Big(\frac{n\pi}{2a}\Big)^2\,, \qquad \mathcal{J}_n(x_2) := \sqrt{\frac{1}{a}}\,\sin\left[E_n(x_2+a)\right]\,, \tag{18.22}$$

while the spectral resolution of $-\Delta^{\mathbb{R}}$ is obtained by the Fourier transform. Then it is easy to see that the heat semigroup e^{-tH_0} is an integral operator with kernel

$$s_0(x,x',t) := \sum_{n=1}^{\infty} e^{-E_n t}\,\mathcal{J}_n(x_2)\,p(x_1,x_1',t)\,\mathcal{J}_n(x_2')\,, \tag{18.23}$$

where

$$p(x_1,x_1',t) := \frac{e^{-(x_1-x_1')^2/(4t)}}{\sqrt{4\pi t}}$$

is the well known heat kernel of $-\Delta^{\mathbb{R}}$.

18.4.2 The decay rate

Concerning the large time behaviour of e^{-tH_0}, it follows from the decomposition (18.21) that

$$\sigma(H_0) = \sigma_{\mathrm{ess}}(H_0) = [E_1,\infty)\,, \tag{18.24}$$

and therefore, as a consequence of (18.20),

$$\big\|e^{-tH_0}\big\|_{L^2(\Omega_0)\to L^2(\Omega_0)} = e^{-E_1 t} \tag{18.25}$$

for each time $t \geq 0$. Consequently, any solution of (18.2) satisfies the global decay estimate $\|u(t)\| \leq e^{-E_1 t}\,\|u_0\|$ for every $t \geq 0$.

However, it is possible to obtain an extra polynomial decay for solutions with initial data decaying sufficiently fast at the infinity of the strip Ω_0. To see it, let us consider the weight function

$$w(x) := e^{x_1^2/4} \tag{18.26}$$

and restrict the class of initial data to those u_0 which belong to the weighted space $L^2_w(\Omega_0)$ defined in the same way as (18.16). Then we have the improved decay estimate $\|u(t)\| \leq C\,t^{-1/4}\,e^{-E_1 t}\,\|u_0\|_w$ for every $t \geq 1$. This is a consequence of the following result.

Proposition 18.3. *There exists a positive constant C such that for every $t \geq 1$,*

$$C^{-1}\,t^{-1/4}\,e^{-E_1 t} \;\leq\; \big\|e^{-tH_0}\big\|_{L^2_w(\Omega_0)\to L^2(\Omega_0)} \;\leq\; C\,t^{-1/4}\,e^{-E_1 t}\,.$$

Moreover, for every bounded set $B \subset \Omega_0$ and $x \in \Omega_0$ there is a constant $C_{B,x}$ such that for $t \geq 1$,

$$C_{B,x}^{-1}\,t^{-1/2}\,e^{-E_1 t} \;\leq\; \mathbb{P}_x\big(X_t \in B, \tau_{\Omega_0} > t\big) \;\leq\; C_{B,x}\,t^{-1/2}\,e^{-E_1 t}\,.$$

Proof. The second assertion is a rather immediate consequence of (18.23). In order to see this, observe that

$$\begin{aligned}\mathbb{P}_x\big(X_t \in B, \tau_{\Omega_0} > t\big) &= \sum_{n=1}^{\infty} e^{-E_n t}\,\mathcal{J}_n(x_2)\int_B p(x_1,x_1',t)\,\mathcal{J}_n(x_2')\,dx \\ &= e^{-E_1 t}\,\mathcal{J}_1(x_2)\int_B p(x_1,x_1',t)\,\mathcal{J}_1(x_2')\,d(x_1',x_2') + R_B(t,x_1,x_2)\,,\end{aligned} \tag{18.27}$$

where $R_B(t,x_1,x_2)$ satisfies $|R_B(t,x_1,x_2)| \le e(x_1,x_2)e^{-E_2 t}$ $(t \ge 1)$ for some locally bounded function $e : \Omega_0 \to \mathbb{R}_+$. Thus there exists $t_0 = t_0(x_1,x_2,B) \ge 1$ such that for every $t \ge t_0$ one has

$$|R_B(t,x_1,x_2)| \le \frac{1}{2} e^{-E_1 t}\, \mathcal{J}_1(x_2) \int_B p(x_1,x_1',t)\, \mathcal{J}_1(x_2')\, dx'.$$

Therefore from (18.27) we conclude that for $t \ge t_0$

$$\begin{aligned} \frac{1}{2} e^{-E_1 t}\, \mathcal{J}_1(x_2) \int_B p(x_1,x_1',t)\, \mathcal{J}_1(x_2')\, dx' &\le \mathbb{P}_x\big(X_t \in B, \tau_{\Omega_0} > t\big) \\ &\le \frac{3}{2} e^{-E_1 t}\, \mathcal{J}_1(x_2) \int_B p(x_1,x_1',t)\, \mathcal{J}_1(x_2')\, dx', \end{aligned}$$

which, using the explicit form of p, gives the assertion for $t \ge t_0$. Adjusting the constants $C_{B,x}$ allows to extend this to $t \ge 1$.

Let us now consider the first assertion. Using the Schwarz inequality, we get

$$\begin{aligned} \big\| e^{-tH_0} u_0 \big\|^2 &\le \|u_0\|_w^2 \int_{\Omega_0 \times \Omega_0} s_0(x,x',t)^2\, w(x')^{-1}\, dx\, dx' \\ &= \|u_0\|_w^2\; e^{-2E_1 t} \sum_{n=1}^{\infty} e^{-2(E_n - E_1)t} \int_{\mathbb{R}\times\mathbb{R}} p(x_1,x_1',t)^2\, e^{-{x_1'}^2/4}\, dx_1\, dx_1' \end{aligned}$$

for every $u_0 \in L^2_w(\Omega_0)$ and $t \ge 0$. Here the sum can be estimated by a constant independent of $t \ge 1$ and the integral (computable explicitly) is proportional to $t^{-1/2}$. This establishes the upper bound of the proposition.

To get the lower bound, we may restrict to the class of initial data of the form $u_0(x) = \varphi(x_1)\, \mathcal{J}_1(x_2)$ with $\varphi \in L^2_w(\mathbb{R})$ (here w is considered as a function on $\mathbb{R}$). Then it is easy to see from (18.23) that

$$\big\| e^{-tH_0} \big\|_{L^2_w(\Omega_0) \to L^2(\Omega_0)} \ge e^{-E_1 t} \big\| e^{t\Delta^{\mathbb{R}}} \big\|_{L^2_w(\mathbb{R}) \to L^2(\mathbb{R})}$$

for every $t \ge 0$. The lower bound with $t^{-1/4}$ is well known for the heat semigroup of $-\Delta^{\mathbb{R}}$ (or can be easily established by taking $\varphi = w^{-\alpha}$ with any $\alpha > 1/2$ and evaluating the integrals with the kernel p explicitly). □

Remark 18.2. It is clear from the proof that the bounds hold in less restrictive weighted spaces. Indeed, it is enough to have a corresponding result for the one-dimensional heat semigroup $e^{t\Delta^{\mathbb{R}}}$.

For the following Corollary we recall the definition of the elementary conditional probability. If the measurable subset B satisfies $\mathbb{P}_x(B) > 0$, then $\mathbb{P}_x(A \mid B) := \mathbb{P}_x(A \cap B)/\mathbb{P}_x(B)$. The concept of conditional probabilities allows to focus on the polynomial decay factors, as the exponential terms cancel each other.

Corollary 18.1. *Let $K = 0$. For every bounded measurable subset $B \in \Omega_0$ and every $x \in \Omega_0$ there exists a constant $c_{B,x} > 0$ such that*

$$c_{B,x}^{-1}\, t^{-\frac{1}{2}} \le \mathbb{P}_x\big(X_t \in B \mid \tau_{\Omega_0} > t\big) \le c_{B,x}\, t^{-\frac{1}{2}}$$

for every $t \ge 1$.

Proof. The inequalities follow from Proposition 18.3 and the fact that for every $x = (x_1,x_2) \in \Omega_0$ by independence of the horizontal and vertical components of (X_t) (in the flat case)

$$\lim_{t\to\infty} e^{E_1 t} \mathbb{P}_x\big(\tau_\Omega > t\big) = \mathcal{J}_1(x_2) \int_{(-a,a)} \mathcal{J}_1(x_2)\, dx_2.$$

From the definition of the conditional probability, we see that the exponential cancel and we remain with the polynomial decay as stated in the assertion. □

As a consequence of this result, we get that conditioned on not hitting the boundary Ω_0 the Brownian particle will escape to infinity.

Proposition 18.3 establishes the decay rate for zero curvature as announced in Table 18.1.

18.4.3 The criticality of the transport

Let us now explain what we mean by the vague statement in Table 18.1 that the transport is 'critical' on flat surfaces.

We say that the transport is *critical* if the spectral threshold of H_K is not 'stable against local attractive perturbations', *i.e.*,

$$\forall V \in C_0^\infty(\Omega_0)\,,\ V \neq 0\,,\ V \leq 0\,, \qquad \inf\sigma(H_K + V) < \lambda_K\,. \tag{18.28}$$

Then we also say that H_K is critical. As a consequence of the spectral mapping theorem, we get

$$\left\|e^{-t(H_K+V)}\right\|_{L^2_f(\Omega_0)\to L^2_f(\Omega_0)} = e^{\gamma t}e^{-\lambda_K t}$$

for each time $t \geq 0$, where $\gamma := \lambda_K - \inf\sigma(H_K + V)$ is positive. That is, the criticality leads to an exponential slow-down in the decay of the perturbed semigroup.

Property (18.28) is well known for H_0 and is equivalent to the fact that the first component of $(X_t)_{t\geq 0}$ – a one-dimensional Brownian motion – is recurrent. For some results concerning this connection in a more abstract context we refer to [30].

Proposition 18.4. *H_0 is critical.*

Proof. By the variational characterization of the spectral threshold, it is enough to construct a test function ψ from $H^1_0(\Omega_0)$ such that

$$Q[\psi] := \|\nabla\psi\|^2 - E_1\|\psi\|^2 - \left\||V|^{1/2}\psi\right\|^2 < 0\,.$$

For every $n \geq 1$, we define $\psi_n(x) := \varphi_n(x_1)\mathcal{J}_1(x_2)$, with $\varphi_n := w^{-n}$, where w is the weight (18.26) (considered as a function on $\mathbb{R}$). Due to the normalization of $\mathcal{J}_1$, we have

$$Q[\psi_n] = \|\dot\varphi_n\|^2_{L^2(\mathbb{R})} - \|v\,\varphi_n\|^2_{L^2(\mathbb{R})}\,,$$

where $v(x_1) := \||V(x_1,\cdot)|^{1/2}\mathcal{J}_1\|^2_{L^2((-a,a))}$. By hypothesis, $v \in L^1(\mathbb{R})$ and the integral $\|v\|_{L^1(\mathbb{R})}$ is positive. Finally, an explicit calculation yields $\|\dot\varphi_n\|_{L^2(\mathbb{R})} \sim n^{-1/4}$. By the dominated convergence theorem, we therefore have

$$Q[\psi_n] \xrightarrow[n\to\infty]{} -\|v\|_{L^1(\mathbb{R})}\,.$$

Consequently, taking n sufficiently large, we can make $Q[\psi_n]$ negative. □

In Section 18.6 we shall show that the spectrum of H_0 is unstable against purely geometric deformations characterized by positive curvature, too.

18.5 Asymptotically flat manifolds

We say that the strip Ω is *asymptotically flat* if its Gauss curvature K vanishes at infinity, *i.e.*,

$$\lim_{|x_1|\to\infty} \operatorname*{ess\,sup}_{x_2\in(-a,a)} |K(x)| = 0\,. \tag{18.29}$$

In this paper, we are interested in a 'locally perturbed traveller' by usually assuming a stronger hypothesis that K is compactly supported, *i.e.*,

$$\operatorname{supp}(K) \cap \Omega_0 \quad \text{is bounded.} \tag{18.30}$$

It follows from (18.30) that there exists a positive R such that $K(x) = 0$ for all $|x_1| > R$. Then, as a consequence of (18.10),

$$|x_1| > R \quad \Longrightarrow \quad f(x) = 1\,. \tag{18.31}$$

Of course, (18.29) trivially holds for the strips satisfying (18.30). Nevertheless, let us state the following result under the more general hypothesis (18.29).

Theorem 18.1. *Assume* (18.14) *and* (18.29). *Then*

$$\sigma_{\rm ess}(H_K) = [E_1, \infty)\,.$$

Proof. The fact that the threshold of the essential spectrum does not descend below the energy E_1 has been proved in [23, Thm. 1] by means of a Neumann bracketing argument. Let us therefore only show that $[E_1, \infty)$ belongs to the essential spectrum of H_K.

Our proof is based on the Weyl criterion adapted to quadratic forms in [4] and applied to quantum waveguides in [27]. By this general characterization of essential spectrum and since the set $[E_1, \infty)$ has no isolated points, it is enough to find for every $\lambda \in [E_1, \infty)$ a sequence $\{\psi_n\}_{n=1}^\infty \subseteq \mathfrak{D}(h_K)$ such that

(i) $\forall n \in \mathbb{N} \setminus \{0\}, \quad \|\psi_n\|_f = 1,$

(ii) $\big\|(H_K - \lambda)\psi_n\big\|_{[\mathfrak{D}(h_K)]^*} \xrightarrow[n\to\infty]{} 0.$

Here $\|\cdot\|_{[\mathfrak{D}(h_K)]^*}$ denotes the norm in the dual space $[\mathfrak{D}(h_K)]^*$ of $\mathfrak{D}(h_K)$ Let $n \in \mathbb{N} \setminus \{0\}$. Given $k \in \mathbb{R}$, we set $\lambda = E_1 + k^2$.

