Dynamic boundary conditions for divergence form operators with Hölder coefficients

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5-6-2024

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Rough sketch of problem

Let $\Omega \subset \mathbb{R}^d$ be open, bounded and connected. Consider parabolic systems with dynamic (Wentzell) boundary conditions

$$\left[\begin{array}{ll} \dfrac{d}{dt}\, u(t,\cdot) = -Bu(t,\cdot) & \text{on } \Omega, \\[2ex] \dfrac{d}{dt}\, \mathrm{Tr}\, u(t,\cdot) = -\beta\, \partial_\nu u(t,\cdot) - \alpha \mathrm{Tr}\, u(t,\cdot) & \text{on } \partial\Omega, \\[2ex] u(0,\cdot) = u_0 & \text{on } \Omega. \end{array}\right.$$

Here B is a second-order elliptic operator,
$\alpha \in L_\infty(\Omega)$,
$\beta \colon \Omega \to (0,\infty)$ bounded measurable with $\operatorname{ess\,inf} \beta > 0$,
∂_ν is the outward co-normal derivative associated with the operator B.

Rough sketch problem, reformulation

On $\Omega \oplus \partial\Omega$ consider

$$\frac{d}{dt}\begin{pmatrix} u(t,\cdot) \\ \varphi(t,\cdot) \end{pmatrix} = -\mathbb{A}\begin{pmatrix} u(t,\cdot) \\ \varphi(t,\cdot) \end{pmatrix}, \quad \begin{pmatrix} u(0,\cdot) \\ \varphi(0,\cdot) \end{pmatrix} = \begin{pmatrix} u_0 \\ \operatorname{Tr} u_0 \end{pmatrix},$$

where

$$\mathbb{A} = \begin{pmatrix} B & 0 \\ \beta\,\partial_\nu & \alpha \end{pmatrix}$$

and formally $D(\mathbb{A}) \subset \{(v,\varphi) : \operatorname{Tr} v = \varphi\}$.

Previous results

$B = -\sum \partial_k \, c_{kl} \, \partial_l$ formally. $\Gamma = \partial\Omega$.

author(s)		Ω	c_{kl}	β	L_1	L_p	$C(\overline{\Omega})$
Amann–Escher	1996	C^2	$C(\overline{\Omega})$	$\mathbb{1}$	C_0	hol	C_0
Favini et al.	2002	C^2	$a\delta_{kl}$	$C^1(\Gamma)$	C_0	hol	C_0
Arendt et al.	2003	Lipschitz	δ_{kl}	$L_\infty(\Gamma)$	C_0	hol	
Arendt et al.	2003	$C^{2+\kappa}$	δ_{kl}	$C(\Gamma)$			C_0
Engel	2003	C^∞	δ_{kl}				hol $\frac{\pi}{2}$
Engel–Fragnelli	2005	C^∞	C^∞	$\mathbb{1}$			hol
Warma	2010	C^∞	δ_{kl}	$C^1(\Gamma)$	hol $\frac{\pi}{2}$		
Favini et al.	2010	C^∞	C^∞	$C^\infty(\Gamma)$	hol	hol	hol
Binz–Engel	2020	$C^{1,1}$	$W^{1,\infty}$	$\mathbb{1}$			hol $\frac{\pi}{2}$

Conditions

Let $\kappa \in (0,1)$.
Let $\Omega \subset \mathbb{R}^d$ be open, bounded connected of class $C^{1,\kappa}$.
Write $\Gamma = \partial\Omega$ with $(d-1)$-dim Hausdorff measure σ.
For all $k, l \in \{1, \ldots, d\}$ let $c_{kl} \in C^\kappa(\Omega, \mathbb{R})$ and let $c_0 \colon \Omega \to \mathbb{R}$ be a bounded measurable function.
Suppose that $c_{kl} = c_{lk}$ for all $k, l \in \{1, \ldots, d\}$.
Let $\alpha \in L_\infty(\Gamma, \mathbb{C})$.
Let $\beta \colon \Gamma \to (0, \infty)$ be bounded measurable with $\operatorname{ess\,inf} \beta > 0$.
Suppose there exists a $\mu > 0$ such that

$$\operatorname{Re} \sum_{k,l=1}^{d} c_{kl}(x)\, \xi_k\, \overline{\xi_l} \geq \mu\, |\xi|^2$$

for all $x \in \Omega$ and $\xi \in \mathbb{C}^d$.

