Essential spectra of Sturm-Liouville operators and their indefinite counterpart

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Overview

SL on $\mathbb{R}$: $$\frac{1}{r}\left(-\frac{d}{dx}p\frac{d}{dx}+q\right) \qquad r, q, p^{-1} \in L^1_{loc}, p > 0,$$

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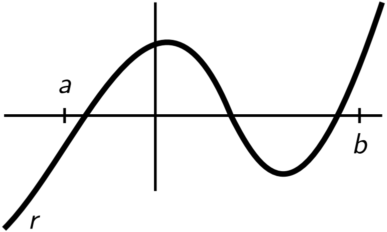

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$$\tau := \frac{1}{r}\left(-\frac{d}{\mathrm{d}x} p \frac{d}{\mathrm{d}x} + q\right) \qquad r, q, p^{-1} \in L^1_{loc}, p > 0 \text{ on } \mathbb{R}$$

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$\tau_{-\infty} \rightsquigarrow$	$K_{-\infty}$	$\mathcal{D}(-\infty, a)$, $f(a) = 0$	$r < 0$	bd from above
$\tau_{a,b} \rightsquigarrow$	$K_{a,b}$	$\mathcal{D}(a, b)$, $f(a) = f(b) = 0$		$\emptyset$
$\tau_{\infty} \rightsquigarrow$	K_{∞}	$\mathcal{D}(b, \infty)$, $f(b) = 0$	$r > 0$	bd from below

$$\mathcal{D}(\mathbb{R}) := \left\{ f \in L^2((\mathbb{R}); |r|) : f, pf' \in \mathcal{AC}(\mathbb{R}), \tau f \in L^2((\mathbb{R}); |r|) \right\},$$

where $\mathcal{AC}$: absolutely continuous functions, $\mathcal{D}(I)$ analogously.

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Theorem

The maximal operators K associated to τ in the weighted L^2-space $L^2(\mathbb{R}; |r|)$ satisfies

- K has non-empty resolvent set
- $$\sigma_{\text{ess}}(K) = \sigma_{\text{ess}}(K_{-\infty}) \cup \sigma_{\text{ess}}(K_{\infty}).$$

Proof is with [Behrndt, Philipp'10].

Perturbation and essential spectrum

Assume we have a second indefinite SL-expression on $\mathbb{R}$

$$\widetilde{\tau} := \frac{1}{\widetilde{r}}\left(-\frac{d}{dx}\widetilde{p}\frac{d}{dx} + \widetilde{q}\right) \quad \widetilde{r}, \widetilde{q}, \widetilde{p}^{-1} \in L^1_{loc}, \widetilde{p} > 0 \text{ on } \mathbb{R}$$

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Then the resolvent sets $\rho(K)$ and $\rho(\widetilde{K})$ are nonempty,and

$$\sigma_{\mathrm{ess}}(K) = \sigma_{\mathrm{ess}}(\widetilde{K}) \subset \mathbb{R}.$$

Proof is with [Behrndt, Schmitz, Teschl, T.'23].

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with the following property (P):There exists $\epsilon > 0$ with

$$(\lambda - \epsilon, \lambda) \subset \rho(K_{\infty}) \text{ and } (\lambda, \lambda + \epsilon) \subset \rho(K_{-\infty})$$

or

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Proof is with locally definitizable operators.

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Remark: This is, one can use accumulation results from Hilbert space self-adjoint operators and receive accumulation results for the indefinite operator K.

Proof is with [Behrndt, Möws, T.'14].

Example I

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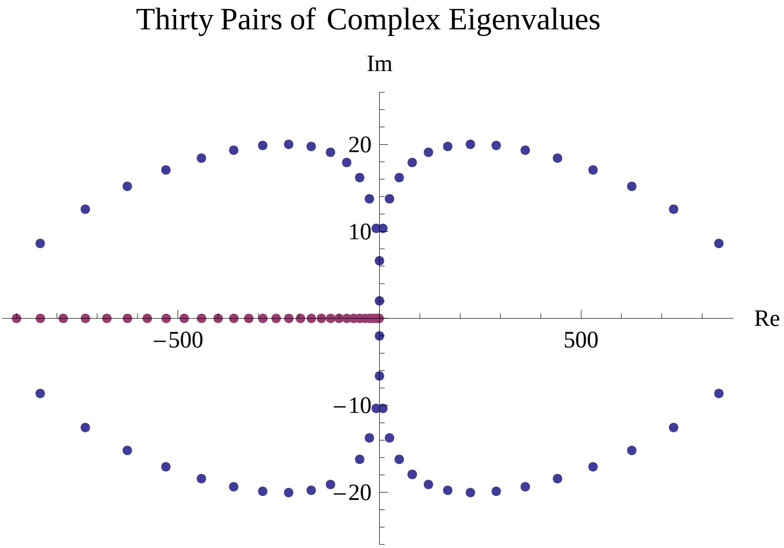
Eigenvalues of both operators do not accumulate to zero, hence

Theorem [Behrndt, Katatbeh, T.'09]

- 0 has property (P).
- A has 30 pairs of non-real eigenvalues.

$q(x) = -30 \cdot 31 \ \mathrm{sech}^2(x).$



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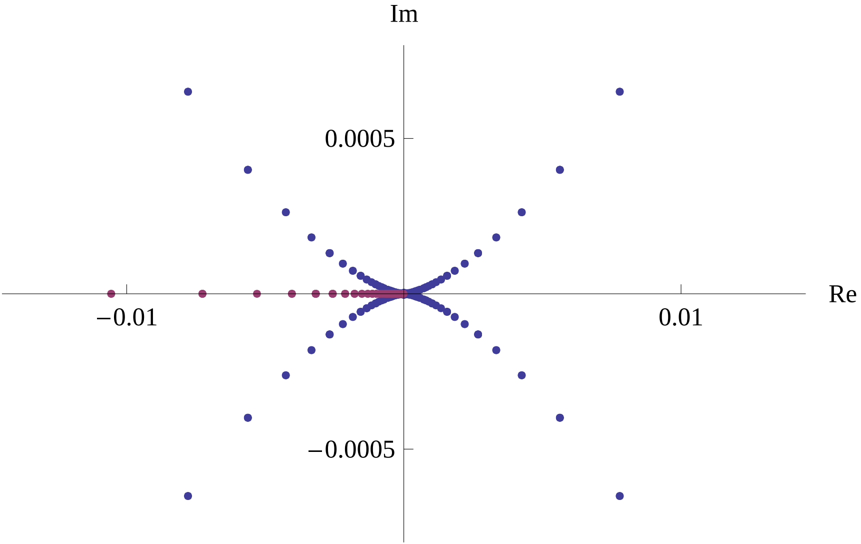
Eigenvalues of both operators accumulate to zero, hence

Theorem [Behrndt, Katatbeh, T.'08]

- K has not property (P).
- Non-real eigenvalues of K accumulate to zero.

$$q(x) = -\frac{1}{1+|x|}$$



Thank You.