Since Ω is asymptotically flat, a good candidate for the sequence are plane waves in the x_1-direction modulated by the ground-state eigenfunction $\mathcal{J}_1$ in the x_2-direction and 'localized at infinity':

$$\psi_n(x) := \varphi_n(x_1)\, \mathcal{J}_1(x_2)\, e^{ikx_1}\,.$$

Here $\varphi_n(x_1) :^{-1/2} \varphi(x_1/n - n)$ with φ being a non-zero C^∞-smooth function with compact support in the interval $(-1, 1)$. Note that $\operatorname{supp}\varphi_n \subset (n^2 - n, n^2 + n)$. We further assume that φ is normalized to 1 in $L^2(\mathbb{R})$, so that the norm of φ_n is 1 as well.

Clearly, $\psi_n \in H^1_0(\Omega_0) = \mathfrak{D}(h_K)$. To satisfy (i), one can redefine ψ_n by dividing it by its norm $\|\psi_n\|_f$. However, since

$$\|\psi_n\|_f^2 \geq 1 - \frac{\|K\|_\infty\, a^2}{1 - \|K\|_\infty\, a^2} > 0$$

due to Lemma 18.1 and the normalizations of φ and $\mathcal{J}_1$, it is enough to verify the condition (ii) directly for our unnormalized functions ψ_n.

By the definition of the dual norm, we have

$$\big\|(H_K - \lambda)\psi_n\big\|_{[\mathfrak{D}(h_K)]^*} = \sup_{\phi \in H^1_0(\Omega_0)\setminus\{0\}} \frac{\big|h_K(\phi, \psi_n) - \lambda(\phi, \psi_n)_f\big|}{\|\phi\|_{\mathfrak{D}(h_K)}}\,. \tag{18.32}$$

An explicit computation using integrations parts yields

$$\begin{aligned} h_K(\phi, \psi_n) - \lambda(\phi, \psi_n)_f = \big(\phi, [-\ddot\varphi_n - 2ik\dot\varphi_n]\, \mathcal{J}_1\, e^{ikx_1}\big) \\ + \big(\partial_1\phi, [f^{-1} - 1]\partial_1\psi_n\big) - k^2\,\big(\phi, [f - 1]\psi_n\big) - \big(\phi, [\partial_2 f]\partial_2\psi_n\big)\,. \end{aligned}$$

Using the Schwarz inequality, we estimate the individual terms on the right hand side of the identity as follows

$$\begin{aligned} \big|\big(\phi, [-\ddot\varphi_n - 2ik\dot\varphi_n]\, \mathcal{J}_1\, e^{ikx_1}\big)\big| &\leq \|\phi\|_{\mathfrak{D}(h_K)}\, \sqrt{\|\ddot\varphi_n\|^2_{L^2(\mathbb{R})} + 4k^2\|\dot\varphi_n\|^2_{L^2(\mathbb{R})}}\, \|f^{1/2}\|_\infty\,, \\ \big|\big(\partial_1\phi, [f^{-1} - 1]\partial_1\psi\big)\big| &\leq \|\phi\|_{\mathfrak{D}(h_K)}\, \|\dot\varphi_n\|_{L^2(\mathbb{R})}\, \operatorname*{ess\,sup}_{\operatorname{supp}\varphi_n} \big(f^{1/2}\,|f^{-1} - 1|\big)\,, \\ \big|\big(\phi, [f - 1]\psi_n\big)\big| &\leq \|\phi\|_{\mathfrak{D}(h_K)}\, \|\varphi_n\|_{L^2(\mathbb{R})}\, \operatorname*{ess\,sup}_{\operatorname{supp}\varphi_n} \big(f^{-1/2}\,|f - 1|\big)\,, \\ \big|\big(\phi, [\partial_2 f]\partial_2\psi_n\big)\big| &\leq \|\phi\|_{\mathfrak{D}(h_K)}\, \|\varphi_n\|_{L^2(\mathbb{R})}\, E_1\, \operatorname*{ess\,sup}_{\operatorname{supp}\varphi_n} \big(f^{-1/2}\,|\partial_2 f|\big)\,. \end{aligned}$$

Hence, the dual norm (18.32) can be bounded from above by a constant multiplied by a sum of terms containing either $\|\dot\varphi_n\|_{L^2(\mathbb{R})}$, $\|\ddot\varphi_n\|_{L^2(\mathbb{R})}$ or the suprema involving f over the support of φ_n. By hypothesis (18.29), the suprema tend to zero as $n \to \infty$ due to Lemma 18.1 and (18.12). The remaining terms tend to zero as $n \to \infty$ because

$$\|\dot\varphi_n\|_{L^2(\mathbb{R})} = n^{-1}\, \|\dot\varphi\|_{L^2(\mathbb{R})}\,, \qquad \|\ddot\varphi_n\|_{L^2(\mathbb{R})} = n^{-2}\, \|\ddot\varphi\|_{L^2(\mathbb{R})}\,.$$

□

Theorem (18.1) implies that we always have $\lambda_K \leq E_1$ for asymptotically flat strips. Therefore, as a consequence of (18.20),

$$\big\|e^{-tH_K}\big\|_{L^2_f(\Omega_0)\to L^2_f(\Omega_0)} \geq e^{-E_1 t}$$

for each time $t \geq 0$.

18.6 Positively curved manifolds

We say that a manifold is *positively curved* if K is non-zero and non-negative (in the sense of a measurable function on the manifold). In this section we give a meaning to the vague statement of Table 18.1 that the 'positive curvature is bad for transport'. It is based on the following result, which we adopt from [23].

Theorem 18.2. *Assume* (18.14) *and* $K \in L^1(\Omega_0)$. *We have*

$$(\mathcal{J}_1, K\mathcal{J}_1)_f > 0 \qquad \Longrightarrow \qquad \inf\sigma(H_K) < E_1\,.$$

Remark 18.3. Recall that $\mathcal{J}_1$ is the first transverse eigenfunction introduced in (18.22). Here, not to burden the notation, we denote by the same symbol $\mathcal{J}_1$ the function $x \mapsto \mathcal{J}_1(x_2)$ on Ω_0.

Proof. The proof of the theorem is very similar to that of Proposition 18.4. By the variational characterization of the spectral threshold of H_K, it is enough to construct a test function ψ from $H_0^1(\Omega_0)$ such that

$$Q_K[\psi] := h_K[\psi] - E_1\|\psi\|_f^2 < 0\,.$$

Using the same sequence of functions $\psi_n(x) = \varphi_n(x_1)\mathcal{J}_1(x_2)$ as in the proof of Proposition 18.4, we arrive at

$$Q_K[\psi_n] = (\partial_1\psi_n, f^{-1}\partial_1\psi_n) - \frac{1}{2}\,(\psi_n, K\psi_n)_f\,. \tag{18.33}$$

Here the first (positive) integral on the right hand side vanishes as $n \to \infty$ because

$$(\partial_1\psi_n, f^{-1}\partial_1\psi_n) \leq \frac{1 - \|K\|_\infty a^2}{1 - 2\|K\|_\infty a^2}\;\|\dot\varphi_n\|^2_{L^2(\mathbb{R})}\,,$$

due to Lemma 18.1 and the normalization of $\mathcal{J}_1$, and $\|\dot\varphi_n\|_{L^2(\mathbb{R})} \sim n^{-1/4}$. Using, at the same time, the dominated convergence theorem in the second integral on the right hand side of (18.33), we finally get

$$Q_K[\psi_n] \xrightarrow[n\to\infty]{} -\frac{1}{2}\,(\mathcal{J}_1, K\mathcal{J}_1)_f\,.$$

Since the limit is negative by hypothesis, we can make $Q[\psi_n]$ negative by taking n sufficiently large. □

Remark 18.4. The integrability of K is just a technical assumption in Theorem 18.2. It is only important to give a meaning to the integral $(\mathcal{J}_1, K\mathcal{J}_1)_f$, the value $+\infty$ being admissible in principle. For instance, it is enough to assume that K is non-trivial and non-negative on Ω_0 for the present proof to work.

Combining Theorem 18.2 with Theorem 18.1, we get that H_K possesses at least one discrete eigenvalue below the essential spectrum under the hypotheses. In view of the criticality notion introduced in Section 18.4.3, the result of Theorem 18.2 can be also interpreted in the sense that H_0 is not stable against geometric perturbations characterized by the presence of positive curvature.

In any case, regardless of whether the spectral threshold of H_K represents an eigenvalue or the bottom of the essential spectrum, Theorem 18.2 implies that the gap $\gamma := E_1 - \lambda_K$ is always positive for positively curved strips. If K vanishes at infinity, then the bottom of the spectrum has to be an isolated eigenvalue. Therefore, as a consequence of (18.20) and [40], we conclude with

Corollary 18.2. *Assume* (18.14), $K \in L^1(\Omega_0)$ *and* $(\mathcal{J}_1, K\mathcal{J}_1)_f > 0$. *Then*

$$\left\|e^{-tH_K}\right\|_{L^2_f(\Omega_0)\to L^2_f(\Omega_0)} = e^{\gamma t}\,e^{-E_1 t}$$

for each time $t \geq 0$, *where* γ *is positive. Moreover, if additionally* (18.29) *is satisfied then there exists a unique non-negative normalized* $\phi_0 \in L^2_f(\Omega_0)$ *such that for every bounded measurable set* $B \subset \Omega_0$ *and every* $x \in \Omega_0$

$$\lim_{t\to\infty} e^{-(\gamma - E_1)t}\,\mathbb{P}_x\big(X_t \in B, \tau_{\Omega_0} > t\big) = \phi_0(x)\int_B \phi_0(y) f(y)\,dy\,.$$

That is, the presence of positive curvature clearly slows down the decay of the heat semigroup, even without the need to work with the weighted space $L^2_{wf}(\Omega_0)$. A Brownian traveller should avoid 'mountains' satisfying $(\mathcal{J}_1, K\mathcal{J}_1)_f > 0$, if he/she wants to make sure that he/she is able to reach is goal early and wants to avoid spending too much time in a given bounded region.

The following Corollary (again a rather direct consequence of Theorems 18.2 and 18.1) shows that in contrast to the flat case the Brownian traveller – conditioned on not-hitting the boundary $\partial\Omega_0$ – might not have been able to have left a bounded region forever.

Corollary 18.3. *Assume that* (18.29) *and the conditions of Theorem 18.2 are satisfied. Then for almost every* $x \in \Omega_0$ *we have*

$$\lim_{t\to\infty} \mathbb{P}_x\big(X_t \in \cdot \mid \tau_{\Omega_0} > t\big) = \frac{\phi_0(y) f(y)\,dy}{\int_{\Omega_0}\phi_0(y) f(y)\,dy},$$

where the convergence is with respect to the total variation distance.

Proof. By definition of the total variation distance we have to prove that

$$\lim_{t\to\infty} \sup_{B\subset\Omega_0} \left| \mathbb{P}_x\big(X_t \in B \mid \tau_{\Omega_0} > t\big) - \frac{\int_B \phi_0(y) f(y)\, dy}{\int_{\Omega_0} \phi_0(y) f(y)\, dy} \right| = 0.$$

Observe that we do not assume that the sets B are bounded and that the assertions of Corollary 18.2 do not suffice to prove the desired assertion.

According to general spectral theory we know, that the eigenfunction $\phi_0 \in L^2_f(\Omega_0)$ does not change sign and that the eigenspace is one-dimensional. In the first step we show that ϕ_0 actually also belongs to $L^1_f(\Omega_0)$, with the notation $L^p_f(\Omega_0) := L^p(\Omega_0, f(x)\,dx)$. This will allow us to interpret the ground state as a probability distribution. Of course, many results concerning the decay properties are known, but we have not been able to find a reference covering our setting. Observe first that due to the probabilistic interpretation the semigroup $(e^{-tH_K})_{t\geq 0}$ in $L^2_f(\Omega_0)$ gives rise to a consistent strongly continuous semigroups $(T^p_t)_{t\geq 0}$ in $L^p_f(\Omega_0)$ for $1 \leq p < \infty$. Moreover, due to the Gaussian bound from Proposition 18.1, these semigroups are analytic with angle $\pi/2$. Let the generators be denoted by H^p_K. Due to the consistence of the semigroups, by taking Laplace transforms of the semigroups, we conclude that the resolvents $F^p(z) := (H^p_K - z)^{-1}$ $(z \in \rho(H^p_K) := \mathbb{C} \setminus \sigma(H^p_K))$ are as well consistent in the sense that for every $z \in \rho(H^p_K) \cap \rho(H^q_K)$

$$F^p(z) \restriction L^p_f(\Omega_0) \cap L^q_f(\Omega_0) = F^q(z) \restriction L^p_f(\Omega_0) \cap L^q_f(\Omega_0).$$

Since according to Theorem 5 in [3] we have $\sigma(H^p_K) = \sigma(H^2_K) (= \sigma(H_K))$ for every $1 \leq p < \infty$ and since λ_K is an isolated eigenvalue for H^2_K, we conclude by Corollary 1.4 in [16] that λ_K is an isolated point of $\sigma(H^1_K)$ and that the eigenvector ϕ_0 of H^2_K is also an eigenvector of H^1_K, *i.e.*, in particular, $\phi_0 \in L^1_f(\Omega_0)$.

