

# Pluriharmonic solutions to Maxwell's equations and Yang-Mills equations

**Marius Beceanu**

The New York Department of Environmental Conservation

**Sachin Munshi**

Grand Canyon University

**Rongwei Yang\***

University at Albany, the State University of New York

*Mathematical aspects of the physics with non-self-adjoint  
operators*

CIRM, June 3 - 7, 2024

# Coauthors



# Motivation

Maxwell's equations (ME) form a foundation of modern physics. They not only describe electromagnetism but also led to special relativity theory and Yang-Mills theory (YM) which are crucial to the standard model for particle physics. Since complex number has become a necessity for physics, it is worthwhile to take another look at ME and YM from a complex analysis point of view.

1. New perspectives
2. New solutions

# Main topics

- ▶ Maxwell's equations (ME) in complex differential forms
- ▶ Pluriharmonic solutions
- ▶ Yang-Mills equations: a quantization of ME
- ▶ Lagrangian and YM-functional
- ▶ Gauge invariance
- ▶ Pluriharmonic connection forms
- ▶ An example

## Maxwell' equations

In electromagnetism,  $\mathbf{E} = (E_1, E_2, E_3)$ : electric field, and  $\mathbf{B} = (B_1, B_2, B_3)$ : magnetic field. Then

$$\nabla \cdot \mathbf{B} = 0, \quad (1)$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (2)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (3)$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}, \quad (4)$$

Poincare lemma  $\Rightarrow$  there exist magnetic vector potential  $\mathbf{A}$  and the electric scalar potential  $\phi$  such that

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad (5)$$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi. \quad (6)$$

## Differential Form Version

$$\omega = \phi dx_0 - A_1 dx_1 - A_2 dx_2 - A_3 dx_3, \text{ (potential 1 - form)}$$

$$J = \rho dx_0 + J_1 dx_1 + J_2 dx_2 + J_3 dx_3, \text{ (current 1 - form)}$$

where  $x_0 = ct$ . *Faraday field strength* or Faraday 2-form:

$$F = -dx_0 \wedge (E_1 dx_1 + E_2 dx_2 + E_3 dx_3) \\ - B_1 dx_2 \wedge dx_3 + B_2 dx_1 \wedge dx_3 - B_3 dx_1 \wedge dx_2.$$

# Differential Form Version

$$\omega = \phi dx_0 - A_1 dx_1 - A_2 dx_2 - A_3 dx_3, \text{ (potential 1 - form)}$$

$$J = \rho dx_0 + J_1 dx_1 + J_2 dx_2 + J_3 dx_3, \text{ (current 1 - form)}$$

where  $x_0 = ct$ . *Faraday field strength* or Faraday 2-form:

$$F = -dx_0 \wedge (E_1 dx_1 + E_2 dx_2 + E_3 dx_3) \\ - B_1 dx_2 \wedge dx_3 + B_2 dx_1 \wedge dx_3 - B_3 dx_1 \wedge dx_2.$$

The *Hodge star operator*  $\star : \Lambda^p(\mathbb{R}^4) \rightarrow \Lambda^{4-p}(\mathbb{R}^4)$ ,  $0 \leq p \leq 4$  such that for  $\eta, \xi \in \Lambda^p$  one has

$$\eta \wedge \star \bar{\xi} = \langle \eta, \xi \rangle_g dx_0 dx_1 dx_2 dx_3.$$

# Complex Electromagnetic Field

Set  $z_1 = x_0 + ix_1$ ,  $z_2 = x_2 + ix_3$ ,  $\partial_j = \frac{\partial}{\partial z_j}$ ,  $j = 1, 2$ . For complex smooth functions  $f_1, f_2, \bar{f}_1, \bar{f}_2$  in  $z_1, z_2$ . Write the complex differential form

$$\omega = f_1 dz_1 + f_2 dz_2 + \bar{f}_1 d\bar{z}_1 + \bar{f}_2 d\bar{z}_2, \text{ and } F_\omega = d\omega. \quad (7)$$

Then

$$\begin{aligned} F_\omega &= (\partial_1 f_2 - \partial_2 f_1) dz_1 \wedge dz_2 + (\bar{\partial}_1 \bar{f}_2 - \bar{\partial}_2 \bar{f}_1) d\bar{z}_1 \wedge d\bar{z}_2 \\ &+ (\partial_1 \bar{f}_2 - \bar{\partial}_2 f_1) dz_1 \wedge d\bar{z}_2 + (\partial_2 \bar{f}_1 - \bar{\partial}_1 f_2) dz_2 \wedge d\bar{z}_1 \\ &+ (\partial_1 \bar{f}_1 - \bar{\partial}_1 f_1) dz_1 \wedge d\bar{z}_1 + (\partial_2 \bar{f}_2 - \bar{\partial}_2 f_2) dz_2 \wedge d\bar{z}_2. \end{aligned}$$

