

Challenges for non-selfadjoint spectral problems in analysis and computation

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[Trefethen/Embree '05]

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2. **Spectral invisibility**:

$\exists \lambda \in \sigma(T)$ that is **no** accumulation point of some $\lambda_n \in \sigma_p(T_n)$, $n \in \mathbb{N}$.

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Ex. 1 Right shift in $\ell_2(\mathbb{Z})$:

$$T := \begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & 0 & 0 & & & \\ & \mathbf{1} & 0 & 0 & & \\ & & \mathbf{1} & 0 & \ddots & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \ddots \end{pmatrix}, \quad T_n := \begin{pmatrix} 0 & 0 & & & & \\ \mathbf{1} & 0 & 0 & & & \\ & \mathbf{1} & 0 & \ddots & & \\ & & \ddots & \ddots & 0 & \\ & & & \ddots & \mathbf{1} & 0 \\ & & & & & 0 \end{pmatrix} \in \mathbb{C}^{n \times n}.$$

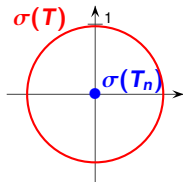
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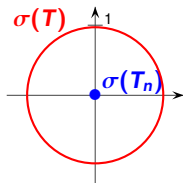
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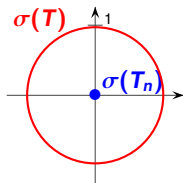
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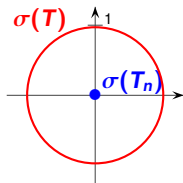
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Problems not restricted to non-selfadjoint operators

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T unbounded – some surprises ... [Bögli/Marletta/T. JFA '20]:

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 - $\overline{W(T)}$ strip $\implies W_e(T) \neq \emptyset, W_e(T)$ strip;
 - $W(T) = \mathbb{C} \implies W_e(T) \neq \emptyset, W_e(T) = W(T) = \mathbb{C}$;
 - $\overline{W(T)}$ half-plane, $W_e(T) \neq \emptyset \implies W_e(T)$ half-plane.

Basic properties (carrying over from bounded case)

- Prop. 1**
- $W_e(T) \subset \overline{W(T)}$;
 - $W_e(T)$ closed, convex;
 - $\text{conv } \sigma_e(T) \subset W_e(T)$,

here $\sigma_e(T) = \sigma_{e2}(T)$ [Edmunds/Evans] with

$$\sigma_{e2}(T) := \left\{ \lambda \in \mathbb{C} : \exists (x_n)_{n \in \mathbb{N}} \subset \text{dom } T, \|x_n\| = 1, x_n \xrightarrow{w} 0, \|(T - \lambda)x_n\| \rightarrow 0 \right\}.$$

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Equivalent characterizations of $W_e(T)$... useful here and elsewhere:

Thm. 1 [Hefti/T. SM '22: W_{e5}] Define

$$W_{e1}(T) := \bigcap_{V \subset \mathcal{H}, \dim V < \infty} \overline{W(T|_{V^\perp \cap \text{dom } T})},$$

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If $\overline{\text{dom } T \cap \text{dom } T^*} = \mathcal{H}$ or if $W(T) \neq \mathbb{C}$, then

$$W_{ei}(T) = W_e(T), \quad i = 1, 2, 3, 4.$$

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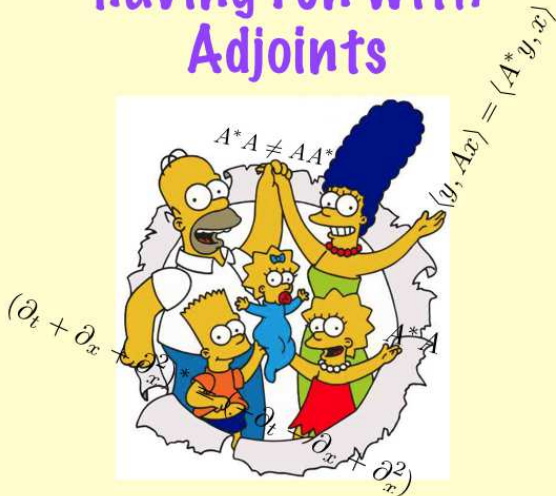
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NOTE: Everything is possible for $\text{dom } T \cap \text{dom } T^*$, see [Arlinskii/T. AIM'20]

