Birman-Schwinger Analysis for Bound States in a Pair of Parallel Quantum Waveguides with a Semitransparent Boundary

David Krejčiřík

Faculty of Mathematics and Physics, Charles University,
V Holešovičkách 2,180000 Prague,
and
Nuclear Physics Institute, Academy of Sciences, Řež

A recent paper [4] presented a detailed analysis of a waveguide consisting of a pair of parallel straight Dirichlet strips in the plane coupled laterally through a window in the common boundary. Here we want introduce a waveguide with a similar geometry. We consider the two infinite strips but they are now separated mutually by an arbitrary semitransparent barrier. We suppose that the barrier is supported by the common line only and that the transparentness is modulated by a function $\alpha = \alpha(x)$ (see Figure 1 for specification of the coordinate system). Under these presumptions we can identify the hamiltonian H_α of such system with the formal expression $-\Delta + \alpha(x)\delta(y)$.

The motivation for this problem is the following. While physically closely related (the waveguide with a window can be regarded as a limit case for $\alpha = \infty$ everywhere except in the window's interval), the two systems differ mathematically. In our case of a semitransparent boundary, the perturbed and unperturbed Hamiltonians have the same form domain. This makes it possible to employ other mathematical methods; in the first place the Birman-Schwinger method to analyze the discrete spectrum.

By means of the technique we will find the perturbative expansion (13) for the ground-state energy when the perturbation of the asymptotic value α_0 is small in the *supremum* norm. Such bound state exists if and only if the effective coupling function $\alpha - \alpha_0$ is not repulsive in the mean (see Theorem 4.2).

However, we could find an analogous expansion in the case that the perturbation is small in the L^1-norm too. This becomes interesting if we want to investigate the threshold behaviour of the ground-state energy for large α_0 because we should approach the results of the article studying the waveguide with a window (*cf.* Sec. VI.1 of [4]).

1 Introduction

1.1 The Hamiltonian

In order to introduce the Hamiltonian describing the singular interaction in a mathematically rigorous method, we define it by means of the boundary conditions. Denote $\mathcal{O} := \mathcal{O}_2 \cup \mathcal{O}_1 = (-d_2, 0) \cup (0, d_1)$, $\Omega := \mathbb{R} \times \mathcal{O}$ and put $\hbar^2/2m = 1$. Then

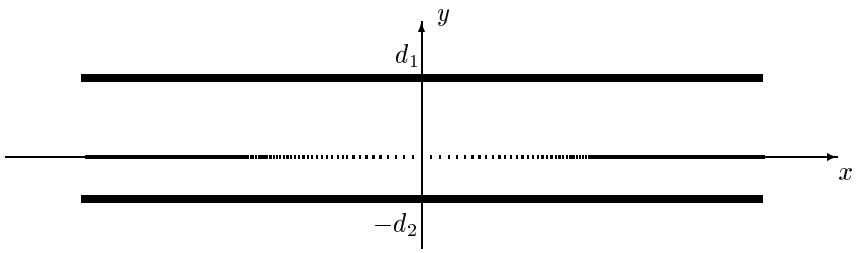

Figure 1: Double waveguide with a δ-barrier

we have $H_\alpha := -\Delta$ with the domain

$$D(H_\alpha) := \left\{ \psi \in W_2^2(\Omega) |\ \forall x \in \mathbb{R}: \begin{array}{l} \psi(x,-d_2) = \psi(x,d_1) = 0 \\ \psi(x,0+) = \psi(x,0-) =: \psi(x,0) \\ \psi_y(x,0+) - \psi_y(x,0-) = \alpha(x)\psi(x,0) \end{array} \right\}. \quad (1)$$

The associated quadratic form is obtained by a simple integration of the Schrödinger equation by parts:

$$\begin{aligned} q_\alpha[\psi] &:= \int_{\mathbb{R}\times\bar{\mathcal{O}}} |\nabla\psi|^2(x,y)dxdy + \int_{\mathbb{R}} \alpha(x)|\psi(x,0)|^2 dx \quad (2) \\ D(q_\alpha) &:= \{\psi \in W_2^1\ (\mathbb{R}\times(-d_2,d_1)) \mid \forall x \in \mathbb{R}: \psi(x,-d_2) = \psi(x,d_1) = 0\}. \end{aligned}$$

It is obviously symmetric; one can check that it is closed and thus really associated with H_α.