Observe now that $(X_t)_{\tau_{\Omega_0}\geq t\geq 0}$ is λ_K-recurrent in the sense of [44] and we also conclude that the measure $\pi(dx) = \phi_0(y) f(y)\, dy$ is finite and due to reversibility with respect to the measure $f(x)\,dx$ satisfies

$$\mathbb{P}_\pi\big(X_t \in A, \tau_{\Omega_0} > t\big) = e^{-\lambda_K t} \int_A \phi_0(x) f(x)\, dx$$

for every measurable set $A \subset \Omega_0$. As $\phi \in L^2_f(\Omega_0)$ we conclude that (X_t) is λ_K-positive recurrent in the sense of [44] (product-critical in the sense of [36]). Applying Theorem 7 in [44], we are thus able to conclude the assertion of the Corollary. More precisely, formula (5.9) in [44] shows that for almost all $x \in \Omega_0$

$$\lim_{t\to\infty} \sup_{B\subset\Omega_0} \left| e^{\lambda_K t} \mathbb{P}_x\big(X_t \in B, \tau_{\Omega_0} > t\big) - \phi_0(x) \int_B \phi_0(y) f(y)\, dy \right| = 0. \tag{18.34}$$

Therefore we have for almost all $x \in \Omega_0$

$$\begin{aligned}
&\sup_{B\subset\Omega_0} \left| \mathbb{P}_x\big(X_t \in B \mid \tau_{\Omega_0} > t\big) - \frac{\int_B \phi_0(z) f(z)\, dz}{\int_{\Omega_0} \phi_0(y) f(y)\, dy} \right| \\
&\quad \leq \big(e^{\lambda_K t} \mathbb{P}_x\big(\tau_{\Omega_0} > t\big)\big)^{-1} \sup_{B\subset\Omega_0} \left| e^{\lambda_K t} \mathbb{P}_x\big(X_t \in B\big) - \phi_0(x) \int_B \phi_0(y) f(y)\, dy \right| \\
&\qquad + \sup_{B\subset\Omega_0} \left| \frac{\phi_0(x) \int_B \phi_0(y) f(y)\, dy}{e^{\lambda_K t} \mathbb{P}_x\big(\tau_{\Omega_0} > t\big)} - \frac{\int_B \phi_0(y) f(y)\, dy}{\int_{\Omega_0} \phi_0(y) f(y)\, dy} \right| \\
&\quad = \big(e^{\lambda_K t} \mathbb{P}_x\big(\tau_{\Omega_0} > t\big)\big)^{-1} \sup_{B\subset\Omega_0} \left| e^{\lambda_K t} \mathbb{P}_x\big(X_t \in B\big) - \phi_0(x) \int_B \phi_0(y) f(y)\, dy \right| \\
&\qquad + \left(\sup_{B\subset\Omega_0} \int_B \phi_0(z)\, dz \right) \left| \frac{\phi_0(x)}{e^{\lambda_K t} \mathbb{P}_x\big(\tau_{\Omega_0} > t\big)} - \frac{1}{\int_{\Omega_0} \phi_0(y) f(y)\, dy} \right|.
\end{aligned}$$

Two applications of (18.34) complete the proof. □

18.7 Negatively curved manifolds

In analogy with positively curved manifolds, we say that a manifold is *negatively curved* if K is non-zero and non-positive. In this section, on the contrary, we show that the presence of negative curvature improves the decay of the heat semigroup, supporting in this way the vague statement of Table 18.1 that the 'negative curvature is good for transport'. First, however, we have to explain why the negative sign of curvature is much more delicate for the study of large time properties of (18.2).

Recall that the positivity of the curvature K pushes the spectrum below E_1 (*cf* Theorem 18.2). The objective of this subsection is to show that the effect of negative curvature is rather opposite: it 'has the tendency' to push the spectrum above E_1. This effect is more subtle because $[E_1, \infty)$ belongs to the spectrum of H_K, irrespectively of the sign of the curvature, as long as the curvature vanishes at infinity (*cf* Theorem 18.1).

The way how to understand this 'repulsive tendency' is to replace the Poincaré-type inequality requirement $H_K - E_1 \geq \text{const} > 0$ (which is false for the asymptotically flat manifolds) by a weaker, *Hardy-type inequality*:

$$H_K - E_1 \;\geq\; \rho > 0\,. \tag{18.35}$$

Here $\rho : \Omega_0 \to (0, \infty)$ is assumed to be merely a *positive function* (necessarily vanishing at the infinity of Ω_0 for the asymptotically flat manifolds).

By Theorem 18.2, (18.35) is false for positively curved manifolds. It is also violated for flat manifolds because of the criticality result of Proposition 18.4. In this subsection, we show that (18.35) typically holds for negatively curved manifolds.

18.7.1 Hardy-type inequality and the large time behaviour

For completeness we first sketch an abstract elementary argument from [21], relating the Hardy inequality to low energy properties of the Hamiltonian and the large time behaviour of the semigroup. If the semigroup is associated to a stochastic process, then the validity of a Hardy-type inequality is related to the concept of R-transience of the stochastic process.

Assume that there exists a positive function ρ, with a locally bounded inverse ρ^{-1}, such that the inequality (18.35) holds true for the self-adjoint non-negative operator $L_K := H_K - E_1$. Then according to Theorem 8.31 in [47] we conclude that for all $\lambda < 0$ and every $h \in L^2_f(\Omega_0)$ we have

$$\big(h, (L_K - \lambda)^{-1} h\big)_f \leq \big(h, (M_\rho - \lambda)^{-1} h\big)_f\,, \tag{18.36}$$

where M_ρ denotes the maximal multiplication operator acting via multiplication with the function ρ. If h satisfies $(h, \rho^{-1} h)_f < \infty$, then (18.36) implies

$$\forall \lambda < 0\,, \qquad \int_{(E_1, \infty)} (\nu - \lambda)^{-1}\, d\big\|E^{L_K}_\nu h\big\|^2_f \leq \big(h, \rho^{-1} h\big)_f < \infty\,, \tag{18.37}$$

where $(E^{L_K}_\nu)_\nu$ denotes the spectral resolution of L_K. Using monotone convergence, we get for all h with $(h, \rho^{-1} h)_f < \infty$ (in particular for all continuous h with compact support in Ω_0)

$$\int_0^\infty \big(h, e^{-t(H_K - E_1)} h\big)_f\, dt < \infty. \tag{18.38}$$

Observe that (18.38) – which in the probabilistic literature such as [44], [46] and [45] might be called E_1-transience – does not hold in the case of positively curved and flat manifolds.

Property (18.37) is related to the low energy behaviour of the spectral measure $E^{L_K}(\cdot)$ in the sense that it implies that for all $r \in (0, 1]$ and $-1 \leq \lambda < 0$

$$\big\|E^{L_K}((0, r)) h\big\|^2_f = \int_0^r d\|E^{L_K}_\nu h\|^2_f \leq \int_0^r \frac{r - \lambda}{\nu - \lambda}\, d\|E^{L_K}_\nu h\|^2_f. \tag{18.39}$$

where we used that $\frac{r-\lambda}{\nu-\lambda} \geq 1$ for $\nu \in (0, r)$ and negative λ. Sending λ to 0 and using (18.37), we conclude that there is $C > 0$ such that for h with $(h, \rho^{-1} h)_f \leq 1$ and $r \in (0, 1)$

$$\big\|E^{L_K}((0, r)) h\big\|^2_f \leq C\, r.$$

This insight can easily be translated into an assertion concerning the large time behaviour.

Proposition 18.5. *Assume that $H_K - E_1$ satisfies the Hardy-type inequality* (18.35) *with a positive function ρ satisfying $\rho^{-1} \in L^\infty_{\mathrm{loc}}(\Omega_0)$. Then*

$$\sup_{(h, \rho^{-1} h)_f < 1} \big\|e^{-t(H_K - E_1)} h\big\|^2_f \leq \frac{1}{t}\left(1/2 + 2e^{-2}\right).$$

Proof. For the proof we again set $L_K := H_K - E_1$ and denote by μ_h the spectral measure corresponding to L_K and h. Via the spectral theorem, integration by parts and (18.39), we obtain

$$\begin{aligned}\left\|e^{-tL_K}h\right\|_f^2 &= \int_0^\infty e^{-2\nu t}\,d\mu_h(\nu) = 2t\int_0^1 e^{-2\nu t}\,\mu_h(\nu)\,d\nu + 2t\int_1^\infty e^{-2\nu t}\,\mu_h(\nu)\,d\nu\\ &\le 2t\int_0^1 e^{-2\nu t}\nu\,d\mu_h + 2t\int_1^\infty e^{-2\nu t}\,\mu_h(\nu)\,d\nu\\ &\le 2t\int_0^\infty e^{-2\nu t}\nu\,d\nu + t\int_1^\infty e^{-2t\nu}\,d\nu \le \frac{\Gamma(1)}{2t} + 2te^{-2t}\\ &= \frac{1}{t}\big(1/2 + 2t^2e^{-2t}\big).\end{aligned}$$

Observing that $\max_{t>0}\big(2t^2e^{-2t}\big) = 2e^{-2}$ yields the desired assertion. □

Observe again that, under weak conditions on the Hardy weight ρ, Proposition 18.5 already gives an accelerated decay rate when compared with the one in the flat case given in Proposition 18.3.

18.7.2 The Hardy inequality for negatively curved manifolds

In this subsection, we show that (18.35) typically holds for negatively curved manifolds.

One way how to establish (18.35) is to generalize the method of [24]. It works as follows:

1. *Transverse ground-state estimate.* Recalling the structure of our operator (18.15), we clearly have

$$H_K - E_1 \;\ge\; -f^{-1}\partial_1 f^{-1}\partial_1 + \mu_K \tag{18.40}$$

   in the form sense on $L^2_f(\Omega_0)$, where $x \mapsto \mu_K(x_1)$ denotes the lowest eigenvalue of the one-dimensional shifted 'transverse' operator $-f^{-1}\partial_2 f\partial_2 - E_1$ on the Hilbert space $L^2\big((-a,a), f(x_1,x_2)\,dx_2\big)$, subject to Dirichlet boundary conditions, with x_1 being considered as a parameter in the one-dimensional eigenvalue problem. More specifically, we have

$$\mu_K(x_1) = \inf_{\varphi\in H^1_0((-a,a))\setminus\{0\}} \frac{\int_{-a}^a |\dot\varphi(x_2)|^2\, f(x_1,x_2)\,dx_2}{\int_{-a}^a |\varphi(x_2)|^2\, f(x_1,x_2)\,dx_2} \;-\; E_1\,. \tag{18.41}$$

   With an abuse of notation, we denote by the same symbol μ_K both the function on $\mathbb{R}$ and its natural extension $x \mapsto \mu_K(x_1)$ to Ω_0.

2. *Longitudinal Hardy-type estimate.* Now we regard the right hand side of (18.40) as a one-dimensional Schrödinger-type operator on the Hilbert space $L^2\big((-a,a), f(x_1,x_2)\,dx_1\big)$, with x_2 being considered as a parameter and μ_K playing the role of potential. We assume that each of the x_2-dependent family of operators satisfies a Hardy-type inequality, so that

$$-f^{-1}\partial_1 f^{-1}\partial_1 + \mu_K \;\ge\; \rho_K > 0 \tag{18.42}$$

   in the form sense on $L^2_f(\Omega_0)$, with some positive function $\rho_K : \Omega_0 \to (0,\infty)$. Then (18.35) holds as a consequence of (18.42) and (18.40).

In this way, we have reduced the problem to ensuring the existence of one-dimensional Hardy-type inequalities (18.42). However, the criticality of one-dimensional Schrödinger operators is well studied, at least if $f = 1$. We present two sufficient conditions which guarantee the validity of (18.42) and confirm thus that (18.35) typically holds for negatively curved manifolds.

Positivity of the ground-state estimates

Since the kinetic part of the Schrödinger-type operator on the left hand side of (18.42) is a non-negative operator, we get a trivial estimate

$$-f^{-1}\partial_1 f^{-1}\partial_1 + \mu_K \;\ge\; \mu_K \tag{18.43}$$

in the form sense on $L^2_f(\Omega_0)$. As a consequence of (18.40), $H_K - E_1 \ge \mu_K$.

This represents a *local* Hardy-type inequality provided that μ_K is non-zero and non-negative. By 'local' we mean that the function μ_K is compactly supported for manifolds with compactly supported curvature K, which is a typical hypothesis of the present paper. Hence it does not fit to the initial definition (18.35), which can be called *global* Hardy-type inequality. However, it is known that local Hardy-type inequalities imply global ones.

Theorem 18.3 (Hardy inequality for non-negative μ_K)**.** *Assume* (18.14)*. If μ_K is non-zero and non-negative in some bounded open subinterval $J \subset \mathbb{R}$, then there exists a positive constant c_K, depending on a and properties of K, such that*

$$-f^{-1}\partial_1 f^{-1}\partial_1 + \mu_K \;\geq\; \frac{c_K}{1+\delta^2} \tag{18.44}$$

in the form sense on $L^2_f(\Omega_0)$. Here $\delta(x) := |x_1 - x_1^0|$, with x_1^0 being the mid-point of J. As a consequence of (18.40)*, the Hardy-type inequality* (18.35) *holds.*

Proof. The proof follows by a modification of the proof of [24, Thm. 3.1] (*cf* also [25, Thm. 6.7]). For the clarity of the exposition, we divide it into several steps.

1. A consequence of the classical Hardy inequality. The main ingredient in the proof is the following Hardy-type inequality for a Schrödinger operator in the strip Ω_0 with a characteristic-function potential:

$$\|(1+\delta^2)^{-1/2}\psi\|^2 \leq 16\,\|\partial_1\psi\|^2 + (2+64/|J|^2)\,\|\chi_J\psi\|^2 \tag{18.45}$$

for every $\psi \in H^1(\Omega_0)$. Here J is any bounded open subinterval of $\mathbb{R}$ and χ_J denotes the characteristic function of the set $J\times(-a,a) \subset \Omega_0$. This inequality can be established quite easily (*cf* [6, Sec. 3.3]) by means of Fubini's theorem and the classical one-dimensional Hardy inequality $\int_0^b s^{-2}|\varphi(s)|^2 ds \leq 4\int_0^b |\dot\varphi(s)|^2 ds$ valid for any $\varphi \in H^1((0,b))$, $b>0$, satisfying $\varphi(0)=0$.