Having Fun with Adjoint



With thanks to Prof. Dwight Barkley, University of Warwick

4. Stability / Perturbation results

Thm. 4 For all $K \in L(\mathcal{H})$ compact,

$$W_e(T) = W_e(T + K), \quad W_{ei}(T) = W_{ei}(T + K), \quad i = 1, 2, 3, 4, 5.$$

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Thm. 6 $T = A + iB$, $K = U + iV$ with A, B, U, V symmetric and one of:

- (i) A selfadjoint and semibounded, U, V A -compact, or
- (ii) B selfadjoint and semibounded, U, V B -compact, or
- (iii) A, B selfadjoint and semibounded, U A -compact, V B -compact.

Then $W_e(T) = W_e(T + K)$.

5. Spectral convergence results

Projection method: $\mathcal{H}_n \subset \text{dom } T$, $n \in \mathbb{N}$, finite dimensional subspaces,
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Thm. 7 Let T be closed, $\overline{\text{dom } T} = \mathcal{H}$. If $P_{\mathcal{H}_n} \xrightarrow{s} I_{\mathcal{H}}$ and $T_n \xrightarrow{gsr} T$, i.e.
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- spurious eigenvalues may only occur in $W_e(T)$;
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[Descloux '81]: T bounded; [Levitin/Shargorodsky '04], [Lewin/Séré '10]:

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$W_e(T)$ is the smallest possible set capturing spectral pollution !!!

Domain truncation for PDEs: Thm. 7 also holds for bdd. nested $\Omega_n \subset \mathbb{R}^d$, $n \in \mathbb{N}$, exhausting \mathbb{R}^d (C-boundary if $d \geq 2$) and **strongly elliptic PDOs T in $L^2(\mathbb{R}^d)$ and T_n in $L^2(\Omega_n)$** induced by the sectorial forms

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with coefficients $Q_{\alpha,\beta} \in L^\infty(\mathbb{R}^d, \mathbb{C})$, $Q_{\alpha,\beta} := q_{\alpha,\beta} \in \mathbb{C}$ for $|\alpha|+|\beta|=2m$ and $\text{Re } p_{2m}(\xi) > 0$, $\xi \neq 0$, for

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Prop. 4 If $Q_{\alpha,\beta}(x) \rightarrow c_{\alpha,\beta} \in \mathbb{C}$, $|x| \rightarrow \infty$, if $|\alpha|+|\beta| < 2m$ and

$$p_\infty(\xi) := p_{2m}(\xi) + \sum_{|\alpha|+|\beta| < 2m} c_{\alpha,\beta} \xi^\alpha, \quad \xi \in \mathbb{R}^d,$$

then

$$W_e(T) = \text{conv} \{ p_\infty(\xi) : \xi \in \mathbb{R}^d \} = \text{conv } \sigma_e(T).$$

Ex. 3 Advection-diffusion type differential operator

$$T := -D^2 + Q_1 D + Q_0, \quad \text{dom}(T) := H^2(\mathbb{R}),$$

with $Q_1, Q_0 \in L^\infty(\mathbb{R}, \mathbb{C})$ and $Q_1(x) \rightarrow -2, Q_0(x) \rightarrow 0, |x| \rightarrow \infty$. Then

