On Molchanov's criterion for the compactness of the resolvent for the non-selfadjoint Sturm–Liouville operator

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Introduction

Consider complex-valued $q \in L_{1,loc}(\mathbb{R}_+)$ and the operators in $L_2(\mathbb{R}_+)$:

$$L_0 \subset L_U \subset L$$

defined by the form:

$$l(y) = -y'' + qy$$

on one of the following domains

$$D = \big\{ y \in L_2(\mathbb{R}_+) \,\big|\, y, y' \in AC_{loc}(\mathbb{R}_+),\ l(y) \in L_2(\mathbb{R}_+) \big\},$$
$$D_0 = \big\{ y \in D \,\big|\, y(0) = y'(0) = 0,\ \exists x_0 > 0\ \forall x \geq x_0\ y(x) = 0 \big\},$$
$$D_U = \big\{ y \in D \,\big|\, U(y) = 0 \big\},$$

$$D_0 \subset D_U \subset D$$

where U is some form of boundary conditions at $x = 0$:

$$U(y) = Ay(0) + By'(0), \quad A, B \in \mathbb{C},\ |A| + |B| > 0.$$

Introduction

The goal is to find for simple conditions for discreteness of the spectrum and compactness of the resolvent

Well known classical result:

Theorem (Molchanov A. M.)

Let $q \geq C$ *— real-valued potential, the following condition on the potential is a criterion for the compactness of the resolvent (hence the discreteness of the spectrum):*

$$\forall a > 0 \ \lim_{x \to +\infty} \int\limits_{x}^{x+a} q(\xi)\, d\xi = +\infty,$$

We consider the case of ordinary differential equations. In a more complex form the criterion was prooved for partial differential equations also.

Introduction

Well-known huge number of generalizations and related issues:

- The case of real-valued q: [I.Brinck] for $\int\limits_t^x q\,d\xi > C$, $|x-t| \le 1$, [Ismagilov] for $\int\limits_t^x q\,d\xi > \alpha(x) - \beta(t)$ + conds. on α, β.
- Complex-valued q: [Lidskii], for $\arg q \in [0, \pi/2]$ or $\arg q \in [-\pi/2, 0]$, [Fortunato] for $q \in L_{2,loc}$, $\arg q \in [0, \pi - \delta]$.
- Case $L_2(\mathbb{R}^n)$ [Maz'ya, Shubin]
- Distributional potentials [Albeverio, Kostenko, Malamud]
- Matrix and operator potentials [Levitan, Suvorchenkova, Ismagilov, Kostuchenko, Dall'Ara]
- Essential spectra [Kwong, Zettl, Evans, Lewis]
- etc...

Molchanov Condition

Definition

Let $q \in L_{1,loc}(\mathbb{R}_+)$. We say that q satisfies the **Molchanov condition** if

$$\forall a > 0 \ \lim_{x \to +\infty} \int_x^{x+a} |q(\xi)| \, d\xi = +\infty,$$

Note

For real semi-bounded potentials and for potential with values in sectors with angles less than π, *this is equivalent to*

$$\forall a > 0 \ \lim_{x \to +\infty} \left| \int_x^{x+a} q(\xi) \, d\xi \right| = +\infty.$$

Rellich's lemma

The following Lemma plays a crucial role in proving the compactness of the resolvent in the case of real potentials:

Lemma (Rellich, for self-adjoint $A \geq 0$)

Consider $A \geq 0$ *in the Hilbert space* $\mathfrak{H}$ *acting on the domain* $\mathscr{D}_A \subset \mathfrak{H}$. *The operator* A *has a compact resolvent if and only if*

$$\left\{\varphi \in \mathscr{D}_A \,\middle|\, (A\varphi, \varphi) \leq 1\right\} \quad \textit{is compact.}$$

Can be easily generalized to the case of **m-sectoral** operators.

Lemma

The m-sectoral operator A *has a compact resolvent if and only if*

$$\left\{\varphi \in \mathscr{D}_A \,\middle|\, \mathrm{Re}(A\varphi, \varphi) \leq 1\right\} \quad \textit{is compact.}$$

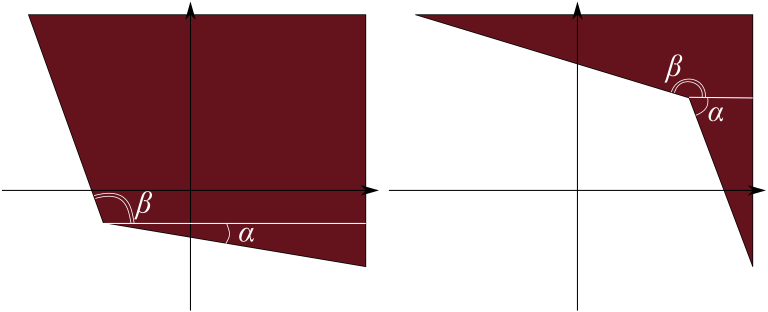
Sectorial potentials with $\mathbb{R}_-$ condition

Definition

We say that q satisfies $\mathbb{R}_-$ **– condition** if for all sufficiently large $x > x_0 \geq 0$ the values of $q(x)$ lie in the sector

$$\alpha \leq \arg(q(x) - q_0) \leq \beta$$

for some $-\pi < \alpha \leq \beta < \pi$ and $q_0 \in \mathbb{C}$.