Using Lemma 18.1, (18.45) can be cast into the form

$$\left\|f^{-1}\partial_1\psi\right\|_f^2 + \left\|\mu_K^{1/2}\psi\right\|_f^2 \geq c\,\|(1+\delta^2)^{-1/2}\psi\|_f^2 - C\,\|\chi_J\psi\|_f^2\,, \tag{18.46}$$

where the constants are given by

$$c := \frac{1-\|K\|_\infty a^2}{16}\,, \qquad C := \left(\frac{1}{8}+\frac{4}{|J|^2}\right)\left[1-\left(\frac{\|K\|_\infty a^2}{1-\|K\|_\infty a^2}\right)^2\right]^{-1}.$$

2. A Poincaré-type inequality in a bounded strip. For every $\psi \in H^1(\Omega_0)$, we have

$$\begin{aligned}\left\|f^{-1}\partial_1\psi\right\|_f^2 + \left\|\mu_K^{1/2}\psi\right\|_f^2 &\geq \left\|\chi_J f^{-1}\partial_1\psi\right\|_f^2 + \left\|\chi_J\mu_K^{1/2}\psi\right\|_f^2 \\ &\geq \lambda_J\left\|\chi_J\psi\right\|_f^2\,,\end{aligned} \tag{18.47}$$

where λ_J denoted the lowest eigenvalue of the operator $-f^{-1}\partial_1 f^{-1}\partial_1 + \mu_K$ on $L^2_f(J\times(-a,a))$, subject to Neumann-type (*i.e.* no in the form setting) boundary conditions at $(\partial J)\times(-a,a)$. We claim that λ_J can be bounded from below by a positive constant which depends exclusively on properties of μ_K. Indeed, assume $\lambda_J = 0$. By the variational characterization of λ_J, it follows that

$$\left\|\chi_J f^{-1}\partial_1\psi_J\right\|_f^2 = 0 \qquad \text{and} \qquad \left\|\chi_J\mu_K^{1/2}\psi_J\right\|_f^2 = 0\,,$$

where $\psi_J \in H^1(J\times(-a,a))$ is an eigenfunction corresponding to λ_J. Recalling Lemma 18.1, we conclude that $\|\mu_K\|_{L^1(J)} = 0$, which contradicts the hypothesis that μ_K is non-trivial on J.

3. Some interpolation. Combining (18.46) with (18.47), we eventually arrive at

$$\left\|f^{-1}\partial_1\psi\right\|_f^2 + \left\|\mu_K^{1/2}\psi\right\|_f^2 \geq c\,\epsilon\,\|(1+\delta^2)^{-1/2}\psi\|_f^2 + [(1-\epsilon)\lambda_J - C\,\epsilon]\;\|\chi_J\psi\|_f^2$$

for every $\psi \in H^1(\Omega_0)$ and any $\epsilon \in (0,1)$. Choosing ϵ in such a way that the term with the square brackets vanishes, we get the Hardy-type inequality of the theorem with $c_K := c\lambda_J/(\lambda_J + C)$. □

On the positivity of the ground-state eigenvalue

Since the fundamental hypothesis of Theorem 18.3 is the non-negativity of μ_K, let us comment on its relation to the non-positivity of K.

We claim that the function μ_K is typically positive for negatively curved manifolds. Indeed, for any fix $x_1 \in \mathbb{R}$, let us make the change of test function $\phi := \sqrt{f(x_1,\cdot)}\,\varphi$ in (18.41). Integrating by parts and using (18.10), one easily arrives at

$$\mu_K(x_1) = \inf_{\phi\in H^1_0((-a,a))\setminus\{0\}} \frac{\int_{-a}^a \left(|\dot\phi(x_2)|^2 - E_1\,|\phi(x_2)|^2 + V(x)\,|\phi(x_2)|^2\right) dx_2}{\int_{-a}^a |\phi(x_2)|^2\,dx_2} \tag{18.48}$$

with

$$V := -\frac{1}{2}\,K + \frac{1}{4}\left(\frac{\partial_2 f}{f}\right)^2 . \tag{18.49}$$

By the Poincaré inequality for the Dirichlet Laplacian in $L^2((-a,a))$, we therefore get

$$\mu_K(x_1) \geq \operatorname*{ess\,inf}_{x_2\in(-a,a)} V(x_1,x_2)\,. \tag{18.50}$$

Let us assume for a moment that K is continuous. Then, for every $x_1 \in \mathbb{R}$ fixed, it follows from (18.10) that

$$\lim_{a\to 0} V(x) = -\frac{1}{2}\,K(x_1,0)\,.$$

Hence, if $K(x_1,x_2)$ is negative for every $x_2 \in (-a,a)$ and x_1 from a compact interval J, there exists a positive half-width a such that $\mu_K(x_1)$ is positive for every $x_1 \in J$. For merely bounded curvature K, we replace the pointwise non-positivity requirement on the curve Γ by the hypothesis that the function

$$k(x_1) := \lim_{a\to 0} \operatorname*{ess\,inf}_{x_2\in(-a,a)} K(x_1,x_2) \tag{18.51}$$

is non-zero and non-positive.

It is less obvious how to get uniform lower bounds, *i.e.* to ensure that, for a given a, $\mu_K(x_1)$ is non-negative for almost every $x_1 \in \mathbb{R}$. An example of manifolds for which the uniform non-negativity is possible to check is given by strips on ruled surfaces studied in [24].

Example 18.1 (Ruled strips)**.** Let Γ be a straight line in $\mathbb{R}^3$; without loss of generality, we may assume that $\Gamma(x_1) = (x_1,0,0)$. Given a C^1-smooth function $\theta : \mathbb{R} \to \mathbb{R}$, let us define $\mathcal{L}(x) := \big(x_1, x_2\cos\theta(x_1), x_2\sin\theta(x_1)\big)$. The image (18.7) is a ruled surface, composed of segments of length $2a$ translated and rotated along Γ. It is straightforward to check that the corresponding metric G admits the block form (18.9) with the explicit formulae

$$f(x) = \sqrt{1+\dot\theta(x_1)^2\,x_2^2}\,, \qquad K(x) = -\frac{\dot\theta(x_1)^2}{f(x)^4}\,.$$

The *ad hoc* defined mapping $\mathcal{L}$ represents an explicit parametrization of Ω via the exponential map (18.6). The hypothesis (18.14) is satisfied for every a provided that we assume that $\dot\theta$ is bounded. The strip Ω is asymptotically flat if $\dot\theta(x_1)$ tends to zero as $|x_1| \to \infty$. Finally, an explicit calculation yields

$$V(x) = \frac{\dot\theta(x_1)^2\big[2-\dot\theta(x_1)^2\,x_2^2\big]}{4\,f(x)^4}\,. \tag{18.52}$$

It follows that V is non-zero and non-negative provided that $\dot\theta$ is non-zero and the half-width a is so small that $\|\dot\theta\|_\infty\, a < \sqrt{2}$. Consequently, under the same assumptions about a and $\dot\theta$, the quantity μ_K is non-zero and non-negative, too. We refer to [24] for more geometric and spectral properties of the ruled strips.

Thin strips

The second sufficient condition which guarantees the validity of (18.42) is based on the ideas of the previous subsection.

Theorem 18.4 (Hardy inequality for thin strips)**.** *Assume* (18.14) *and* (18.30)*. Let the function k defined in* (18.51) *be non-zero and non-positive. Then there exists a positive number a_0, depending on properties of K, such that*

$$-\,f^{-1}\partial_1 f^{-1}\partial_1 + \mu_K \;\geq\; \frac{\tilde c_K}{1+x_1^2} \tag{18.53}$$

holds in the form sense on $L^2_f(\Omega_0)$ for all $a \leq a_0$ with some constant $\tilde c_K$ depending on properties of K. As a consequence of (18.40)*, the Hardy-type inequality* (18.35) *holds for all $a \leq a_0$.*

Proof. In view of (18.50), Lemma 18.1, (18.12) and (18.31), it is easy to show that

$$\mu_K(x_1) \geq -\frac{1}{2}\,k - C(\|K\|_\infty\, a^2)\,\chi_{[-R,R]}(x_1)\,,$$

for almost every $x_1 \in \mathbb{R}$, where

$$C(\xi) := \frac{1}{4}\xi^2\left(1+\frac{\xi^2}{1-\xi^2}\right)^2\left(1-\frac{\xi^2}{1-\xi^2}\right)^{-2}.$$

Hence, $\mu_K \to \mu_K^0 := -\frac{1}{2}k$ as $a \to 0$. For every $\psi \in H^1(\Omega_0)$, we write

$$\left\|f^{-1}\partial_1\psi\right\|_f^2 + \left(\psi, \mu_K\psi\right)_f = \left\|f^{-1}\partial_1\psi\right\|_f^2 + \left(\psi, \mu_K^0\psi\right)_f + \left(\psi, [\mu_K - \mu_K^0]\psi\right)_f .$$

Applying Theorem 18.3 to the first two terms on the right hand side of this identity, we get

$$\left\|f^{-1}\partial_1\psi\right\|_f^2 + \left(\psi, \mu_K\psi\right)_f \geq \int_{\Omega_0} \left[\frac{c_K}{1+x_1^2} - C(\|K\|_\infty a^2)\,\chi_{[-R,R]}(x_1)\right] |\psi(x)|^2\, f(x)\, dx\,.$$

It is important to notice that c_K can be bounded from below by a positive constant independent of a (*cf* proof of Theorem 18.3). On the other hand, $C(\|K\|_\infty\, a^2)$ tends to zero as $a \to 0$. Then the result follows by estimating the characteristic function by $(1+x_1^2)^{-1}$ multiplied by a constant smaller than c_K for all sufficiently small a. □

Remark 18.5. The positive function ρ on the right hand side of (18.35) can in principle vanish on the boundary of $\partial\Omega_0$. The objective of this remark is to show that, if (18.35) holds, with an arbitrary positive function ρ, there is also an inequality of the type (18.42) with the right hand side which is independent of the 'transverse' variable x_2. This can be seen as follows. Assume (18.42) and (18.30). For any $\psi \in H_0^1(\Omega_0)$ and $\epsilon \in (0,1)$, we write

$$\begin{aligned} h_K[\psi] - E_1\,\|\psi\|_f^2 &= \epsilon\left(h_K[\psi] - E_1\,\|\psi\|_f^2\right) + (1-\epsilon)\left(h_K[\psi] - E_1\,\|\psi\|_f^2\right) \\ &\geq \epsilon\left(\|\partial_2\psi\|_f^2 - E_1\,\|\psi\|_f^2\right) + (1-\epsilon)\,\|\rho^{1/2}\psi\|_f^2 \\ &= \epsilon\left(\|\partial_2\phi\|^2 - E_1\,\|\phi\|^2 + (\phi, V\phi)\right) + (1-\epsilon)\,\|\rho^{1/2}\phi\|^2 \\ &\geq \epsilon\left(\psi, [V+\lambda_\epsilon]\psi\right). \end{aligned}$$

Here the last equality follows by the change of test function $\phi := \sqrt{f}\,\psi$, as in Section 18.7.2, and $x \mapsto \lambda_\epsilon(x_1)$ denotes the lowest eigenvalue of the one-dimensional operator $-\partial_2^2 - E_1 + (1-\epsilon)\epsilon^{-1}\rho(x_1,\cdot)$ on $L^2((-a,a))$, subject to Dirichlet boundary conditions, with x_1 considered as a parameter. More specifically, we have

$$\lambda_\epsilon(x_1) := \inf_{\varphi \in H_0^1((-a,a))\setminus\{0\}} \frac{\int_{-a}^a \left(|\dot\varphi(x_2)|^2 - E_1\,|\varphi(x_2)|^2 + \frac{1-\epsilon}{\epsilon}\rho(x_1,x_2)\,|\varphi(x_2)|^2\right) dx_2}{\int_{-a}^a |\phi(x_2)|^2\, dx_2}\,.$$

Since K has bounded support, it is also true for V, *cf* (18.31). On the other hand, since $\rho(x)$ is positive for almost every $x \in \Omega_0$, $\lambda_\epsilon(x_1)$ is positive for almost every $x_1 \in \mathbb{R}$. Furthermore, $\lambda_\varepsilon(x_1)$ tends to infinity as $\epsilon \to 0$ for almost every $x_1 \in \mathbb{R}$. Consequently, for sufficiently small ϵ, $V + \lambda_\epsilon$ can be bounded from below by a positive function which depends on x_1 only.

Finally, let us emphasize that Theorem 18.4 covers a very general class of manifolds, not necessarily negatively curved. It is only important that the manifold is 'negatively curved in the vicinity of the reference curve' Γ, *cf* (18.51).

18.7.3 The fine decay rate

As in the flat case in Proposition 18.3, we again restrict the class of initial data to the weighted spaces of $L^2_{wf}(\Omega_0) \subset L^2_f(\Omega_0)$ and consider the following (polynomial) *decay rate* quantity:

$$\Gamma_K := \sup\Big\{\Gamma \in \mathbb{R} \;\Big|\; \exists C_\Gamma > 0,\, \forall t \geq 0,\, \left\|e^{-(H_K - E_1)t}\right\|_{L^2_{wf}(\Omega_0)\to L^2_f(\Omega_0)} \leq C_\Gamma\,(1+t)^{-\Gamma}\Big\}. \quad (18.54)$$

Sections 18.7.1 and 18.7.2 already imply that the heat semigroup decays faster than in the flat case provided that the Hardy-type inequality (18.35) holds. It follows from Proposition 18.3 that we have $\Gamma_0 = 1/4$ (*i.e.* for $K = 0$), whereas Proposition 18.5 gives $\Gamma_K \geq 1/2$ if (18.35) is satisfied.

The abstract arguments leading to Proposition 18.5 do not give the precise additional polynomial decay rate. The objective of the following subsections is to show that Γ_K is in fact three times bigger whenever the curvature K is non-zero and non-positive.

In probabilistic terms we are interested in the precise decay exponent

$$\gamma_K(x,B) := \sup\Big\{\gamma\in\mathbb{R}\;\Big|\;\exists\tilde{C}_\gamma>0,\,\forall t\geq 0,\\ \mathbb{P}_x\big(X_t\in B,\tau_{\Omega_0}>t\big)\leq C_\gamma\,(1+t)^{-\Gamma}\Big\}. \quad (18.55)$$

where $x\in\Omega_0, B\subset\subset\Omega_0$. Again we find that the non-zero and non-positive situation differs from the straight manifold by a factor 3. This is the meaning of the last item in Table 18.1.