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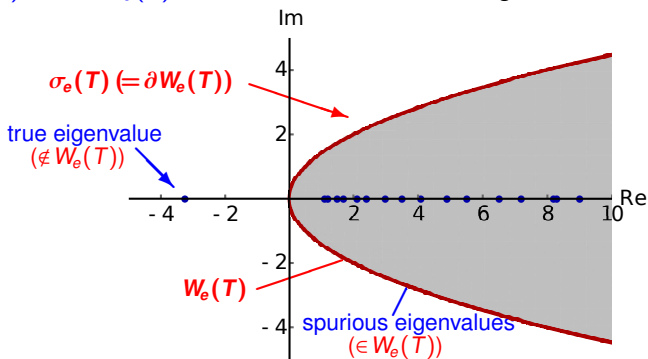
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$Q_1(x) = -2, Q_0(x) = 20 e^{-x^2} \sin x$: truncated eigenvalues on $\Omega_n = [-9, 9]$



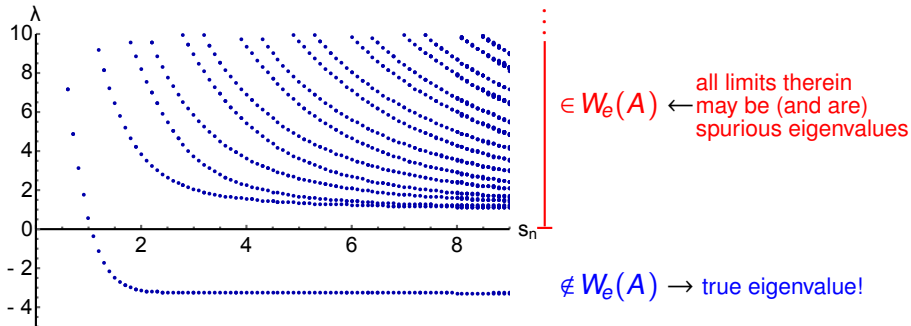
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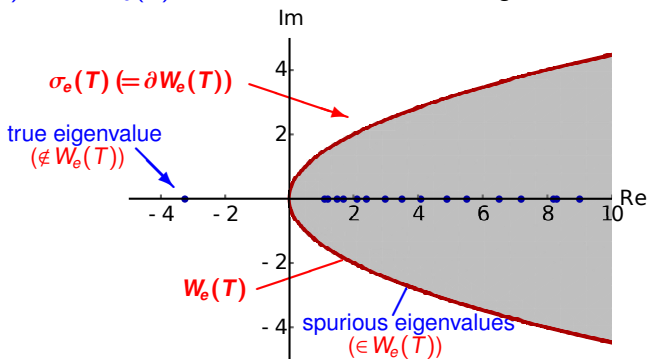
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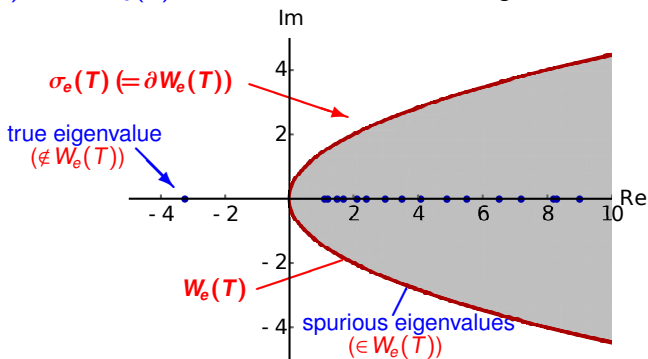
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A more challenging application ... [Bögli/Ferraresso/Marletta/T. '23]:

Domain truncation for Maxwell's equation

$\Omega \subseteq \mathbb{R}^3$ with Lipschitz boundary, nested $\Omega_n \subset \Omega$, $n \in \mathbb{N}$, exhausting Ω :

$$\begin{aligned} -i\sigma E + i \operatorname{curl} H &= \omega \varepsilon E \\ -i \operatorname{curl} E &= \omega \mu H \end{aligned} \quad \text{in } \Omega, \quad \nu \times E = 0 \quad \text{on } \partial\Omega,$$

with $\varepsilon, \mu, \sigma \in L^\infty(\Omega, \mathbb{R}^{3 \times 3})$, $\varepsilon, \mu > 0$, $\sigma \geq 0$, and its truncations to Ω_n , $n \in \mathbb{N}$ \rightsquigarrow