Notation

Define form $\mathfrak{b}\colon W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{C}$ by

$$\mathfrak{b}(u,v) = \sum_{k,l=1}^{d} \int_\Omega c_{kl}\,(\partial_k u)\,\overline{\partial_l v} + \int_\Omega c_0\, u\, \overline{v}.$$

Further define $\mathcal{B}\colon W^{1,2}(\Omega) \to \mathcal{D}'(\Omega)$ by

$$\langle \mathcal{B}u, \tau\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = \mathfrak{b}(u,\tau).$$

If $u \in W^{1,2}(\Omega)$ and $\psi \in L_2(\Gamma)$, then we say that $u \in D(\partial_\nu^C)$ and $\partial_\nu^C u = \psi$ if $\mathcal{B}u \in L_2(\Omega)$ and

$$\mathfrak{b}(u,v) - \int_\Omega (\mathcal{B}u)\,\overline{v} = \int_\Gamma \psi\, \overline{\operatorname{Tr} v}\, \mathrm{d}\sigma$$

for all $v \in W^{1,2}(\Omega)$ (co-normal derivative).

Define $\mathbb{L}_p = L_p(\Omega) \times L_p(\Gamma)$.

The operator in L_2 ([AmE], [AMPR])

Define the form $\mathfrak{a}\colon W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}(u,v) = \sum_{k,l=1}^{d} \int_\Omega c_{kl}\,(\partial_k u)\,\overline{\partial_l v} + \int_\Omega c_0\, u\,\overline{v} + \int_\Gamma \frac{\alpha}{\beta}\,(\mathrm{Tr}\, u)\,\overline{\mathrm{Tr}\, v}\,\mathrm{d}\sigma.$$

Define $j\colon W^{1,2}(\Omega) \to \mathbb{L}_2$ by

$$j(u) = (u, \mathrm{Tr}\, u).$$

Then $\mathfrak{a}$ is a continuous j-elliptic form and j is continuous and has dense range.
Let $\mathbb{A}$ to be the m-sectorial operator in $\mathbb{L}_2$ associated with $(\mathfrak{a}, j)$.
Then $-\mathbb{A}$ is the generator of a C_0-semigroup which is holomorphic in the right half-plane.

The operator in L_2, part 2

Lemma.

Let $(u,\varphi),(f,\eta) \in \mathbb{L}_2$. The following are equivalent.

- $(u,\varphi) \in D(\mathbb{A})$ and $\mathbb{A}(u,\varphi) = (f,\eta)$.
- $$\left[\begin{array}{l} u \in W^{1,2}(\Omega), \\ \varphi = \operatorname{Tr} u, \text{ and} \\ \mathfrak{a}(u,v) = \int_\Omega f\,\overline{v} + \int_\Gamma \eta\,\overline{\operatorname{Tr} v}\,\frac{d\sigma}{\beta} \quad \text{for all } v \in W^{1,2}(\Omega). \end{array}\right.$$
- $u \in D(\partial_\nu^C)$, $\varphi = \operatorname{Tr} u$, $f = \mathcal{B}u$ and $\eta = \beta\,\partial_\nu^C u + \alpha \operatorname{Tr} u$.

Operator in $W^{1,2}(\Omega)$ (Favini et al. (2003))

Via the operator $j\colon W^{1,2}(\Omega) \to \mathbb{L}_2$ by given by $j(u) = (u, \operatorname{Tr} u)$ transfer the form to $j(W^{1,2}(\Omega))$ and then take the part in $W^{1,2}(\Omega)$.

Proposition.
Define the operator A in the Hilbert space $W^{1,2}(\Omega)$ by

$$D(A) = \{u \in D(\partial_\nu^C) : \mathcal{B}u \in W^{1,2}(\Omega) \text{ and } \beta\, \partial_\nu^C u = \operatorname{Tr} \mathcal{B}u - \alpha \operatorname{Tr} u\}$$

and $Au = \mathcal{B}u$. Then $-A$ generates a holomorphic C_0-semigroup on $W^{1,2}(\Omega)$.

The operator in $\mathbb{L}_p$

Theorem.

Let $p \in [1, \infty)$. The semigroup generated by $-\mathbb{A}$ extends consistently to a C_0-semigroup on $\mathbb{L}_p$ which is holomorphic with angle $\frac{\pi}{2}$.
Moreover, if $p \in (1, \infty)$, then its generator has maximal L_r-regularity on $\mathbb{L}_p$ for all $r \in (1, \infty)$.

Previous results on maximal L_r-regularity on $\mathbb{L}_p$ for non-divergence operators by Denk–Prüss–Zacher (2008) and Goldstein et al. (2020).

The operator on continuous functions

Define

$$X_c = \{(u, \varphi) \in C(\overline{\Omega}) \times C(\Gamma) : u|_\Gamma = \varphi\}.$$

Then X_c is naturally isomorphic with $C(\overline{\Omega})$.
Let $\mathbb{A}_c$ be the part of $\mathbb{A}$ in X_c.

Theorem.
The operator $-\mathbb{A}_c$ is the generator of a C_0-semigroup in X_c which is holomorphic with angle $\frac{\pi}{2}$ and consistent with the semigroup generated by $-\mathbb{A}$.