1.2 The Unperturbed System

If $\alpha(x) = \alpha$ is a constant function, we can solve the Schrödinger equation by separation of variables. In this situation we can find the spectral condition $-\alpha = k(\cot kd_1 + \cot kd_2)$, which gives arise to the increasing sequence of eigenvalues of the transverse operator; we denote it by $\{\nu_n(\alpha)\}_{n=1}^\infty$. The corresponding eigenfunctions will be:

$$\chi_n(y;\alpha) = \begin{cases} N_n \sin\sqrt{\nu_n}d_1 \sin\sqrt{\nu_n}(y+d_2) & \text{if} \quad y \in \mathcal{O}_2 \\ -N_n \sin\sqrt{\nu_n}d_2 \sin\sqrt{\nu_n}(y-d_1) & \text{if} \quad y \in \mathcal{O}_1. \end{cases} \quad (3)$$

N_n is the appropriate normalization factor chosen in such a way that χ_n would be a unit vector in $L^2(-d_2,d_1)$. Furthermore, the Green's function of the Hamiltonian can be written down explicitly:

$$G_\alpha(x,y,x',y';k) = \sum_{n=1}^\infty \frac{i}{2k_n} e^{ik_n|x-x'|}\ \chi_n(y;\alpha)\,\bar{\chi}_n(y';\alpha)\,, \quad (4)$$

where $k_n := \sqrt{k^2 - \nu_n(\alpha)}$.

2 The Existence of a Bound State

One of the most interesting situations is when $\lim_{|x|\to\infty} \alpha(x) = \alpha_0$, so the function α approaches an unperturbed value α_0 at large x-distances. Then by a simple bracketing argument (as in [4], Sec. II) employing the *minimax principle* we can localize the essential spectrum, $\sigma_{ess}(H_\alpha) = [\nu_1(\alpha_0), \infty)$.

In [4] it is proved that an arbitrary "window" leads to the existence of a bound state. For our case one can show that it is sufficient to make the effective coupling function smaller in the mean in order to create a locally stronger tunelling between the both channels.

This general existence result is given by the following theorem. Its variational proof is based on the idea of [5], Sec. II. (see also Thm.2.1. of [3] and [4], Sec. III).

Theorem 2.1 *Assume*

(i) $\alpha - \alpha_0 \in L^1_{loc}(\mathbb{R})$

(ii) $\alpha(x) - \alpha_0 = \mathcal{O}(|x|^{-1-\varepsilon})$ *for some* $\varepsilon > 0$ *as* $|x| \to \infty$.

If $\int_{\mathbb{R}} (\alpha(x) - \alpha_0) dx < 0$, *then* H_α *has at least one isolated eigenvalue below its essential spectrum.*

3 The Resolvent

To learn more about the discrete spectrum we would like to know the resolvent of our Hamiltonian. In [2] the resolvent of a Hamiltonian with a singular interaction was found on condition that the configuration space is all Euclidean space. The procedure can be modified for our case when the configuration space is the Dirichlet strip and the δ-interaction is supported by a line.

As a result we get an analogous resolvent formula

$$R^k(\alpha) = R^k(\alpha_0) - R^k_{m,dx}(\alpha_0) \left[I + (\alpha - \alpha_0) R^k_{m,m}(\alpha_0) \right]^{-1} (\alpha - \alpha_0) R^k_{dx,m}(\alpha_0) \quad (5)$$

for any $k \in \mathbb{C}^+ := \{k \in \mathbb{C} \mid \operatorname{Im} k > 0\}$. $R^k_{\mu,\nu}(\alpha_0)$ is the integral operator from $L^2(\Omega, d\mu)$ to $L^2(\Omega, d\nu)$ with the kernel $G_{\alpha_0}(\cdot, \cdot; k)$. In particular, we have

$$(R^k_{\mu,\nu}(\alpha_0)\, \psi)(x) = \int_\Omega G_{\alpha_0}(x, y; k) \psi(y) d\mu(y) \quad (6)$$

for all $\psi \in D(R^k_{\mu,\nu}) \subset L^2(\Omega, d\mu)$. (There is no ν in the definition; we compute the right-hand-side and interpret it as values of a function in $L^2(\Omega, d\nu)$.) μ, ν are positive Radon measures without a discrete component; especially, dx is the standard Lebesgue measure and m denotes the δ-measure (formally, $dm(x, y) = dx\, \delta(y) dy$).