In view of (18.54), it is more convenient to study the shifted heat equation

$$\begin{cases}\partial_t u + H_K u - E_1 u = 0 & \text{in}\quad \Omega_0\times(0,\infty)\,,\\ u = u_0 & \text{on}\quad \Omega_0\times\{0\}\,,\end{cases} \quad (18.56)$$

in the functional setting on $L^2_f(\Omega_0)$ as explained in Section 18.3.2. Indeed, (18.56) is obtained from (18.2) by the replacement $u(t)\mapsto e^{-E_1 t}u(t)$, with help of the Fermi coordinates.

18.7.4 The self-similarity transformation

Our method to study the asymptotic behaviour of the heat equation (18.2) in the presence of curvature is to adapt the technique of self-similar solutions used in the case of the heat equation in the whole Euclidean space by Escobedo and Kavian [7] to the present problem. We closely follow the approach of the recent papers [28, 52], where the technique is applied to twisted waveguides in three and two dimensions, respectively.

We perform the self-similarity transformation in the first (longitudinal) space variable only, while keeping the other (transverse) space variable unchanged. More precisely, given $s\in(0,\infty)$, let us consider the change of function defined by

$$(U_s\psi)(y) := e^{s/4}\psi(e^{s/2}y_1,y_2)\,.$$

It defines a unitary transformation from $L^2_f(\Omega_0)$ to $L^2_{f_s}(\Omega_0)$, where

$$f_s(y) := f(e^{s/2}y_1,y_2)\,. \quad (18.57)$$

Now we associate to every solution $u\in L^2_{\mathrm{loc}}\big((0,\infty),dt;L^2_f(\Omega_0)\big)$ of (18.2) a 'self-similar' solution $\tilde{u}(s) := U_s[u(e^s-1)]$ in a new s-time weighted space $L^2_{\mathrm{loc}}\big((0,\infty),e^s ds;L^2_{f_s}(\Omega_0)\big)$. We have

$$\tilde{u}(y_1,y_2,s) = e^{s/4}u(e^{s/2}y_1,y_2,e^s-1) \quad (18.58)$$

and the inverse change of variables is given by

$$u(x_1,x_2,t) = (t+1)^{-1/4}\,\tilde{u}\big((t+1)^{-1/2}x_1,x_2,\log(t+1)\big)\,.$$

Note that the original space-time variables (x,t) are related to the 'self-similar' space-time variables (y,s) via the relations

$$\begin{aligned}(x_1,x_2,t) &= (e^{s/2}y_1,y_2,e^s-1)\,,\\ (y_1,y_2,s) &= \big((t+1)^{-1/2}x,y_2,\log(t+1)\big)\,.\end{aligned} \quad (18.59)$$

Hereafter we consistently use the notation for respective variables to distinguish the two space-times.

It is easy to check that this change of variables transfers the weak formulation of (18.56) to the evolution problem

$$\big\langle\tilde{v},\tilde{u}'(s)-\tfrac{1}{2}\,y_1\partial_1\tilde{u}(s)\big\rangle_{f_s} + \tilde{a}_s\big(\tilde{v},\tilde{u}(s)\big) = 0\,, \quad (18.60)$$

for each $\tilde{v}\in H^1_0(\Omega_0)$ and a.e. $s\in[0,\infty)$, with $\tilde{u}(0)=\tilde{u}_0 := U_0u_0 = u_0$. Here $\langle\cdot,\cdot\rangle_{f_s}$ stands for the pairing of $H^1_0(\Omega_0,G_s)$ and its dual $[H^1_0(\Omega_0,G_s)]^*$, where G_s is the metric of the form (18.9) with f being replaced by f_s, and $\tilde{a}_s(\cdot,\cdot)$ denotes the sesquilinear form associated with

$$\begin{aligned}\tilde{a}_s[\tilde{u}] &:= \|f_s^{-1}\partial_1\tilde{u}\|^2_{f_s} + e^s\,\|\partial_2\tilde{u}\|^2_{f_s} - e^s\,E_1\,\|\tilde{u}\|^2_{f_s} - \frac{1}{4}\,\|\tilde{u}\|^2_{f_s}\,,\\ \tilde{u}\in\mathfrak{D}(\tilde{a}_s) &:= H^1_0(\Omega_0)\,.\end{aligned} \quad (18.61)$$

More specifically, $H^1_0(\Omega_0,G_s)$ denotes the completion of $C^\infty_0(\Omega_0)$ with respect to the norm $\|\cdot\|_{\mathfrak{D}(h_{K_s})} := (h_{K_s}[\cdot]+\|\cdot\|^2_{f_s})^{1/2}$, where h_{K_s} is defined as (18.17) with f being replaced by f_s.

Remark 18.6. Note that (18.60) is a parabolic equation with s-time-dependent coefficients. The same occurs and has been previously analysed for the heat equation in the twisted waveguides [28, 29], for the heat equation in the plane with magnetic field [26] and also for a convection-diffusion equation in the whole space but with a variable diffusion coefficient [8, 5]. A careful analysis of the behaviour of the underlying elliptic operators as s tends to infinity leads to a sharp decay rate for its solutions. An important difference of the present problem with respect to the previous works is that also the Hilbert space becomes time-dependent after the self-similarity transformation, which makes the analysis substantially more difficult.

18.7.5 The setting in weighted Sobolev spaces

Since U_s acts as a unitary transformation, it preserves the space norm of solutions of (18.56) and (18.60), *i.e.*,

$$\|u(t)\|_f = \|\tilde{u}(s)\|_{f_s}\,. \tag{18.62}$$

This means that we can analyse the asymptotic time behaviour of the former by studying the latter.

However, the natural space to study the evolution (18.60) is not $L^2_{f_s}(\Omega_0)$ but rather the weighted space $L^2_{wf_s}(\Omega_0)$ with the Gaussian weight (18.26). Following the approach of [28] based on a theorem of J. L. Lions [1, Thm. X.9] about weak solutions of parabolic equations with time-dependent coefficients, it can be shown that (18.60) is well posed in the scale of Hilbert spaces

$$H^1_0(\Omega_0, w\,G_s) \subset L^2_{wf_s}(\Omega_0) \subset \big[H^1_0(\Omega_0, w\,G_s)\big]^*\,. \tag{18.63}$$

Here $H^1_0(\Omega_0, w\,G_s)$ denotes the completion of $C^\infty_0(\Omega_0)$ with respect to the norm $(h_{K_s}[w^{1/2}\cdot] + \|\cdot\|^2_{wf_s})^{1/2}$.

More precisely, choosing $\tilde{v} := wv$ for the test function in (18.60), where $v \in C^\infty_0(\Omega_0)$ is arbitrary, we can formally cast (18.60) into the form

$$\big\langle v, \tilde{u}'(s)\big\rangle_w + a_s\big(v, \tilde{u}(s)\big) = 0\,. \tag{18.64}$$

Here $\langle\cdot,\cdot\rangle_w$ denotes the pairing of $H^1_0(\Omega_0, w\,G_s)$ and $[H^1_0(\Omega_0, w\,G_s)]^*$ and $a_s(\cdot,\cdot)$ denotes the sesquilinear form associated with

$$\begin{aligned} a_s[\tilde{u}] \;&:=\; \|f_s^{-1}\partial_1\tilde{u}\|^2_{wf_s} + e^s\,\|\partial_2\tilde{u}\|^2_{wf_s} - e^s\,E_1\,\|\tilde{u}\|^2_{wf_s} - \frac{1}{4}\,\|\tilde{u}\|^2_{wf_s} \\ &\quad + \frac{1}{2}\,\big(y_1\tilde{u}, [f_s^{-2}-1]\,\partial_1\tilde{u}\big)_{wf_s}\,, \\ \mathfrak{D}(a_s) \;&:=\; H^1_0(\Omega_0, w)\,, \end{aligned} \tag{18.65}$$

with $H^1_0(\Omega_0, w)$ denoting the closure of $C^\infty_0(\Omega_0)$ with respect to the weighted Sobolev norm $(\|\nabla\cdot\|^2_w + \|\cdot\|^2_w)^{1/2}$. Note the appearance of the extra term with respect to (18.61) (it makes the form a_s non-symmetric if the Hilbert space $L^2_{wf}(\Omega_0)$ is considered to be complex).

By 'formally' we mean that the formulae are meaningless in general, because the solution $\tilde{u}(s)$ and its derivative $\tilde{u}'(s)$ may not belong to $H^1_0(\Omega_0, w\,G_s)$ and $[H^1_0(\Omega_0, w\,G_s)]^*$, respectively. The justification of (18.60) being well posed in the scale (18.63) consists basically in checking the boundedness and a coercivity of the form a_s defined on $\mathfrak{D}(a_s)$ and in noticing that the time-dependent spaces $L^2_{wf_s}(\Omega_0)$ and $H^1_0(\Omega_0, w\,G_s)$ coincide with $L^2_w(\Omega_0)$ and $H^1_0(\Omega_0, w)$, respectively, as vector spaces. It is straightforward by using (18.14) and Lemma 18.1.

18.7.6 Reduction to a spectral problem

Choosing $v := \tilde{u}(s)$ in (18.64), we arrive at the identity

$$\frac{1}{2}\frac{d}{ds}\|\tilde{u}(s)\|^2_{wf_s} = -\hat{l}_s[\tilde{u}(s)]\,, \tag{18.66}$$

where $\hat{l}_s[\tilde{u}] := \Re\{a_s[\tilde{u}]\}$, $\tilde{u} \in \mathfrak{D}(\hat{l}_s) := \mathfrak{D}(a_s) = H^1_0(\Omega_0, w)$ (independent of s as a vector space). It remains to analyse the coercivity of $\hat{l}_s$.

More precisely, as usual for energy estimates, we replace the right hand side of (18.66) by the spectral bound, valid for each fixed $s \in [0,\infty)$,

$$\forall\tilde{u} \in \mathfrak{D}(\hat{l}_s)\,, \qquad \hat{l}_s[\tilde{u}] \geq \nu_K(s)\,\|\tilde{u}\|^2_{wf_s}\,, \tag{18.67}$$

where $\nu_K(s)$ denotes the lowest point in the spectrum of the self-adjoint operator $\hat{L}_s$ associated on $L^2_{wf_s}(\Omega_0)$ with $\hat{l}_s$; it depends on the curvature K through the dependence on f. Then (18.66) together with (18.67) implies the exponential bound

$$\forall s \in [0,\infty)\,, \qquad \|\tilde{u}(s)\|_{wf_s} \leq \|\tilde{u}_0\|_{wf_0}\; e^{-\int_0^s \nu_K(r)\,dr}\,. \tag{18.68}$$

Finally, recall that the exponential bound in s transfers to a polynomial bound in the original time t, *cf* (18.59). In this way, the problem is reduced to a spectral analysis of the family of operators $\{\hat{L}_s\}_{s\geq 0}$.

18.7.7 Removing the weight

In order to investigate the operator $\hat{L}_s$ on $L^2_{wf_s}(\Omega_0)$, we first map it into a unitarily equivalent operator $L_s := \mathcal{U}\hat{L}_s\mathcal{U}^{-1}$ on $L^2_{f_s}(\Omega_0)$ via the unitary transform

$$\mathcal{U}\,\tilde{u} := w^{1/2}\,\tilde{u}\,.$$

By definition, L_s is the self-adjoint operator associated on $L^2_{f_s}(\Omega_0)$ with the quadratic form $l_s[v] := \hat{l}_s[\mathcal{U}^{-1}v]$, $v \in \mathfrak{D}(l_s) := \mathcal{U}\,\mathfrak{D}(\hat{l}_s)$. A straightforward calculation yields

$$\begin{aligned} l_s[v] \;&= \|f_s^{-1}\partial_1 v\|^2_{f_s} + e^s\,\|\partial_2 v\|^2_{f_s} - e^s\,E_1\,\|v\|^2_{f_s} - \frac{1}{4}\,\|v\|^2_{f_s} \\ &\quad - \frac{1}{2}\,(y_1 v, \partial_1 v)_{f_s} + \frac{1}{16}\,\big(y_1 v, [2 - f_s^{-2}]\,y_1 v\big)_{f_s}\,. \end{aligned} \tag{18.69}$$

Here and in the sequel, we assume that v is real, which is justified by the positivity preserving property of the heat equation as explained in Section 18.3.2.

For everywhere vanishing curvature, *i.e.* $K = 0$, we have that f is identically equal to one. Consequently, $f_s = 1$ for all $s \geq 0$. Then, integrating by parts in the first term on the second line of (18.69), we get that l_s coincides with the form l^0_s on $L^2(\Omega_0)$ defined by

$$\begin{aligned} l^0_s[v] &:= \|\partial_1 v\|^2 + e^s\,\|\partial_2 v\|^2 - e^s\,E_1\,\|v\|^2 + \frac{1}{16}\,\|y_1 v\|^2\,, \\ \mathfrak{D}(l^0_s) &:= H^1_0(\Omega_0) \cap L^2(\Omega_0, y_1^2\,dy)\,. \end{aligned} \tag{18.70}$$

In order to specify the domain of l_s for any curvature, we assume (18.30) and consider l_s as a perturbation of l^0_s. It follows from (18.31) that

$$|y_1| > e^{-s/2}R \quad \Longrightarrow \quad f_s(y) = 1\,. \tag{18.71}$$

In particular, $f_s(y) = 1$ for all $|y_1| > R$.