**Note:** complex electromagnetic components  $\mathbf{E}, \mathbf{B} \in \mathbb{C}^3$  with inner product

$$\langle \mathbf{E}, \mathbf{B} \rangle = E_1 \bar{B}_1 + E_2 \bar{B}_2 + E_3 \bar{B}_3.$$

## Self-dual and Anti-self-dual 2-Forms

- ▶ In Euclidean metric:  $\star^2 = I$  on 2-forms  $\Rightarrow \sigma(\star) = \{\pm 1\}$  with corresponding eigenspaces

$$\Lambda_+^2 = \text{span} \{ dz_1 \wedge dz_2, d\bar{z}_1 \wedge d\bar{z}_2, dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 \},$$

$$\Lambda_-^2 = \text{span} \{ dz_1 \wedge d\bar{z}_2, dz_2 \wedge d\bar{z}_1, dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2 \}.$$

- ▶ In Minkowski metric:  $\star^2 = -I$   $\Rightarrow \sigma(\star) = \{\pm i\}$  with corresponding eigenspaces

$$\Lambda_+^2 = \text{span} \{ dz_1 \wedge dz_2 + idz_2 \wedge d\bar{z}_1, dz_1 \wedge d\bar{z}_1 + idz_2 \wedge d\bar{z}_2, \\ dz_1 \wedge d\bar{z}_2 + id\bar{z}_1 \wedge d\bar{z}_2 \},$$

$$\Lambda_-^2 = \text{span} \{ dz_1 \wedge dz_2 - idz_2 \wedge d\bar{z}_1, dz_1 \wedge d\bar{z}_1 - idz_2 \wedge d\bar{z}_2, \\ dz_1 \wedge d\bar{z}_2 - id\bar{z}_1 \wedge d\bar{z}_2 \}.$$

## Maxwell's Equations: $dF = 0$ , $\star d \star F = J$

**Note:** In vacuum, if  $\omega$  is a 1-form such that  $F := d\omega$  is self-dual (SD) or anti-self-dual (ASD), then  $\omega$  is a solution to ME.

**Observations.** Under the Euclidean metric,  $F = d\omega$  is selfdual if and only if

$$\begin{aligned}\partial_1 f_1 - \bar{\partial}_1 f_1 &= \partial_2 f_2 - \bar{\partial}_2 f_2, \\ \partial_1 f_2 - \bar{\partial}_2 f_1 &= \partial_2 f_1 - \bar{\partial}_1 f_2 = 0;\end{aligned}$$

and it is anti-self-dual if and only if

$$\begin{aligned}\partial_1 f_1 - \bar{\partial}_1 f_1 &= -(\partial_2 f_2 - \bar{\partial}_2 f_2), \\ \partial_1 f_2 - \partial_2 f_1 &= \bar{\partial}_1 f_2 - \bar{\partial}_2 f_1 = 0.\end{aligned}$$

**Note:** Holomorphic functions provide natural SD solutions to ME.

## Pluriharmonic solutions: Euclidean background

Consider  $\omega(z) = f_1 dz_1 + f_2 dz_2 + \bar{f}_1 d\bar{z}_1 + \bar{f}_2 d\bar{z}_2$  and  $F_\omega = d\omega$ .

### Proposition

*Assume  $f_i, \bar{f}_i, i = 1, 2$  are holomorphic. Then  $F_\omega$  is a self-dual solution to Maxwell's equations in vacuum. Furthermore, in this case*

$$F_\omega = (\partial_1 f_2 - \partial_2 f_1) dz_1 \wedge dz_2 + (\bar{\partial}_1 \bar{f}_2 - \bar{\partial}_2 \bar{f}_1) d\bar{z}_1 \wedge d\bar{z}_2$$

## Pluriharmonic solutions: Euclidean background

Consider  $\omega(z) = f_1 dz_1 + f_2 dz_2 + \bar{f}_1 d\bar{z}_1 + \bar{f}_2 d\bar{z}_2$  and  $F_\omega = d\omega$ .

### Proposition

*Assume  $f_i, \bar{f}_i, i = 1, 2$  are holomorphic. Then  $F_\omega$  is a self-dual solution to Maxwell's equations in vacuum. Furthermore, in this case*

$$F_\omega = (\partial_1 f_2 - \partial_2 f_1) dz_1 \wedge dz_2 + (\bar{\partial}_1 \bar{f}_2 - \bar{\partial}_2 \bar{f}_1) d\bar{z}_1 \wedge d\bar{z}_2$$

**Def.** The 1-form  $\omega$  is said to be *harmonic* if all the functions  $f_1, f_2, \bar{f}_1, \bar{f}_2$  are harmonic.