The formula (5) can be used as a basis of the Birman-Schwinger analysis in analogy with [2]. However, one can derive the symmetrized expression:

$$R^k(\alpha) = R^k(\alpha_0) - R^k_{m,dx}(\alpha_0)\,(\alpha-\alpha_0)^{\frac{1}{2}}\cdot \cdot\left[I + |\alpha-\alpha_0|^{\frac{1}{2}}\,R^k_{m,m}(\alpha_0)\,(\alpha-\alpha_0)^{\frac{1}{2}}\right]^{-1} |\alpha-\alpha_0|^{\frac{1}{2}} R^k_{dx,m}(\alpha_0)\,, \quad (7)$$

which gives better results. We have introduced here the common notation $(\alpha-\alpha_0)^{\frac{1}{2}} := |\alpha-\alpha_0|^{\frac{1}{2}}\,\mathrm{sgn}(\alpha-\alpha_0)$.

For the integral operator $K^k_\alpha := |\alpha-\alpha_0|^{\frac{1}{2}}\,R^k_{m,m}(\alpha_0)\,(\alpha-\alpha_0)^{\frac{1}{2}}$ we can derive the following assertion which is the basis of Birman-Schwinger principle.

Proposition 3.1 $\forall k \in \mathbb{C}^+ :\ \dim \mathrm{Ker}(H_{\alpha m} - k^2) = \dim \mathrm{Ker}\left(I + K^k_\alpha\right)$.

In particular, these results imply the crucial equivalence stated in the following corollary.

Corollary 3.1 $\forall k \in \mathbb{C}^+ :\ k^2 \in \sigma_{disc}(H_\alpha) \Leftrightarrow K^k_\alpha \psi = -\psi$

It tells us that to solve the partial differential equation is equivalent to find out when the integral operator K^k_α has eigenvalue -1. Notice that its kernel is analogous in dependency on the coupling function to the one in classical Birman-Schwinger principle (*cf.* (1) in [6]).

4 The Adiabatic Limit

These preliminaries we can use for investigation of bound states in the case of the weakly coupled Hamiltonian. In this situation α differs from α_0 slightly and it means that we can replace $\alpha-\alpha_0$ by $\lambda(\alpha-\alpha_0)$ and consider small λ.

Similarly as in [6], [1] and [3], Thm. 4.2., we find necessary and sufficient conditions for H_α to have a (unique) bound state for all sufficiently small positive λ. In addition we shall derive expansion of the bound-state energy at $\lambda = 0$.

Henceforth we assume that

(a1) $\alpha(x)-\alpha_0 \in L^{1+\varepsilon}(\mathbb{R}, dx)$ for some small $\varepsilon > 0$,

(a2) $\alpha(x)-\alpha_0 \in L^1(\mathbb{R}, |x|\,dx)$.

As a consequence of *(a1), (a2)*; the function $\alpha(x)-\alpha_0$ belongs to $L^1(\mathbb{R}, dx)$ too.

4.1 Preliminaries

The Green's function G_{α_0} in (6) is given by (4). In the following we consider k^2 as an eigenvalue of the weakly coupled Hamiltonian given by the replacement $\alpha-\alpha_0 \mapsto \lambda(\alpha-\alpha_0)$, and look for the function $\lambda \mapsto k^2$. Since we are interested in

the discrete spectrum, we must take the Green's function for $k^2 < \nu_1(\alpha_0)$. Thus we have

$$K_\alpha(x,x';k) = |\alpha(x)-\alpha_0|^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{|\chi_n(0;\alpha_0)|^2}{2\kappa_n} e^{-\kappa_n|x-x'|} \, (\alpha(x')-\alpha_0)^{\frac{1}{2}}, \qquad (8)$$

where $\kappa_n := \sqrt{\nu_n(\alpha_0) - k^2}$.