Lemma 18.2. *Assume* (18.14) *and* (18.30). *Then*

$$\mathfrak{D}(l_s) = \mathfrak{D}(l^0_s) = H^1_0(\Omega_0) \cap L^2(\Omega_0, y_1^2\,dy)\,.$$

Proof. Using some rearrangement and integration by parts, it is convenient to rewrite (18.69) as follows

$$l_s[v] = \|f_s^{-1}\partial_1 v\|^2_{f_s} + e^s\,\|\partial_2 v\|^2_{f_s} - e^s\,E_1\,\|v\|^2_{f_s} + \frac{1}{16}\,\|y_1 v\|^2_{f_s} + r_s[v]\,, \tag{18.72}$$

where

$$r_s[v] := -\frac{1}{4}\,\big(v, [f_s - 1]v\big) - \frac{1}{2}\,\big(y_1 v, [f_s - 1]\partial_1 v\big) - \frac{1}{16}\,\big(y_1 v, [f_s^{-1} - f_s]\,y_1 v\big)\,.$$

Using Lemma 18.1, it is easy to see that for each $s \geq 0$ there exists a positive constant $C = C(s, \|K\|_\infty, a)$ such that

$$C^{-1}\,l^0_s[v] \leq l_s[v] - r_s[v] \leq C\,l^0_s[v]$$

for every $v \in \mathfrak{D}(l^0_s)$. Consequently (see, *e.g.*, [2, Corol. 4.4.3]), the quadratic form $l_s - r_s$ is closed on the domain $\mathfrak{D}(l^0_s)$ given by (18.70). It remains to show that r_s is a relatively bounded perturbation of l^0_s with relative bound smaller than one. It is clear for the first term of r_s which is in fact a bounded perturbation in view of Lemma 18.1. We employ (18.71) to deal with the remaining terms. For the second term we have,

$$\begin{aligned} \big|\big(y_1 v, [f_s - 1]\partial_1 v\big)\big| &\leq \|f - 1\|_\infty \int_{\{|y_1| < e^{-s/2}R\}} |y_1||v(y)||\partial_1 v(y)|dy \\ &\leq \|f - 1\|_\infty\,e^{-s/2}R\,\|v\|\,\|\partial_1 v\| \\ &\leq \|f - 1\|_\infty\,R\,\big(\epsilon^{-1}\|v\|^2 + \epsilon\,\|\partial_1 v\|^2\big) \end{aligned}$$

for every $v \in \mathfrak{D}(l^0_s)$. and any $\epsilon \in (0, 1)$. Similarly,

$$\big|\big(y_1 v, [f_s^{-1} - f_s]\,y_1 v\big)\big| \leq \|f^{-1} - f\|_\infty\,R^2\,\|v\|^2\,.$$

for every $v \in \mathfrak{D}(l^0_s)$. □

Remark 18.7. The proof of the lemma represents a direct way how to show that the form (18.69) is closed on the domain $\mathfrak{D}(l^0_s)$. In view of the unitary equivalence $\mathcal{U}$, it also *a posteriori* establishes the closedness of the form (18.65).

As a consequence of Lemma 18.2, we get that L_s (and therefore $\hat{L}_s$) has compact resolvent and thus purely discrete spectrum for all $s \geq 0$. In particular, $\nu_K(s)$ represents the lowest eigenvalue of L_s.

18.7.8 The strong-resolvent convergence

In order to study the decay rate via (18.68), we need information about the limit of the eigenvalue $\nu_K(s)$ as the time s tends to infinity. This can be deduced from the asymptotic properties of the resolvent of L_s for large s.

In view of (18.30), the function $y \mapsto f_s(y)$ converges to one locally uniformly in $|y_1| > 0$, $y_2 \in (-a, a)$, as $s \to \infty$. Moreover, the scaling of the transverse variable in (18.69) corresponds to considering the operator L_0 in the shrinking strip $\mathbb{R} \times (-e^{-s/2}a, e^{-s/2}a)$. This suggests that L_s will converge, in a suitable sense, to the one-dimensional harmonic-oscillator operator

$$h := -\frac{d^2}{dy_1^2} + \frac{1}{16}\, y_1^2 \qquad \text{on} \qquad L^2(\mathbb{R}) \tag{18.73}$$

(*i.e.* the Friedrichs extension of this operator initially defined on $C_0^\infty(\mathbb{R})$), potentially subjected to an extra condition at the origin. For further purposes, let us note that the spectrum of h is known explicitly (see any book on quantum mechanics, *e.g.*, [11, Sec. 2.3])

$$\sigma(h) = \left\{ \frac{1}{2}\left(n + \frac{1}{2}\right) \right\}_{n=0}^{\infty} . \tag{18.74}$$

We shall see that the difference between the negatively curved and flat case consists in that the limit operator for the former is subjected to an extra Dirichlet boundary condition at $y_1 = 0$. Thus, simultaneously to h introduced in (18.73), let us consider the self-adjoint operator h_D in $L^2(\mathbb{R})$ whose quadratic form acts in the same way as that of h but has a smaller domain

$$\mathfrak{D}(h_D^{1/2}) := \left\{ \varphi \in \mathfrak{D}(h^{1/2}) \mid \varphi(0) = 0 \right\} .$$

To make this singular operator limits mentioned above rigorous (L_s and h act on different spaces), we decompose the Hilbert space $L^2(\Omega_0)$ into an orthogonal sum

$$L^2(\Omega_0) = \mathfrak{H}_1 \oplus \mathfrak{H}_1^\perp , \tag{18.75}$$

where the subspace $\mathfrak{H}_1$ consists of functions of the form $\psi_1(y) = \varphi(y_1)\mathcal{J}_1(y')$. Recall that $\mathcal{J}_1$ denotes the positive eigenfunction of $-\Delta_D^{(-a,a)}$ corresponding to E_1, normalized to 1 in $L^2((-a,a))$, *cf* (18.22). Given any $\psi \in L^2(\Omega_0)$, we have the decomposition $\psi = \psi_1 + \psi^\perp$ with $\psi_1 \in \mathfrak{H}_1$ as above and $\psi^\perp \in \mathfrak{H}_1^\perp$. The mapping $\pi : \varphi \mapsto \psi_1$ is an isomorphism of $L^2(\mathbb{R})$ onto $\mathfrak{H}_1$. Hence, with an abuse of notations, we may identify any operator h on $L^2(\mathbb{R})$ with the operator $\pi h \pi^{-1}$ acting on $\mathfrak{H}_1 \subset L^2(\Omega_0)$.

Finally, we mention that the Hilbert spaces $L^2(\Omega_0)$ and $L^2_{f_s}(\Omega_0)$ can be identified as vector sets because their norms are equivalent. More specifically, in view of Lemma 18.1 and the definition (18.57), we have

$$1 - \frac{\|K\|_\infty a^2}{1 - \|K\|_\infty a^2} \le \frac{\|\psi\|^2_{f_s}}{\|\psi\|^2} \le 1 + \frac{\|K\|_\infty a^2}{1 - \|K\|_\infty a^2} , \tag{18.76}$$

for every non-zero $\psi \in L^2(\Omega_0)$ and all $s \ge 0$.

In the flat case, *i.e.* $K = 0$, it is readily seen that the operator L_s^0 associated with the form (18.70) can be identified with the decomposed operator

$$h \otimes 1 + 1 \otimes (-e^s\, \Delta_D^{(-a,a)} - e^s\, E_1) \qquad \text{in} \qquad L^2(\mathbb{R}) \otimes L^2((-a,a)) , \tag{18.77}$$

where 1 denotes the identity operators in the appropriate spaces. Using (18.74), it follows that $\nu_0(s) = 1/4$ for all $s \in [0, \infty)$. Consequently,

$$\nu_0(\infty) := \lim_{s \to \infty} \nu_0(s) = 1/4 . \tag{18.78}$$

Moreover, (18.77) can be used to show that L_s^0 converges to $h \oplus 0^\perp$ in the norm-resolvent sense as $s \to \infty$, where $0^\perp$ denotes the zero operator on $\mathfrak{H}_1^\perp$.

It is more difficult (and more interesting) to establish the asymptotic behaviour of $\nu_K(s)$ for $K \neq 0$. A fine analysis of its limit leads to the key observation of the paper, ensuring a gain of $1/2$ in the decay rate in the negatively curved case. This can be understood from the following proposition, which represents the main auxiliary result of the present paper.

Proposition 18.6. *Assume* (18.14) *and* (18.30). *Let the Hardy-type inequality* (18.35) *holds. Then the operator L_s converges to $h_D \oplus 0^\perp$ in the strong-resolvent sense as $s \to \infty$,* i.e.*,*

$$\forall F \in L^2(\Omega_0) , \qquad \lim_{s \to \infty} \left\| \left(L_s + i\right)^{-1} F - \left[\left(h_D + i\right)^{-1} \oplus 0^\perp \right] F \right\| = 0 .$$

Proof. For the clarity of the exposition, we divide the proof into several steps. The equivalence of norms (18.76) and other consequences of Lemma 18.1 are widely used in the present proof.

1. The resolvent equation for L_s. Let $F \in L^2(\Omega_0)$. Then also $F \in L^2_{f_s}(\Omega_0)$ for every $s \geq 0$ due to (18.76). Let z be a sufficiently large positive number to be specified later. We set $\psi_s := (L_s + z)^{-1}F$. In other words, ψ_s satisfies the resolvent equation

$$\forall v \in \mathfrak{D}(l_s)\,, \qquad l_s(v, \psi_s) + z\,(v, \psi_s)_{f_s} = (v, F)_{f_s}\,. \tag{18.79}$$

In particular, choosing ψ_s for the test function v in (18.79), we have

$$\forall v \in \mathfrak{D}(l_s)\,, \qquad l_s[\psi_s] + z\,\|\psi_s\|^2_{f_s} = (\psi_s, F)_{f_s} \leq \|\psi_s\|_{f_s}\|F\|_{f_s}\,. \tag{18.80}$$

2. Boundedness of ψ_s. Our primary objective is to deduce from (18.80) that $\{\psi_s\}_{s\geq 0}$ is a bounded family in the space $\mathfrak{D}_0 := H^1_0(\Omega_0) \cap L^2(\Omega_0, y_1^2\,dy)$ equipped with the intersection topology.

We search a lower bound to the operator $L_s + z$. Using the convenient form (18.72) for $l_s[\psi_s]$ and proceeding as in the proof of Lemma 18.2, we easily check that

$$|r_s[\psi_s]| \leq C\left(\epsilon\,\|f_s^{-1}\partial_1\psi_s\|^2_{f_s} + \epsilon^{-1}\|\psi_s\|^2_{f_s}\right). \tag{18.81}$$

with any $\epsilon \in (0,1)$, where C is a positive constant depending on $\|K\|_\infty\, a^2$ and R. Hence

$$\begin{aligned} l_s[\psi_s] + z\,\|\psi_s\|^2_{f_s} \;&\geq (1-2\epsilon)\,\|f_s^{-1}\partial_1\psi_s\|^2_{f_s} + e^s\,\|\partial_2\psi_s\|^2_{f_s} - e^s\,E_1\,\|\psi_s\|^2_{f_s} \\ &\quad + \epsilon\,\|f_s^{-1}\partial_1\psi_s\|^2_{f_s} + \frac{1}{16}\,\|y_1\psi_s\|^2_{f_s} + (z - C\,\epsilon^{-1})\,\|\psi_s\|^2_{f_s}\,. \end{aligned} \tag{18.82}$$

If we choose z larger than $C\,\epsilon^{-1}$, all the terms on the second line are non-negative.

To get a non-negative lower bound to the first line on the right hand side of (18.82), we introduce a new function u_s by $\psi_s(y) = e^{s/4}u_s(e^{s/2}y_1, y_2)$ (*cf* the self-similarity transformation (18.58)). Making the change of variables $(x_1, x_2) = (e^{s/2}y_1, y_2)$, recalling the definition (18.41) and using the Hardy-type inequality (18.35), we obtain

$$\begin{aligned} &(1-2\epsilon)\,\|f_s^{-1}\partial_1\psi_s\|^2_{f_s} + e^s\,\|\partial_2\psi_s\|^2_{f_s} - e^s\,E_1\,\|\psi_s\|^2_{f_s} \\ &\quad = e^s\,(1-2\epsilon)\,\|f^{-1}\partial_1 u_s\|^2_f + e^s\,\|\partial_2 u_s\|^2_f - e^s\,E_1\,\|u_s\|^2_f \\ &\quad \geq e^s\,(1-2\epsilon)\,\|\rho^{1/2}u_s\|^2_f + e^s\,2\epsilon\,(u_s, \mu_K u_s)_f\,. \end{aligned} \tag{18.83}$$

Here ρ is a positive function and, as pointed out in Remark 18.5, we may assume that it depends on x_1 only. On the other hand, μ_K has compact support due to (18.31). Hence, we can choose ϵ sufficiently small so that the new Hardy weight $\tilde\rho(x_1) := (1-2\epsilon)\rho(x_1) + 2\epsilon\mu_K(x_1)$ is positive for almost every $x_1 \in \mathbb{R}$. Coming back to our coordinates y, we thus conclude from (18.83)

$$(1-2\epsilon)\,\|f_s^{-1}\partial_1\psi_s\|^2_{f_s} + e^s\,\|\partial_2\psi_s\|^2_{f_s} - e^s\,E_1\,\|\psi_s\|^2_{f_s} \geq e^s\,\|\tilde\rho_s^{1/2}\psi_s\|^2_f\,,$$

where $\tilde\rho_s(y_1) := \rho(e^{s/2}y_1)$.

Using the last inequality in (18.82) and employing Lemma 18.1, we eventually arrive at

$$l_s[\psi_s] + z\,\|\psi_s\|^2_{f_s} \geq c\left(e^s\,\|\tilde\rho_s^{1/2}\psi_s\|^2 + \|\partial_1\psi_s\|^2 + \|y_1\psi_s\|^2 + (z - C\,\epsilon^{-1})\,\|\psi_s\|^2\right), \tag{18.84}$$

where c is a positive constant depending on $\|K\|_\infty\, a^2$. Comparing this inequality with (18.80), we see that there exists a constant z_0, depending on a and properties of K, such that for all $z \geq z_0$

$$\|\psi_s\| \leq C\,\|F\|\,, \qquad \|y_1\psi_s\| \leq C\,\|F\|\,, \qquad \|\partial_1\psi_s\| \leq C\,\|F\|\,, \tag{18.85}$$

and

$$e^s\,\|\tilde\rho_s^{1/2}\psi_s\|^2 \leq C\,\|F\|^2\,, \tag{18.86}$$

with some constant C depending on a and properties of K. Furthermore, directly from (18.82) and (18.80) with help of (18.85), we also get

$$\|\partial_2\psi_s\| \leq C\,\|F\|\,, \tag{18.87}$$

The estimate (18.84) also shows that $L_s + z$ is invertible for all $z \geq z_0$. This, *a posteriori*, justifies the definition of ψ_s as the unique solution of (18.79).