### Theorem

*A harmonic form  $\omega$  is a solution to the Maxwell's equations in vacuum if and only if  $\bar{\partial}_1 f_1 + \bar{\partial}_2 f_2 + \partial_1 \bar{f}_1 + \partial_2 \bar{f}_2$  is constant.*

# Minkowski background

## Theorem

Let  $f_i$  be holomorphic and  $f_{\bar{i}} = \overline{f_i}$ ,  $i = 1, 2$ . Then  $F_\omega$  is a SD solution to Maxwell's equations in vacuum if and only if  $\partial_2 f_1 - \partial_1 f_2$  is independent of the variable  $z_1$ .

**Note:** Such solutions are time-invariant.

# Minkowski background

## Theorem

Let  $f_i$  be holomorphic and  $f_{\bar{i}} = \overline{f_i}$ ,  $i = 1, 2$ . Then  $F_\omega$  is a SD solution to Maxwell's equations in vacuum if and only if  $\partial_2 f_1 - \partial_1 f_2$  is independent of the variable  $z_1$ .

**Note:** Such solutions are time-invariant.

## Proposition

If  $F$  is any self-dual or anti-self-dual solution to the Maxwell's equations in vacuum, then  $\langle \mathbf{E}, \mathbf{B} \rangle \in i\mathbb{R}$ . In particular, if  $\omega$  is a real self-dual or anti-self-dual solution then  $\langle \mathbf{E}, \mathbf{B} \rangle = 0$ .

**Note.** This explains the natural phenomenon that  $\mathbf{E} \perp \mathbf{B}$ .

# A Quantization of Maxwell's Equations

Replace  $\mathbb{C}$  with a unital  $C^*$ -alg.  $\mathcal{A}$  equipped with a faithful normalized trace  $\text{Tr}$ .

Inner product on  $\mathcal{A}$  :  $\langle a, b \rangle = \text{Tr}(ab^*)$ ,  $a, b \in \mathcal{A}$ .

The Hodge star operator  $\star$  on  $\mathcal{A}$ -valued  $p$ -forms:

$$\text{Tr}(\mu \wedge \star(\eta^*)) = \langle \mu, \eta \rangle \text{vol}. \quad \mu, \eta \in \Lambda^p.$$

Further, if  $\mu$  and  $\eta$  are  $\mathcal{A}$ -valued  $L^2$  differential  $p$ -forms, then an inner product can be defined by

$$(\mu, \eta) = \int_{\mathbb{C}^2} \langle \mu, \eta \rangle \text{vol} = \int_{\mathbb{C}^2} \text{Tr}(\mu \wedge \star(\eta^*)). \quad (8)$$

# Yang-Mills Equations

Assume  $\mathcal{H}$  is an  $\mathcal{A}$ -module, and  $A \in \Lambda^1(\mathcal{A})$ .

## Definition

The covariant derivative  $D_A : \Lambda^p(\mathcal{H}) \rightarrow \Lambda^{p+1}(\mathcal{H})$  is defined as

$$D_A(\eta) = d\eta + A \wedge \eta.$$

$$D_A^2 \eta = (dA + A \wedge A) \wedge \eta \Rightarrow \text{Curvature field } F_A = dA + A \wedge A.$$

## Proposition

In Euclidean metric, the covariant co-derivative  $D_A^* := -\star D_{-A}\star$ .

# Yang-Mills Equations

Assume  $\mathcal{H}$  is an  $\mathcal{A}$ -module, and  $A \in \Lambda^1(\mathcal{A})$ .

## Definition

The covariant derivative  $D_A : \Lambda^p(\mathcal{H}) \rightarrow \Lambda^{p+1}(\mathcal{H})$  is defined as

$$D_A(\eta) = d\eta + A \wedge \eta.$$

$$D_A^2 \eta = (dA + A \wedge A) \wedge \eta \Rightarrow \text{Curvature field } F_A = dA + A \wedge A.$$

## Proposition

In Euclidean metric, the covariant co-derivative  $D_A^* := -\star D_{-A}\star$ .