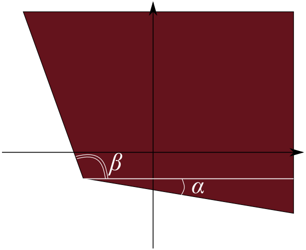
Theorem 1

Theorem

Let $q \in L_{1,loc}(\mathbb{R}_+)$ sectorial potentials with $\mathbb{R}_-$ condition, and $\beta - \alpha < \pi$. In this case the **Molchanov condition**

$$\forall a > 0 \lim_{x \to +\infty} \int\limits_{x}^{x+a} |q(\xi)|\, d\xi = +\infty,$$

is the criteria for the compactness of the resolvent of L_U.

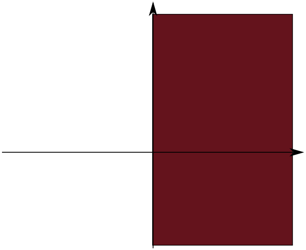


Theorem 2

The condition $\beta - \alpha < \pi$ is essential

Theorem

There exist potentials $q \in L_{1,loc}(\mathbb{R}_+)$, $\operatorname{Re} q \geq 0$ *satisfying the* **Molchanov condition** *such that the corresponding minimal operators does not have extensions with the compact resolvent.*



Above mentioned operators with Dirichlet or Neumann boundary condition have a bounded non-compact resolvent in the left half-plane.

Theorem 3

Theorem

Let $q \in L_{1,loc}(\mathbb{R}_+)$*. Without any additional conditions on the potential, the* **Molchanov condition**

$$\forall a > 0 \ \lim_{x \to +\infty} \int\limits_{x}^{x+a} |q(\xi)| \, d\xi = +\infty,$$

is necessary for the compactness of the resolvent of L_U*.*

Theorem 3a

Let $p_1 \equiv C_0 = \text{const}$, $p_j \in L_{1,loc}(\mathbb{R}_+)$, $j = 2, \dots, n$ — complex-valued. Consider the minimal operator in $L_2(\mathbb{R}_+)$:

$$L_0(y) = \frac{d^n}{dx^n}y + \sum_{j=1}^{n} p_j \frac{d^{n-j}}{dx^{n-j}}y$$

on the domain

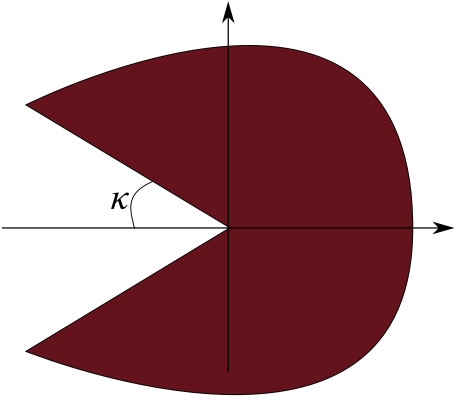
$$\mathscr{D}_0 = \big\{ y \in L_2(\mathbb{R}_+) \,\big|\, y, y', \dots, y^{(n-1)} \in AC_{loc}(\mathbb{R}_+),\ L_0(y) \in L_2(\mathbb{R}_+),$$
$$y(0) = \dots = y^{(n-1)}(0) = 0,\ \exists x_0 > 0\ \forall x \geq x_0\ y(x) = 0 \big\}.$$

Theorem

The necessary condition for the existence of an extension L_0 with a compact resolvent is as follows:

$$\forall a > 0 \ \lim_{x \to +\infty} \int_{x}^{x+a} \sum_{j=2}^{n} |p_j(\xi)|\, d\xi = +\infty.$$

Any sufficient condition for the case $\beta - \alpha > \pi$?



The image of $q(x)$, $\beta = -\pi + \kappa$, $\alpha = \pi - \kappa$, $0 < \kappa < \pi$.

Theorem 4

Theorem

Let for some $x_0 > 0$ *for all* $x \geq x_0 > 0 \quad |q(x)| \geq 1$ *and additionally:*

- $q \in AC_{loc}[x_0, +\infty)$,
- *for some* $0 < \kappa < \pi$

$$-\pi + \kappa < \arg q(x) < \pi - \kappa, \quad x \geq x_0,$$

- *and for some* $0 < \delta < 1$

$$\left| \frac{q'(x)}{q^{3/2}(x)} \right| < 4\delta \sin \frac{\kappa}{2}, \quad x \geq x_0.$$

Under these conditions, for the resolvent of L_U *to be compact, it is sufficient that for any* $a > 0$

$$\lim_{x \to +\infty} \int\limits_{x}^{x+a} |q(\xi)|^{1/2} \, d\xi = \infty.$$

Examples

$$q(x) = x^{\alpha} \exp\bigl(i\beta \cos \gamma x^{1+\alpha/2}\bigr),$$

with arbitrary $\alpha > 0$, $0 \le \beta < \pi$ and γ small enough to satisfy the condition

$$\left| \frac{q'(x)}{q^{3/2}(x)} \right| < 4\delta \sin \frac{\kappa}{2}, \quad x \ge x_0.$$

for some $0 < \delta < 1$.

Tnank you!!!