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Spectral enclosures on Ω and Ω_n & resolvent estimates:

Thm. 7. $\sigma(V) \subset i \left[-\frac{\sigma_{\max}}{\varepsilon_{\min}}, 0 \right] \cup \left\{ \omega \in \mathbb{C} \setminus i\mathbb{R} : \operatorname{Im} \omega \in \left[-\frac{1}{2} \frac{\sigma_{\max}}{\varepsilon_{\min}}, -\frac{1}{2} \frac{\sigma_{\min}}{\varepsilon_{\max}} \right], \right.$

$$\left. (\operatorname{Re} \omega)^2 - 3(\operatorname{Im} \omega)^2 + 2 \frac{\sigma_{\max}}{\varepsilon_{\min}} |\operatorname{Im} \omega| \geq \frac{\lambda_{\min}^\Omega}{\varepsilon_{\max} \mu_{\max}} \right\}$$

with $\lambda_{\min}^\Omega := \min \sigma(\operatorname{curl} \operatorname{curl}_0 |_{H(\operatorname{Div} 0, \Omega)}) \geq 0$ & resolvent estimates **outside**.

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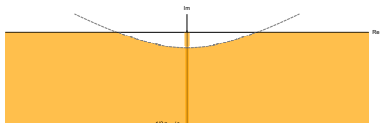
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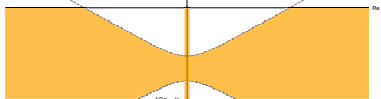
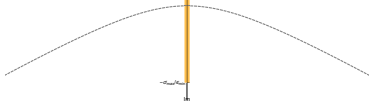
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Possible shapes of enclosure in Thm. 7: $\lambda_{\min}^\Omega = 0 \rightsquigarrow$ **enclosure**, else:



$$\lambda_{\min}^{\Omega} > 0$$

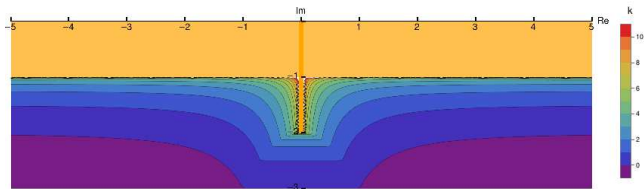


$$\lambda_{\min}^{\Omega} > \frac{1}{4} \frac{\sigma_{\max}^2 \varepsilon_{\max} \mu_{\max}}{\varepsilon_{\min}^2}$$

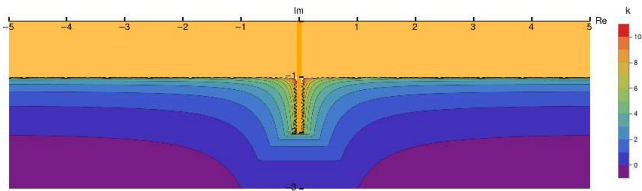


$$\lambda_{\min}^{\Omega} > \frac{1}{3} \frac{\sigma_{\max}^2 \varepsilon_{\max} \mu_{\max}}{\varepsilon_{\min}^2}$$

Level curves of the resolvent **outside** for $\varepsilon_{\min} = \mu_{\min} = 1$, $\sigma_{\max} = 2$:



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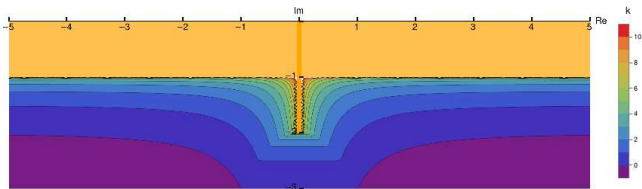
Essential spectrum and spectral approximation