Recall that α and β are not continuous in general.

Theorem.
Define the operator A_c in $C(\overline{\Omega})$ by

$$D(A_c) = \{u \in C(\overline{\Omega}) \cap D(\partial_\nu^C) : \mathcal{B}u \in C(\overline{\Omega}) \text{ and } (\mathcal{B}u)|_\Gamma = \beta\, \partial_\nu^C u + \alpha\, u|_\Gamma \text{ a.e.}\}$$

and $A_c u = \mathcal{B}u$ for all $u \in D(A_c)$. Then $-A_c$ is the generator of a C_0-semigroup in $C(\overline{\Omega})$ which is holomorphic with angle $\frac{\pi}{2}$.

Similarity, part 1

Let B_2^D be the operator with Dirchlet boundary conditions associated with $\mathfrak{b}|_{W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)}$. Recall

$$\mathfrak{b}(u,v) = \sum_{k,l=1}^{d} \int_\Omega c_{kl}\,(\partial_k u)\,\overline{\partial_l v} + \int_\Omega c_0\, u\, \overline{v}.$$

Assume that $0 \notin \sigma(B_2^D)$.
Let $\gamma\colon H^{1/2}(\Gamma) \to H^1(\Omega)$ be the harmonic lifting, so if $\varphi \in H^{1/2}(\Gamma)$, then $\gamma(\varphi) = u \in W^{1,2}(\Omega)$, where $\mathrm{Tr}\, u = \varphi$ and $\mathcal{B}u = 0$.
The map γ extends uniquely to a continuous map $\gamma_2\colon L_2(\Gamma) \to L_2(\Omega)$.

Similarity, part 2 (Cf. Casarino et al.)

Then

$$\begin{pmatrix} I & -\gamma_2 \\ 0 & I \end{pmatrix} \mathbb{A} \begin{pmatrix} I & \gamma_2 \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} B_2^D & 0 \\ 0 & \beta \mathcal{N} \end{pmatrix} + \begin{pmatrix} \gamma_2 \, M_\beta \, \gamma_{2'}^* \, B_2^D & -\gamma_2 \, M_\beta \, \mathcal{N} - \gamma_2 \, M_\alpha \\ -M_\beta \, \gamma_{2'}^* \, B_2^D & M_\alpha \end{pmatrix},$$

where $\mathcal{N}$ is the Dirichlet-to-Neumann operator, which is the operator on $L_2(\Gamma)$ associated with the form $\mathfrak{b}$ and the trace map.
Specifically, if $\varphi, \psi \in L_2(\Gamma)$, then $\varphi \in D(\mathcal{N})$ and $\mathcal{N}\varphi = \psi$ if there exits a $u \in W^{1,2}(\Omega)$ such that $\mathrm{Tr}\, u = \varphi$, $\mathcal{B}u = 0$ and $\partial_\nu^C u = \psi$.

The operator $\beta\,\mathcal{N}$

Theorem.

The operator $-\beta\mathcal{N}$ is the generator of a C_0-semigroup S which is holomorphic with angle $\frac{\pi}{2}$. Moreover, S has a kernel K and for all $\theta \in (0, \frac{\pi}{2})$ there are $c, \omega > 0$ such that

$$|K_z(w_1, w_2)| \leq \frac{c\,|z|^{-(d-1)}\,e^{\omega|z|}}{\left(1 + \dfrac{|w_1 - w_2|}{|z|}\right)^d}$$

for all $z \in \mathbb{C} \setminus \{0\}$ and $w_1, w_2 \in \Gamma$ with $|\arg z| \leq \theta$.

For all $p \in [1, \infty)$ the semigroup $(e^{-t\beta\mathcal{N}})_{t>0}$ extends consistently to a holomorphic semigroup on $L_p(\Gamma)$ with angle $\frac{\pi}{2}$.

For all $p, r \in (1, \infty)$ the operator $\beta\mathcal{N}_p$ has maximal L_r-regularity on $L_p(\Gamma)$.

Define $T_t = e^{-t\beta\mathcal{N}}|_{C(\Gamma)} \colon C(\Gamma) \to C(\Gamma)$ for all $t > 0$. Then T is a C_0-semigroup which is holomorphic with angle $\frac{\pi}{2}$.

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Favini et al.	2010	C^∞	C^∞	$C^\infty(\Gamma)$	hol	hol	hol
Binz–Engel	2020	$C^{1,1}$	$W^{1,\infty}$	$\mathbb{1}$			hol $\frac{\pi}{2}$
Binz–tE		$C^{1,\kappa}$	C^κ	$L_\infty(\Gamma)$	hol $\frac{\pi}{2}$	hol $\frac{\pi}{2}$	hol $\frac{\pi}{2}$