From (18.85) and (18.87), we conclude that $\{\psi_s\}_{s\geq 0}$ is a bounded family in $\mathfrak{D}_0$. Therefore it is precompact in the weak topology of $\mathfrak{D}_0$. Let ψ_∞ be a weak limit point, *i.e.*, for an increasing sequence of positive numbers $\{s_n\}_{n\in\mathbb{N}}$ such that $s_n \to \infty$ as $n \to \infty$, $\{\psi_{s_n}\}_{n\in\mathbb{N}}$ converges weakly to ψ_∞ in $\mathfrak{D}_0$. Actually, we may assume that it converges strongly in $L^2(\Omega_0)$ because $\mathfrak{D}_0$ is compactly embedded in $L^2(\Omega_0)$.

3. Transverse mode decomposition of ψ_s. Now we employ the Hilbert space decomposition (18.75) and write $\psi_s(y) = \varphi_s(y_1)\mathcal{J}_1(y_2) + \psi_s^\perp(y)$, where $\psi_s^\perp \in \mathfrak{H}_1^\perp$, *i.e.*,

$$\big(\mathcal{J}_1, \psi_s^\perp(y_1,\cdot)\big)_{L^2((-a,a))} = 0 \tag{18.88}$$

for a.e. $y_1 \in \mathbb{R}$. It follows from (18.85), (18.87) and (18.88) that also $\{\psi_s^\perp\}_{s\geq 0}$ is a bounded family in $\mathfrak{D}_0$ and that $\{\varphi_s\}_{s\geq 0}$ is a bounded family in $H^1(\mathbb{R}) \cap L^2(\mathbb{R}, y_1^2\, dy_1)$ equipped with the intersection topology. We denote by $\psi_\infty^\perp$ and φ_∞ the respective limit points.

We come back to (18.80) with (18.82) and focus on the inequality

$$e^s \,\|\partial_2\psi_s\|_{f_s}^2 - e^s\, E_1 \,\|\psi_s\|_{f_s}^2 \leq C\,\|F\|^2 \tag{18.89}$$

we have already used to get (18.87). In the same way as we proceeded to get (18.48), we write $\phi_s := \sqrt{f_s}\,\psi_s$ and obtain

$$\|\partial_2\psi_s\|_{f_s}^2 - E_1\,\|\psi_s\|_{f_s}^2 = \|\partial_2\phi_s\|^2 - E_1\,\|\phi_s\|^2 + (\phi_s, V_s\,\phi_s)\,, \tag{18.90}$$

where V_s is defined in the same way as (18.49) but with K and f being replaced by K_s and f_s, respectively. Using (18.71), it is possible to check that $\{\phi_{s_n}\}_{n\in\mathbb{N}}$ is strongly converging in $L^2(\Omega_0)$; in fact,

$$\lim_{n\to\infty} \|\phi_{s_n} - \psi_\infty\| = 0\,. \tag{18.91}$$

Using the fact that the scaled potential V_s in (18.90) vanishes for all $|y_1| > e^{-s/2}R$ together with the strong convergence of $\{\phi_{s_n}\}_{n\in\mathbb{N}}$, it is easy to see that the integral containing the potential tends to zero as $n \to \infty$, after passing to the subsequence $\{s_n\}_{n\in\mathbb{N}}$. Multiplying (18.89) by e^{-s_n} and putting the asymptotically vanishing integral on the right hand side of the inequality, we thus get

$$\lim_{n\to\infty} \big(\|\partial_2\phi_{s_n}\|^2 - E_1\,\|\phi_{s_n}\|^2\big) = 0\,. \tag{18.92}$$

Using in addition the Hilbert space decomposition (18.75) of ϕ_s, *i.e.* $\phi_s(y) = \eta_s(y_1)\mathcal{J}_1(y_2) + \phi_s^\perp(y)$, we see that the same limit (18.92) holds for $\phi_{s_n}^\perp \in \mathfrak{H}_1^\perp$ as well. In that limit, we use the estimate $\|\partial_2\phi_{s_n}^\perp\|^2 \geq E_2\,\|\phi_{s_n}^\perp\|^2$, where $E_2 = 4E_1$ denotes the second eigenvalue of $-\Delta_D^{(-a,a)}$, and conclude that $\|\phi_{s_n}^\perp\|$ tends to zero as $n \to \infty$. The latter together with (18.91) finally yields

$$\lim_{n\to\infty} \|\psi_{s_n}^\perp\| = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|\eta_{s_n} - \varphi_\infty\|_{L^2(\mathbb{R})} = 0\,. \tag{18.93}$$

That is, $\psi_\infty \in \mathfrak{H}_1$.

4. The Dirichlet condition at zero. Now we come back to the inequality (18.86). Recall that $\tilde\rho_s(y_1) = \tilde\rho_s(e^{s/2}y_1)$ and that $\tilde\rho$ is positive (although necessarily vanishing at infinity). Without loss of generality, we may assume that $\tilde\rho$ in (18.86) belongs to $L^1(\mathbb{R})$ (since we can always replace the estimate using a smaller function). Then $e^{s/2}\tilde\rho_s$ converges in the sense of distributions on $\mathbb{R}$ to a Dirac delta at $y_1 = 0$. We want to use this heuristic consideration to show that $\varphi_\infty(0) = 0$.

To do so, first, we use the Hilbert space decomposition (18.75) of ψ_s and notice that the left hand side of (18.86) splits into a sum of two non-negative parts, the mixed term being zero due to (18.88). Second, multiplying the obtained inequality for the term involving the $\mathfrak{H}_1$-part of ψ_s by $e^{-s/2}$, passing to the subsequence $\{s_n\}_{n\in\mathbb{N}}$ and taking the limit $n \to \infty$, we arrive at

$$|\varphi_\infty(0)|^2 \int_{\mathbb{R}} \tilde\rho(x_1)\, dx_1 = 0\,.$$

The limiting procedure is justified by recalling that $\{\varphi_{s_n}\}_{n\in\mathbb{N}}$ converges weakly in $H^1(\mathbb{R})$ and therefore strongly in $H^1(J)$, which is compactly embedded in $C^{0,\lambda}(J)$ for every $\lambda \in (0, 1/2)$, where J is any bounded interval of $\mathbb{R}$.

Since the integral of $\tilde\rho$ is positive by our hypotheses, we thus verify that φ_∞ satisfies the extra Dirichlet condition

$$\varphi_\infty(0) = 0\,.$$

5. The resolvent equation for L_s as $s \to \infty$. Let us summarize our results. We have obtained that the solutions ψ_{s_n} of (18.79) converge in the weak topology of $\mathfrak{D}_0$ and in the strong topology of $L^2(\Omega_0)$ to some ψ_∞.

Moreover, the limiting solution ψ_∞ belongs to $\mathfrak{H}_1$, so that $\psi_\infty(y) = \varphi_\infty(y_1)\mathcal{J}_1(y_2)$ with some $\varphi_\infty \in H^1(\mathbb{R}) \cap L^2(\mathbb{R}, y_1^2\,dy_1) = \mathfrak{D}(h)$. Finally, $\varphi_\infty(0) = 0$, so that actually $\varphi_\infty \in \mathfrak{D}(h_D)$.

Recall that the set $C_0^\infty(\mathbb{R}\setminus\{0\})$ is dense in $\mathfrak{D}(h_D)$. Let $\varphi \in C_0^\infty(\mathbb{R}\setminus\{0\})$ be arbitrary. We take $v(y) := \varphi(y_1)\mathcal{J}_1(y_2)$ as the test function in (18.79) and note that φ and $f_s - 1$ have disjoint support for sufficiently large s due to (18.71). Sending n to infinity in (18.79) with s being replaced by s_n, we thus easily check that

$$(\dot\varphi, \dot\varphi_\infty)_{L^2(\mathbb{R})} + \frac{1}{16}\,(y_1\varphi, y_1\varphi_\infty)_{L^2(\mathbb{R})} + z\,(\varphi, \varphi_\infty)_{L^2(\mathbb{R})} = (\varphi, f)_{L^2(\mathbb{R})}\,,$$

where $f(y_1) := (\mathcal{J}_1, F(y_1,\cdot))_{L^2((-a,a))}$. That is, $\varphi_\infty = (h_D + z)^{-1} f$, for *any* weak limit point of $\{\varphi_s\}_{s\geq 0}$.

In conclusion, we have shown that ψ_s converges strongly to ψ_∞ in $L^2(\Omega_0)$ as $s \to \infty$, where $\psi_\infty(y) := \varphi_\infty(y_1)\mathcal{J}_1(y_2) = \big[(h_D + z)^{-1} \oplus 0^\perp\big]F(y)$.

6. The strong convergence for other values of the spectral parameter. Finally, let us argue that we can replace the real number z by any non-real number. This is actually a consequence of [18, Thm. VIII.1.3], the fact that L_s is self-adjoint on $L^2_{f_s}(\Omega_0)$ and the equivalence of this Hilbert space with $L^2(\Omega_0)$, to which we consider the limit of the strong convergence, due to (18.76). □

Remark 18.8. The crucial step in the proof is certainly the usage of the Hardy inequality (18.35). Indeed, it enables us, first, to ensure the non-negativity of the right hand side of (18.82) and, second, to establish the extra Dirichlet condition at zero.

18.7.9 Spectral consequences

Assume for a moment that Proposition 18.6 stated that the operator L_s converges to $h_D \oplus 0^\perp$ in the *norm-resolvent* sense as $s \to \infty$. Then we would immediately know that $\nu_K(s)$ converges to the first eigenvalue of h_D as $s \to \infty$. In view of the symmetry, the first eigenvalue of h_D coincides with the second eigenvalue of h, which is $3/4$ due to (18.74). Hence, under the hypotheses of proposition 18.6, we would have that the limit of $\nu_K(s)$ as $s \to \infty$ is three-times larger than the same limit in the flat case (18.78).

Unfortunately, the strong-resolvent convergence of Proposition 18.6 is not sufficient to guarantee the convergence of spectra. In general, this is true for eigenvalues of the limiting operator which are *stable* under the perturbation (*cf* [18, Sec. VIII.1]). In our case, however, the spectral convergence can be established directly using the fact that both L_s and h_D are operators with compact resolvents. Using the compactness, the convergence of eigenvalues follow by a straightforward modification of the proof of Proposition 18.6. In particular, we have the following result for the lowest eigenvalue, exactly as we would have under the norm-resolvent convergence described above.

Corollary 18.4. *Under the hypotheses of Proposition 18.6, one has*

$$\nu_K(\infty) := \lim_{s\to\infty} \nu_K(s) = 3/4\,.$$

Proof. First of all, let us notice that $\nu_K(s)$ remains bounded as $s \to \infty$. This is easily seen by the Rayleigh-Ritz variational formula for the lowest eigenvalue of L_s, in which we use the trial function of the form $\psi(y) := \varphi(y_1)\mathcal{J}_1(y_2)$, where $\varphi \in C_0^\infty(\mathbb{R})$ is supported outside $\mathrm{supp}(f-1) \supseteq \mathrm{supp}(f_s - 1)$ (*cf* (18.71)). Indeed,

$$\nu_K(s) \leq \frac{l_s[\psi]}{\|\psi\|^2_{f_s}} = \frac{\|\dot\varphi\|^2_{L^2(\mathbb{R})} + \frac{1}{16}\,\|y_1\varphi\|^2_{L^2(\mathbb{R})}}{\|\varphi\|^2_{L^2(\mathbb{R})}}\,, \tag{18.94}$$

irrespectively of the properties of K.

Now, let ψ_s be the positive eigenfunction of L_s corresponding to $\nu_K(s)$, normalized to 1 in $L^2_{f_s}(\Omega_0)$. ψ_s is a solution of the problem (18.79) with $F := (\nu_K(s) + z)\psi_s$. It is important that F is uniformly bounded in s as an element of $L^2_{f_s}(\Omega_0)$, due to (18.94) and the normalization of ψ_s. Then we can proceed exactly as in the proof of Proposition 18.6.

We show that $\{\psi_s\}_{s\geq 0}$ contains a subsequence $\{\psi_{s_n}\}_{n\in\mathbb{N}}$ which is weakly converging to some ψ_∞ in $\mathfrak{D}_\circ$. Since $\mathfrak{D}_\circ$ is compactly embedded in $L^2(\Omega_0)$, we know that $\{\psi_{s_n}\}_{n\in\mathbb{N}}$ converges to ψ_∞ strongly in $L^2(\Omega_0)$. In particular, $\|\psi_\infty\| = 1$, so that we know that ψ_∞ is non-trivial. At the same time, we show that $\psi_\infty \in \mathfrak{H}_1$ and that $\varphi_\infty(y_1) := (\mathcal{J}_1, \psi_\infty(y_1,\cdot))_{L^2((-a,a))}$ vanishes at $y_1 = 0$.

Taking $v(y) := \varphi(y_1)\mathcal{J}_1(y_2)$ with $\varphi \in C_0^\infty(\mathbb{R}\setminus\{0\})$ as the test function in the weak formulation of the eigenvalue problem (18.79), with s being replaced by s_n, and sending n to infinity, we eventually find that φ_∞ is an eigenfunction of $h_D + z$ with the eigenvalue $\nu_K(\infty) + z$. Since ψ_∞ is obtained as a limit of positive functions, we know that φ_∞ is positive as well. Hence, $\nu_K(\infty)$ represents the lowest eigenvalue of h_D.