Next we note the elementary fact that $\lim_{\lambda\to 0} k^2 = \nu_1$ (*i.e.* $\kappa_1 \to 0$ as $\lambda \to 0$).

The key idea of the further process is that K^k_α has a well-behaved limit as $k^2 \to \nu_1$ except for a divergent rank-one piece. The singularity is contained in the first term of the expansion (8) and can be singled out by taking (as in [1], [3]) $K^k_\alpha = Q_\alpha + P_\alpha = Q_\alpha + A_\alpha + N_\alpha$ where

$$\begin{aligned}
Q_\alpha(x,x') &= |\alpha(x)-\alpha_0|^{\frac{1}{2}} e^{-\kappa_1|x|} \, \frac{|\chi_1(0)|^2}{2\kappa_1} e^{-\kappa_1|x'|} \, (\alpha(x')-\alpha_0)^{\frac{1}{2}} \\
A_\alpha(x,x') &= |\alpha(x)-\alpha_0|^{\frac{1}{2}} \, \frac{|\chi_1(0)|^2}{\kappa_1} e^{-\kappa_1|x|_>} \sinh \kappa_1 |x|_< \, (\alpha(x')-\alpha_0)^{\frac{1}{2}} \qquad (9) \\
N_\alpha(x,x') &= |\alpha(x)-\alpha_0|^{\frac{1}{2}} \sum_{n=2}^{\infty} \frac{|\chi_n(0)|^2}{2\kappa_n} e^{-\kappa_n|x-x'|} (\alpha(x')-\alpha_0)^{\frac{1}{2}} .
\end{aligned}$$

Here we have introduced $|x|_< := \max\{0, \min\{|x|,|x'|\}\operatorname{sgn}(xx')\}$ and $|x|_> := \max\{|x|,|x'|\}$.

Next we define

$$\begin{aligned}
A_0(x,x') &:= |\alpha(x)-\alpha_0|^{\frac{1}{2}} \, |\chi_1(0)|^2 \, |x|_< \, (\alpha(x')-\alpha_0)^{\frac{1}{2}} \\
N_0^\beta(x,x') &:= |\alpha(x)-\alpha_0|^{\frac{1}{2}} \sum_{n=2}^{\infty} \frac{|\chi_n(0)|^2}{2} \, \frac{e^{-\beta\sqrt{\nu_n-\nu_1}|x-x'|}}{\beta\sqrt{\nu_n-\nu_1}} \, (\alpha(x')-\alpha_0)^{\frac{1}{2}}
\end{aligned}$$

($\beta > 0$), then one has

Proposition 4.1 *Assume that the assumptions (a1), (a2) are valid, then*

$$\lim_{\kappa_1\to 0} \|A_\alpha - A_0\|_{HS} = 0 \qquad \text{and} \qquad \lim_{\kappa_1\to 0} \|N_\alpha - N_0^1\|_{HS} = 0 . \qquad (10)$$

PROOF: **(a)** A_0 is Hilbert-Schmidt since

$$\begin{aligned}
\|A_0\|^2_{HS} &= \int_{\mathbb{R}^2} |A_0(x,x')|^2 \, dx \, dx' \\
&= |\chi_1(0)|^4 \int_{\mathbb{R}^2} |\alpha(x)-\alpha_0| \, |x|^2_< \, |\alpha(x')-\alpha_0| \, dx \, dx' \\
&\le |\chi_1(0)|^4 \left(\int_{\mathbb{R}} |x| \, |\alpha(x)-\alpha_0| \, dx \right)^2 < \infty
\end{aligned}$$

by assumption. Next we have $\lim_{\kappa_1 \to 0} A_\alpha = A_0$ and $|A_\alpha(x,x';\kappa_1)| \le |A_0(x,x')|$. This ensures that the assumptions of the dominated convergence theorem are valid, and thus we immediately get the first claim of the proposition.