## Definition

The Yang-Mills equations are given by

$$D_A F_A = 0, \quad D_A^* F_A = J. \quad (9)$$

**Note:** If  $A^* = -A$  (skew-Hermitian), then every SD or ASD  $F_A$  is a solution to YM in vacuum. (“instantons”)

## Definition

The Yang-Mills Lagrangian is defined as

$$L(A) = -A \wedge \star J^* + \frac{1}{2} F_A \wedge \star F_A^*,$$

and the Yang-Mills functional is defined as

$$Y(A) = \int_{\mathbb{C}^2} \text{Tr} L(A) = -(A, J) + \frac{1}{2} (F_A, F_A)$$

for all  $A \in \Lambda^1(\mathbb{C}^2, \mathcal{A})$  such that the integral is convergent.

## Theorem

$F_A$  is a solution to the YM equations if and only if  $A$  is a critical connection to the YM functional.

## Gauge invariance

$GL(\mathcal{A})$ : the set of invertible elements in  $\mathcal{A}$ ;

$g : \mathbb{C}^2 \rightarrow GL(\mathcal{A})$ : smooth function.

**gauge transform** (change of reference frame):

$$A \rightarrow A_g := g^{-1}Ag + g^{-1}dg.$$

**Facts:** a)  $F_{A_g} = g^{-1}F_Ag$ ;      b)  $D_{A_g}^* F_{A_g} = g^{-1}Jg$ .

### Proposition

*Solutions to YM equations in vacuum are gauge invariant.*

# Gauge invariance

$GL(\mathcal{A})$ : the set of invertible elements in  $\mathcal{A}$ ;

$g : \mathbb{C}^2 \rightarrow GL(\mathcal{A})$ : smooth function.

**gauge transform** (change of reference frame):

$$A \rightarrow A_g := g^{-1}Ag + g^{-1}dg.$$

**Facts:** a)  $F_{A_g} = g^{-1}F_Ag$ ;      b)  $D_{A_g}^* F_{A_g} = g^{-1}Jg$ .

## Proposition

*Solutions to YM equations in vacuum are gauge invariant.*

## Lemma

a) Every skew-Hermitian form  $A \in \Lambda^1(\mathcal{A})$  is of the kind

$A = \eta - \eta^*$ , where  $\eta = A_1 dz_1 + A_2 dz_2$ .

b) If  $A$  is skew-Hermitian and  $g$  is unitary-valued, then  $A_g$  is skew-Hermitian.

# Main result

## Theorem

Assume  $\eta(z) = A_1(z)dz_1 + A_2(z)dz_2$  is holomorphic and normal, and  $A = \eta - \eta^*$ . Then

a) Under the Euclidean metric,

- ▶  $F_A$  is SD if and only if  $A_1(z)$  and  $A_2(z)$  commute for each  $z \in \mathbb{C}^2$ . In this case,  $F_A = dA$ . (back to the ME case)
- ▶  $F_A$  is ASD if and only if it is gauge equivalent to 0.

# Main result

## Theorem

Assume  $\eta(z) = A_1(z)dz_1 + A_2(z)dz_2$  is holomorphic and normal, and  $A = \eta - \eta^*$ . Then

a) Under the Euclidean metric,

- ▶  $F_A$  is SD if and only if  $A_1(z)$  and  $A_2(z)$  commute for each  $z \in \mathbb{C}^2$ . In this case,  $F_A = dA$ . (back to the ME case)
- ▶  $F_A$  is ASD if and only if it is gauge equivalent to 0.

b) Under the Minkowski metric,

- ▶  $F_A$  is SD if and only if  $[A_1^* - iA_1, A_2] = i(\partial_1 A_2 - \partial_2 A_1) = 0$ . In this case,  $F_A = \omega - \omega^*$ , where  $\omega = [A_2, A_1^*](idz_1 \wedge dz_2 - dz_2 \wedge d\bar{z}_1)$ .
- ▶ The ASD case is parallel.

**Conclusion:** There exists nontrivial pluriharmonic YM instantons in the Minkowski background.

## An example

Consider any holomorphic function  $h(z)$  on  $\mathbb{C}^2$  and set

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad H(z_1, z_2) := \int_0^{z_1} \partial_2 h(\xi, z_2) d\xi.$$

We define

$$A_1 = \begin{pmatrix} h & 0 \\ 0 & h + 1 - i \end{pmatrix}, \quad A_2 = U^* \begin{pmatrix} H + \cos z_2 & 0 \\ 0 & H + i \sin z_2 \end{pmatrix} U.$$

Then  $[A_1^* - iA_1, A_2] = 0 = i(\partial_1 A_2 - \partial_2 A_1)$ , showing that  $F_A$  is a SD instanton case with Minkowski background. Furthermore,

$$[A_2, A_1^*] = \frac{1}{\sqrt{2}} e^{-i(z_2 - \frac{\pi}{4})} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

**Note:**  $F_A$  does not depend on the variable  $z_1$ , implying that  $F_A$  is a time-invariant.

# Thank You!