It remains to recall that the first eigenvalue of h_D coincides with the second eigenvalue of h, which is $3/4$ due to (18.74). □

18.7.10 A spectral bound to the decay rate

We come back to (18.68). Assume $K = 0$ or that there exists a Hardy-type inequality (18.35). Recalling (18.78) and Corollary 18.4, we know that for arbitrarily small positive number ε there exists a (large) positive time s_ε such that for all $s \geq s_\varepsilon$, we have $\nu_K(s) \geq \nu_K(\infty) - \varepsilon$. Hence, fixing $\varepsilon > 0$, we have

$$\begin{aligned} -\int_0^s \nu_K(r)\,dr &\leq -\int_0^{s_\varepsilon} \nu_K(r)\,dr - [\nu_K(\infty) - \varepsilon](s - s_\varepsilon) \\ &\leq \int_0^{s_\varepsilon} |\nu_K(r)|\,dr + [\nu_K(\infty) - \varepsilon]s_\varepsilon - [\nu_K(\infty) - \varepsilon]s \end{aligned}$$

for all $s \geq s_\varepsilon$. At the same time, assuming $\varepsilon \leq 1/4$, we trivially have

$$-\int_0^s \nu_K(r)\,dr \leq \int_0^{s_\varepsilon} |\nu_K(r)|\,dr + [\nu_K(\infty) - \varepsilon]s_\varepsilon - [\nu_K(\infty) - \varepsilon]s$$

also for all $s \leq s_\varepsilon$. Summing up, for every $s \in [0, \infty)$, we have

$$\|\tilde{u}(s)\|_{wf_s} \leq C_\varepsilon\, e^{-[\nu_K(\infty)-\varepsilon]s}\, \|\tilde{u}_0\|_{wf_0}\,, \tag{18.95}$$

where $C_\varepsilon := e^{\int_0^{s_\varepsilon} |\nu_K(r)|\,dr + [\nu_K(\infty)-\varepsilon]s_\varepsilon}$.

Now we return to the original variables (x, t) via (18.59). Using (18.62) together with the point-wise estimate $1 \leq w$, and recalling that $f_0 = f$ and $\tilde{u}_0 = u_0$, it follows from (18.95) that

$$\|u(t)\|_f = \|\tilde{u}(s)\|_{f_s} \leq \|\tilde{u}(s)\|_{wf_s} \leq C_\varepsilon\, (1+t)^{-[\nu_K(\infty)-\varepsilon]}\, \|u_0\|_{wf}$$

for every $t \in [0, \infty)$. Consequently, we conclude with

$$\left\| e^{-(H_K - E_1)t} \right\|_{L^2_{wf}(\Omega_0) \to L^2_f(\Omega_0)} = \sup_{u_0 \in L^2_{wf}(\Omega_0)\setminus\{0\}} \frac{\|u(t)\|_f}{\|u_0\|_{wf}} \leq C_\varepsilon\, (1+t)^{-[\mu_\theta(\infty)-\varepsilon]}$$

for every $t \in [0, \infty)$. Since ε can be made arbitrarily small, this bound implies

$$\Gamma_K \geq \nu_K(\infty)\,. \tag{18.96}$$

18.7.11 The improved decay rate

Now we arrive to the main result of this paper. It follows from Proposition 18.3 that $\Gamma_0 = 1/4$ (*i.e.* $K = 0$). The lower bound $\Gamma_0 \geq 1/4$ alternatively follows from (18.96) using (18.78). The following theorem states that the decay rate is three times better in the presence of a Hardy-type inequality (18.35).

Theorem 18.5. *Assume* (18.14) *and* (18.30). *If* (18.35) *holds, then*

$$\Gamma_K = 3/4\,.$$

Proof. The assertion $\Gamma_K \geq 3/4$ follows from (18.96) using Corollary 18.4. In order to prove the $\Gamma_K \leq 3/4$ it is sufficient to show, that for some suitable function $\varphi \in C_0^\infty(\Omega_0)$, some constant $c_\varphi > 0$ and some constant $t_0 \geq 0$

$$\forall t \geq t_0, \quad \left\| e^{-tH_K}\varphi \right\|_{L^2_f(\Omega_0)} \geq c_\varphi\, t^{-3/4} e^{E_1 t}. \tag{18.97}$$

Due to (18.30) the support of f is contained in a rectangle $\Omega_R := (-R, R) \times (-a, a)$ for $R > 0$. We choose $\varphi \in C_0^\infty(\Omega_0)$ such that $\mathrm{supp}(\varphi) \subset \Omega_0 \setminus \Omega_R$. Recall that $\mathbb{E}_x$ (respectively, $\mathbb{P}_x$) denote the expectation (respectively, probability measure) corresponding to the Markov process $(X_t)_{t\geq 0}$ associated to the Dirichlet form h_k. Define the stopping times τ_{Ω_0} and τ_{Ω_R} by

$$\tau_{\Omega_0} = \inf\{t \geq 0 \mid X_t \in \Omega_0\} \quad \text{and} \quad \tau_{\Omega_R} = \inf\{t \geq 0 \mid X_t \in \partial\Omega_R\}.$$

The process $(X_t)_{0\leq t<\tau_{\Omega_0}}$ is called Brownian motion on Ω_0 killed at the boundary. For every $x = (x_1, x_2) \in Q_0$ we then conclude

$$e^{-tH_K}\varphi(x) = \mathbb{E}_x\big[\varphi(X_t), ; \tau_{\Omega_0}\big] \geq \mathbb{E}_x\big[\varphi(X_t), \tau_{\Omega_R} \wedge \tau_{\Omega_0} > t\big], \tag{18.98}$$

where $\tau_{\Omega_R} \wedge \tau_{\Omega_0}$ denotes the minimum of the stopping times τ_{Ω_0} and τ_{Ω_R}. Integration of (18.98) and using $K \restriction \Omega_0 \setminus \Omega_R = 0$ and hence – by Lemma 18.1 – $f = 1$ in $\Omega_0 \setminus \Omega_R$ yields

$$\begin{aligned} \left\| e^{-tH_K} \varphi \right\|^2_{L^2_f(\Omega_0)} &\geq \int_{(R,\infty)\times(-a,a)} \left| \mathbb{E}_x \big[\varphi(X_t), \tau_{\Omega_R} \wedge \tau_{\Omega_0} > t \big] \right|^2 dx \\ &= \int_{(R,\infty)\times(-a,a)} \left| \mathbb{E}_x \big[\varphi(X_t), \tau_{\Omega_{0R}} > t \big] \right|^2 dx \end{aligned} \tag{18.99}$$

where $\Omega_{0R} := (R, \infty) \times (-a, a)$ and

$$\tau_{\Omega_{0R}} = \inf\{t \geq 0 \mid X_t \in \partial\Omega_{0R}\}.$$

Due to $f = 1$ in $\Omega_0 \setminus \Omega_R$ the stochastic process $(X_t)_{\tau_{\Omega_{0R}} > t \geq 0}$ is a (deterministically time changed by the factor 2) Brownian motion killed, when exiting the set Ω_{0R}. Due to independence of the coordinates we have

$$\mathbb{E}_x \big[\varphi(X_t), \tau_{\Omega_R} \wedge \tau_{\Omega_0} > t \big] = \sum_{n=1}^{\infty} e^{-E_n t} \mathcal{J}_n(x_1) \int_{\Omega_{0,R}} p_0(t, x_1, y_1) \mathcal{J}_n(2) \varphi(y_1, y_2)\, dy, \tag{18.100}$$

where

$$p_0(t, x, y) := \frac{1}{\sqrt{4\pi t}} \big(e^{-\frac{(x-y)^2}{4t}} - e^{\frac{(x+y)^2}{4t}} \big)$$

is the transition function of a one-dimensional Brownian motion killed when hitting 0. Using (18.99) and (18.100) an elementary calculation gives assertion (18.97). □

Observe that the proof of Theorem 18.5 demonstrates that the 'transient' effect of negative curvature on the survival properties of a Brownian particle is as strong as if we kill a particle when entering the curved region.

18.7.12 From normwise to pointwise bounds

Theorem 18.5 may be reformulated in terms of certain pointwise assertions.

Corollary 18.5. *Assume* (18.14), (18.30) *as well as the* (18.35). *Let* $x \in \Omega_0$, $\delta > 0$ *and a measurable bounded subset* $B \subset \Omega_0$ *be given. Then there exists a constant* $C_{B,\delta,x} > 0$ *such that*

$$\mathbb{P}_x \big(X_t \in B, \tau_{\Omega_0} > t \big) \leq C_{B,\delta,x}\, e^{-E_1 t}\, t^{-\frac{3}{2}+\delta}\,.$$

Proof. We use that according to Proposition 18.1 the integral kernel $e^{-tH_K}(x, y)$ of e^{-tH_K} satisfies the following Gaussian upper bound

$$e^{-tH_K}(x, y) \leq \frac{c_1}{\sqrt{4\pi t}}\, e^{-\frac{(x-y)^2}{4c_2 t}}$$

for some constants $c_1, c_2 > 0$. For fixed x set

$$\psi_{x,\varepsilon}(y) = \frac{c_1}{\sqrt{4\pi\varepsilon}}\, e^{-\frac{(x-y)^2}{4c_2\varepsilon}},$$

where ε is chosen small enough such that $\psi_{x,\varepsilon} \in L^2_{wf}(\Omega_0)$. Then for $t > \varepsilon$ we have for some constant $C_\delta > 0$

$$\begin{aligned} \mathbb{P}_x \big(X_t \in B, \tau_{\Omega_0} > t \big) &= e^{-tH_K} \chi_B(x) = e^{-\varepsilon H_K} e^{-(t-\varepsilon)H_K} \chi_B(x) \\ &= \big(\psi_{x,\varepsilon}, e^{-(t-\varepsilon)H_K} \chi_B \big)_f \\ &= \big(e^{-\frac{t-\varepsilon}{2} H_K} \psi_{x,\varepsilon}, e^{-\frac{t-\varepsilon}{2} H_K} \chi_B \big)_f \\ &\leq \|\psi_{x,\varepsilon}\|_{wf}\, \|\chi_B\|_{wf} \left[C_\delta \left(\frac{t-\varepsilon}{2} \right)^{-(\frac{3}{4}-\delta/2)} e^{-E_1 \frac{t-\varepsilon}{2}} \right]^2, \end{aligned}$$

where the last inequality follows using the Cauchy-Schwarz inequality and Theorem 18.5 have been used. This implies the assertion of the Corollary. □

Remark 18.9. In the case of positively curved manifolds satisfying hypotheses (18.14) and (18.30), the decay rate of $\mathbb{P}_x \big(X_t \in B, \tau_{\Omega_0} > t \big)$ is exactly exponential, whereas in the situation of a flat manifold on has $t^{-1/2} e^{-E_1 t}$.

In terms of Tweedie's R-theory (see [45] and [46]) one can therefore conclude that a Brownian particle in a positively curved tube satisfying condition (18.14) and (18.29) is E_1-positive recurrent, in a flat manifold the Brownian particle is E_1-null recurrent and in the negatively curved tube satisfying (18.14) and (18.30) the Brownian motion is E_1-transient.

Let us finally reformulate our findings in the negatively curved case in another way using conditional probabilities, again.

Corollary 18.6. *Assume* (18.14) *and* (18.30). *Let* $x \in \Omega_0$, $\delta > 0$ *and a measurable bounded subset* $B \subset \Omega_0$ *be given. Then there exists a constant* $\tilde{C}_{B,\delta,x} > 0$ *such that*

$$\mathbb{P}_x\big(X_t \in B \mid \tau_{\Omega_0} > t\big) \leq \tilde{C}_{b,\delta,x} t^{-1+\delta}$$

Proof. This follows directly from Corollary 18.5 together with the fact that for a suitable constant c_x

$$\mathbb{P}_x\big(\tau_{\Omega_0} > t\big) \geq c_x, .e^{-\lambda_K t}\, t^{-\frac{1}{2}}.$$

The latter assertion can be proved by adding a Dirichlet boundary as was done in the proof of Theorem 18.5. □

18.8 Conclusions

The objective of this paper was to investigate the interplay between the curvature and the properties of Brownian motion in the simplest non-trivial case, when the ambient space is two-dimensional and the motion in fact quasi-one-dimensional. More precisely, we were interested in the large time behaviour of the solution to the heat equation in tubular neighbourhoods of unbounded geodesics in a two-dimensional Riemannian manifold, subject to Dirichlet boundary conditions.

Our results are schematically summarized in Table 18.1. The corresponding precise statements can be found in: Propositions 18.3 and 18.4 for flat manifolds; Corollary 18.2 for positively curved manifolds; and Theorem 18.5 for negatively curved manifolds. The moral of the story is that the negative curvature is 'better for travelling', in the sense that the heat semigroup gains an extra polynomial, geometrically induced decay rate. The latter is in fact a consequence of the existence of Hardy-type inequalities in negatively curved manifolds, which play a central role in our proof. Though the proofs are mainly analytic some effort has been made in order to connect our findings with notions and results available in the probabilistic literature, *e.g.* on Markov chains.

The present paper can be considered as a contribution to recent works on the consequences of the existence of Hardy inequality on large-time behaviour of the heat semigroup for quantum waveguides [28, 29, 14, 21] and magnetic Schrödinger operators [22, 26]. More generally, recall that we expect that there is always an improvement of the decay rate for the heat semigroup of an operator satisfying a Hardy-type inequality (*cf* [28, Conjecture in Sec. 6] and [9, Conjecture 1]). The present paper confirms the general conjecture in the particular case of the Dirichlet Laplace-Beltrami operator in the strip-like surfaces. As pointed out in the body of the paper, the Hardy inequality is essentially equivalent to transience properties. Thus it is reasonable to expect that a combination of available probabilistic and analytic methods might be necessary in order to make progress towards a solution of the above mentioned conjectures.

Open Problem: One of the characteristic hypotheses of the present paper was that the curvature K has compact support. We expect the same decay rates if this assumption is replaced by its fast decay at infinity. However, it is quite possible that a slow decay of curvature at infinity will improve the decay of the heat semigroup even further. In particular, can Γ_K be strictly greater than $3/4$ if K decays to zero very slowly at infinity?

Acknowledgement

M.K. would like to thank Laurent Saloff-Coste for clarifying remarks concerning his work [37] and the continuity of transition kernels. This work has been partially supported by the Czech Ministry of Education, Youth, and Sports within the project LC06002 and by the GACR grant No. P203/11/0701.

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