(b) N_0^1 has a logarithmic singularity as $x' \to x$. Nevertheless, its Hilbert-Schmidt norm is still finite because by Hölder inequality we have

$$\begin{aligned}
\|N_0^1\|_{HS}^2 = & \\
= & \sum_{m,n=2}^{\infty} \frac{|\chi_n(0)\,\chi_m(0)|^2}{4\sqrt{\nu_n-\nu_1}\sqrt{\nu_m-\nu_1}} \int_{\mathbb{R}^2} |\alpha(x)-\alpha_0|\; e^{-\frac{1}{2}(\sqrt{\nu_n-\nu_1}+\sqrt{\nu_m-\nu_1})|x-x'|}\cdot \\
& \qquad \cdot\, |\alpha(x')-\alpha_0|\; e^{-\frac{1}{2}(\sqrt{\nu_n-\nu_1}+\sqrt{\nu_m-\nu_1})|x-x'|}\, dx\, dx' \\
\le & \; C(\varepsilon)\, \|\alpha-\alpha_0\|_{1+\varepsilon}^2 \sum_{m,n=2}^{\infty} \frac{1}{\sqrt{\nu_n-\nu_1}\sqrt{\nu_m-\nu_1}\,(\sqrt{\nu_n-\nu_1}+\sqrt{\nu_m-\nu_1})^{2\varepsilon'}}
\end{aligned}$$

ε' is the abbreviation for $\frac{\varepsilon}{1+\varepsilon} > 0$. The monotone convergence theorem justifies the interchange of summation and integration we have made. The last sum now converges because the sequence $\{\sqrt{\nu_n}\}$ grows linearly with n as $n \to \infty$. The assumptions of the dominated convergence theorem are fulfilled again because $\lim_{\kappa_1 \to 0} N_\alpha = N_0^1$ and $|N_\alpha(x,x';\kappa_1)| \le |N_0^1(x,x')|$. ∎

We will also need the boundedness of A_α, N_α for complex $z := \kappa_1$ but it is facile to prove

Lemma 4.1 *1.* $\forall z \in \mathbb{C}, \operatorname{Re} z \ge 0, |z| < \frac{\pi}{2} :\ |A_\alpha(z)| \le \sqrt{2}\,|A_0|$

2. $\exists C, \beta > 0 \quad \forall z \in \mathbb{C}, \operatorname{Re} z \ge 0, |z| < \sqrt{\nu_n - \nu_1} :\ |N_\alpha(z)| \le C|N_0^\beta|$.

4.2 The Main Theorem

In this section we want to prove the following technical result.

Theorem 4.1 *Assume that the hypotheses (a1), (a2) are valid. Then, for λ small, H_α has at most one simple eigenvalue $E(\lambda) < \nu_1$, and this happens if and only if*

$$\sqrt{\nu_1 - E} \equiv \kappa_1 = -\frac{\lambda}{2}\,|\chi_1(0)|^2 \left(e^{-\kappa_1|\cdot|}\,(\alpha-\alpha_0)^{\frac{1}{2}}, (I+\lambda P_\alpha)^{-1}\, e^{-\kappa_1|\cdot|}\,|\alpha-\alpha_0|^{\frac{1}{2}} \right) \tag{11}$$

has a solution $\kappa_1 > 0$.

PROOF: From the proof of Proposition 4.1 it is clear that $\|P_\alpha\| \le \|A_0\|_{HS} + \|N_0\|_{HS} < \infty$, and thus for λ sufficiently small $\|\lambda P_\alpha\| < 1$, so $I + \lambda P_\alpha$ is invertible and we may write

$$(I+\lambda K_\alpha^k)^{-1} = \left[I + (I+\lambda P_\alpha)^{-1}\lambda Q_\alpha\right]^{-1} (I+\lambda P_\alpha)^{-1}.$$

Hence λK_α has eigenvalue -1 if and only if the same is true for $(I+\lambda P_\alpha)^{-1}\lambda Q_\alpha$. Since Q_α is a rank-one operator (see (9)), $(I+\lambda P_\alpha)^{-1}\lambda Q_\alpha = (\psi, \cdot)\,\varphi$ with $\psi :=$

$\lambda \frac{|\chi_1(0)|^2}{2\kappa_1} e^{-\kappa_1|\cdot|} (\alpha(\cdot)-\alpha_0)^{\frac{1}{2}}$ and $\varphi := (I+\lambda P_\alpha)^{-1} e^{-\kappa_1|\cdot|} |\alpha(\cdot)-\alpha_0|^{\frac{1}{2}}$; it has just one eigenvalue (ψ,φ). Putting it equal to -1 we get the condition (11).