Suppose ε , μ and σ have limits $\varepsilon_{\infty}\text{id}$, $\mu_{\infty}\text{id}$ and $\mathbf{0}$, respectively, “at ∞ ”, $\varepsilon_{\infty}, \mu_{\infty} > 0$, and define quadratic operator polynomial $L_{\infty}(\omega)$, $\omega \in \mathbb{C}$, by

$$L_{\infty}(\omega) := \frac{1}{\mu_{\infty}} \text{curl curl}_0 - \varepsilon_{\infty} \omega^2 \text{id}, \quad (= \omega \mathbf{S}_{1,\infty}(\omega))$$

$$\text{dom}(L_{\infty}(\omega)) := \{u \in H_0(\text{curl}, \Omega) \cap H(\text{Div } \mathbf{0}, \Omega) : \text{curl } u \in H(\text{curl}, \Omega)\}.$$

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Thm. 8. $\sigma_e(V) \cap \mathbb{R} = \sigma_e(L_{\infty})$ and $\sigma_e(V) \cap i\mathbb{R} \subset i \left[-\frac{\sigma_{\max}}{\varepsilon_{\min}}, 0 \right]$ with

$$\sigma_e(L_{\infty}) \setminus \{0\} = \left(-\frac{1}{\varepsilon_{\infty} \mu_{\infty}} \sigma_e(\text{curl curl}_0)^{1/2} \right) \cup \left(\frac{1}{\varepsilon_{\infty} \mu_{\infty}} \sigma_e(\text{curl curl}_0)^{1/2} \right).$$

Thm. 9. If $\lambda_{e,\min}^\Omega := \min \sigma_e(\text{curl curl}_0 |_{H(\text{Div } 0, \Omega)}) (\geq 0)$, then

i) $\sigma_{\text{poll}}((V_n)_{n \in \mathbb{N}}) \subset W_e(L_\infty) = \left(-\infty, -\left(\frac{\lambda_{e,\min}^\Omega}{\varepsilon_\infty \mu_\infty}\right)^{1/2}\right] \cup \left[\left(\frac{\lambda_{e,\min}^\Omega}{\varepsilon_\infty \mu_\infty}\right)^{1/2}, \infty\right)$

with $W_e(L_\infty) := \{\omega \in \mathbb{C} : 0 \in W_e(L_\infty(\omega))\}$ (*essential numerical range of L_∞*);

ii) every isolated $\omega \in \sigma_p(V)$ outside of $W_e(L_\infty) \cup i \left[-\frac{\sigma_{\max}}{\varepsilon_{\min}}, 0\right]$ is approximated by the domain truncation $(V_n)_{n \in \mathbb{N}}$.

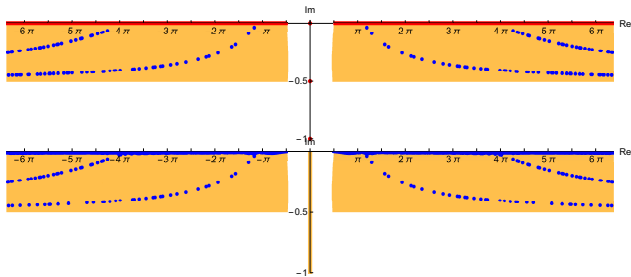
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Ex. 4 $\Omega = (0, \infty) \times (0, 1) \times (0, 2)$, $\Omega_n = (0, n) \times (0, 1) \times (0, 2)$, $n \in \mathbb{N}$, and $\varepsilon = \mu = \text{id}$, $\sigma = \chi_{\Omega_1} \text{id}$:



$\sigma_e(V)$ and $\sigma_p(V)$

$\sigma_p(V_{50})$ for $n=50$

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- [HT'22] N. Hefti, C. Tretter, *Essential numerical range and C-numerical range for unbounded operators*; Studia Math. 264 (2022), 28 p.
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THANK YOU FOR YOUR ATTENTION!