This proves the entire theorem except for the assertion that (11) has *at most* one positive solution for λ small and fixed. Since there is a one-to-one correspondence between eigenvalues of H_α and solutions of (11) and the number of eigenvalue can only go up with replacement $\alpha-\alpha_0 \mapsto -|\alpha-\alpha_0|$, we need only show that (11) has at most one solution when $\alpha-\alpha_0 \leq 0$. In this case, (11) is equivalent to $z = G(z,\lambda)$ where

$$G(z,\lambda) := \frac{\lambda}{2} |\chi_1(0)|^2 \left(e^{-z|\cdot|} |\alpha-\alpha_0|^{\frac{1}{2}}, (I+\lambda P_\alpha)^{-1} e^{-z|\cdot|} |\alpha-\alpha_0|^{\frac{1}{2}} \right). \qquad (12)$$

To complete the proof, we need the following lemmas.

4.2.1 Auxiliary Results

Lemma 4.2 *If $\alpha-\alpha_0 \not\equiv 0$, then $|z|^{-1} \leq C_1\lambda^{-1}$ for λ small.*

PROOF: From (12), we see that any solution of $z = G(z,\lambda)$ for λ small must obey $z = \frac{\lambda}{2}|\chi_1(0)|^2 \int_{\mathbb{R}} |\alpha(x)-\alpha_0|\,dx + \mathcal{O}(\lambda^2)$, what gives us the assertion provided $\alpha-\alpha_0$ is not identically zero. ∎

Lemma 4.3 *For sufficiently small z, $\left\| \frac{\partial P_\alpha}{\partial z} \right\| < \frac{C_2}{|z|}$.*

PROOF: Let us choose a contour in complex plane, $\varphi : s = z + ze^{it}$; $t \in [0, 2\pi]$. The operator-valued fuction $P_\alpha(z)$ is real-analytic in the region $\mathrm{Re}\, z > 0$ and has a bounded limit as $z \to 0+$. Hence Cauchy integral formula together with Lemma 4.1 gives

$$\left| \frac{\partial P_\alpha}{\partial z} \right| = \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{P_\alpha(s)\, i\, z\, e^{it} dt}{z^2\, e^{2it}} \right| \leq \frac{\sqrt{2}|A_0| + C|N_0^\beta|}{|z|},$$

but A_0, N_0^β have finite HS norms. ∎

Lemma 4.4 $\forall z_0 \in \mathbb{R} \quad \exists C_3 > 0 \quad \forall z \in [0, z_0]$:

$$\left| \left(e^{-z|\cdot|} |\alpha-\alpha_0|^{\frac{1}{2}}, \left[2\,|\cdot| + \frac{\partial P_\alpha}{\partial z} \right] e^{-z|\cdot|} |\alpha-\alpha_0|^{\frac{1}{2}} \right) \right| \leq C_3$$

PROOF: An explicit calculation shows that the partial derivative is finite and remains bounded as $z \to 0+$. The whole scalar product is then finite because of the assumptions *(a1), (a2)*. ∎

Lemma 4.5

$$\exists C_4 > 0 \ : \qquad \left| \left(e^{-z|\cdot|} |\alpha-\alpha_0|^{\frac{1}{2}}, \left[|\cdot| P_\alpha + P_\alpha |\cdot| \right] e^{-z|\cdot|} |\alpha-\alpha_0|^{\frac{1}{2}} \right) \right| \leq \frac{C_4}{|z|}$$

PROOF: Since $\forall x > 0 : x\, e^{-x} \leq e^{-1}$, using Schwarz inequality we infer that the expression is bounded by $\frac{e^{-1}}{|z|} \|\alpha-\alpha_0\|_1 \|P_\alpha\|_{HS}$. ∎

4.2.2 Completion of the Proof

By the elementary inequality $(I+\lambda P_\alpha)^{-1} \leq \frac{1-\lambda P_\alpha}{1-\lambda} \leq d\,(1-\lambda P_\alpha)$ held for λ small enough and by the preceding lemmas we have for $|z^{-1}| \leq C_1\lambda^{-1}$ and all sufficiently small λ:

$$\left|\frac{\partial G}{\partial z}\right| \leq C\lambda\,.$$

Let z_1, z_2 are two solutions of $z = G(z,\lambda)$. Because they have to fulfil

$$|z_1 - z_2| = \left|\int_{z_1}^{z_2} \frac{\partial G}{\partial z} dz\right| \leq \left|\int_{z_1}^{z_2} \left|\frac{\partial G}{\partial z}\right| dz\right| \leq C\lambda|z_1 - z_2|,$$

we have the uniqueness ensured for $\lambda < C^{-1}$. ∎

4.3 Condition for a Bound State

Now we introduce necessary and sufficient condition for there to be a bound state when λ is small.

Theorem 4.2 *Let α obeys hypotheses (a1), (a2) and $\alpha - \alpha_0 \not\equiv 0$. Then H_α has an eigenvalue $E(\lambda) < \nu_1$ for all small $\lambda > 0$ if and only if $\int_{\mathbb{R}}(\alpha(x)-\alpha_0)\,dx \leq 0$. If this condition holds, then the eigenvalue is unique and simple and obeys*

$$\begin{aligned}\sqrt{\nu_1 - E(\lambda)} = &-\frac{\lambda}{2}\,|\chi_1(0)|^2 \int_{\mathbb{R}} (\alpha(x)-\alpha_0)\,dx \\ &-\frac{\lambda^2}{4}\Bigg\{|\chi_1(0)|^4 \int_{\mathbb{R}^2} (\alpha(x)-\alpha_0)\,|x-x'|\,(\alpha(x')-\alpha_0)\,dx\,dx' \\ &-|\chi_1(0)|^2 \sum_{n=2}^{\infty} |\chi_n(0)|^2 \int_{\mathbb{R}^2} (\alpha(x)-\alpha_0)\,\frac{e^{-\sqrt{\nu_n-\nu_1}|x-x'|}}{\sqrt{\nu_n-\nu_1}}\,(\alpha(x')-\alpha_0)\,dx\,dx'\Bigg\} \\ &+\mathcal{O}(\lambda^3). \qquad (13)\end{aligned}$$

PROOF: By the implicit-function theorem we see that (11) has a unique solution for λ small and it is given by (13).

It remains to prove that such solution is strictly positive for λ small. It is clearly fulfilled, if $\int(\alpha(x)-\alpha_0)dx < 0$. If this integral is zero, the first term of order λ in the expansion (13) vanishes and we have to prove then that the term of order λ^2 is strictly positive. One can do it by the cunning trick as in the proof of Thm. 2.5 in [6] (see also the end of the proof of Thm. 4.2 in [3]). ∎

The work has been partially supported by the GA AS 1048801.

References

[1] R. Blackenbecler, M. L. Goldberger, and B. Simon, *The bound states of weakly coupled longe-range one-dimensional quantum hamiltonians*, Ann. Phys. **108** (1977), 69–78.

[2] J. F. Brasche, P. Exner, Yu. A. Kuperin, and P. Šeba, *Schrödinger operators with singular interactions*, J. Math. Anal. Appl. **184** (1994), 112–139.

[3] P. Duclos and P. Exner, *Curvature-induced bound states in quantum waveguides in two and three dimensions*, Rev. Math. Phys. **7** (1995), 73–102.

[4] P. Exner, P. Šeba, M. Tater, and D. Vaněk, *Bound states and scattering in quantum waveguides coupled laterally through a boundary window*, J. Math. Phys. **37** (1996), 4867–4887.

[5] J. Goldstone and R. L. Jaffe, *Bound states in twisting tubes*, Phys. Rev. **B 45** (1992), 14100–14107.

[6] B. Simon, *The bound state of weakly coupled Schrödinger operators in one and two dimensions*, Ann. Phys. **97** (1976), 279–288.