Spectral decomposition and decay to grossly determined solutions for a simplified BGK model

Alim Sukhtayev
Miami University

joint work with

Kevin Zumbrun
Indiana University Bloomington

CIRM conference on
Mathematical aspects of the physics with non-self-adjoint operators

partially supported by NSF: DMS-1910820

The Boltzmann equation in 3D

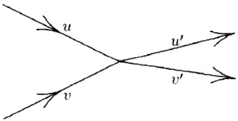
Motion of a rarefied gas: $f = f(t, x, v)$ - density of particles, $v \in \mathbb{R}^3$-velocity, $x \in \mathbb{R}^3$.

No collisions: a particle with speed v located at x at $t = 0$ will move to $x + \tau v$ at a later time $\tau \Rightarrow f(\tau, x, v) = f(0, x - \tau v, v) \Rightarrow$

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = 0.$$

The Boltzmann equation:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \mathcal{Q}(f).$$



$$\begin{aligned}\mathcal{Q}(f)(v) &= \mathcal{Q}_+(f)(v) - \mathcal{Q}_-(f)(v) \\ &= \int_{\mathbb{R}^3} \int_{S^2} (f(v')f(u') - f(v)f(u)) C(v - u, \mathbf{n}) d\mathbf{n} du\end{aligned}$$

Collision invariants

$$\frac{\partial f}{\partial t} = \mathcal{Q}(f), \text{ where } f = f(t, v) \text{ is independent of x.}$$

Seek functionals of the form:

$$\Phi(f) = \int_{\mathbb{R}^3} \phi(v) f(v) dv, \text{ which remain constant along trajectories.}$$

Sufficient condition: $\int_{\mathbb{R}^3} \phi(v)\mathcal{Q}(f) dv = 0$.

- $\int_{\mathbb{R}^3} \mathcal{Q}(f) dv = 0 \Rightarrow$ Conservation of $\int_{\mathbb{R}^3} f(v) dv$ (mass)
- $\int_{\mathbb{R}^3} v_i \mathcal{Q}(f) dv = 0 \Rightarrow$ Conservation of $\int_{\mathbb{R}^3} v_i f(v) dv$ (momentum)
- $\int_{\mathbb{R}^3} |v|^2 \mathcal{Q}(f) dv = 0 \Rightarrow$ Conservation of $\int_{\mathbb{R}^3} |v|^2 f(v) dv$ (energy)

$\phi_0(v) = 1, \phi_i(v) = v_i, \phi_4(v) = |v|^2$ are the collision invariants.

$$\mathcal{Q}(f) = 0 \Leftrightarrow f(v) = M(v) = \frac{\rho}{(2\pi T)^{3/2}} e^{-|v-\xi|^2/2T} \text{ is a Maxwellian.}$$

The 3D BGK equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = M - f.$$

$$\int_{\mathbb{R}^3} \phi_i(v) M(v) dv = \int_{\mathbb{R}^3} \phi_i(v) f(v) dv.$$

The 3D linearized form:

$$Lh = \Sigma_i (h, \phi_i)\phi_i - h, \; h \text{ is a pertubation.}$$

One-dimensional simplified BGK Model:
one collision invariant corresponding to the conservation of mass

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = (f, \phi_0)\phi_0 - f.$$

Partial integro-differential equation in 1D:

$$\frac{\partial f}{\partial t}(t,x,v) + v\frac{\partial f}{\partial x}(t,x,v) = -f(t,x,v) + \int_{\mathbb{R}} w(r) f(t,x,r)dr,$$

where w is the probability density function $w(v) = e^{-v^2}/\sqrt{\pi}$ and the unknown function $f(t,x,v)$ represents the molecular density function of a monotomic gas.

Goals:

1.) To derive the associated generalized Fourier transforms and Parseval's identity.

2.) To show that solutions decay to a subclass of the grossly determined solutions.

grossly determined solutions = determined by gross conditions (mass density, velocity, temperature).

Fourier transform and Parseval's identity.
Example: $L = -i\frac{d}{dx}$, $\sigma_{ess}(L) = \mathbb{R}$,

- $Lu = ku \Rightarrow e^{ikx}$ is a generalized eigenfunction correspoding to k.

- Rigged space (Gelfand triple): $\Phi \subset H \subset \Phi^*$.

$\Phi=$ 'space of test functions', $\Phi^* =$ 'space of corresponding distributions'.

- $(\mathfrak{F}f)(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-ikx}dx$, $f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\mathfrak{F}f)(k)e^{ikx}dk$

- Parseval's identity:

$\int_{\mathbb{R}} f(x)\overline{f(x)}dx = \int_{\mathbb{R}} (\mathfrak{F}f)(k)\overline{(\mathfrak{F}f)(k)}dk$.

- Generalizable to other operators L?

Associated Spectral Problem

Fourier transform:

$$\frac{\partial \hat{f}}{\partial t}(t,\xi,v) = -vi\xi \hat{f}(t,\xi,v) - \hat{f}(t,\xi,v) + \int_{\mathbb{R}} w(r)\hat{f}(t,\xi,r)dr.$$

$$\begin{gathered}(L_\xi g)(v) := (-vi\xi - 1)g(v) + \int_{\mathbb{R}} w(r)g(r)dr, \\ g \in \operatorname{dom}(L_\xi) = \{h \in L^2_w(\mathbb{R}; dv) : L_\xi g \in L^2_w(\mathbb{R}; dv)\}.\end{gathered}$$

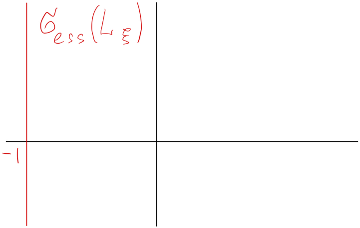
We also decompose L_ξ into the sum M and V, i.e.

$$\begin{aligned} L_\xi &:= M_\xi + V, \\ (M_\xi g)(v) &:= (-vi\xi - 1)g(v), \\ (Vg)(v) &:= (g, \mathbb{1})_{L^2_w} \mathbb{1}. \end{aligned}$$

Notice that $V^2 = V^* = V$.

Essential spectrum

$$\sigma_{ess}(L_\xi) = \sigma_{ess}(M_\xi) = \{\lambda \in \mathbb{C} : \lambda = -1 + i\omega, \omega \in \mathbb{R}\},\ \xi \neq 0,$$
$$\sigma_{ess}(L_0) = \sigma_{ess}(M_0) = \{-1\}.$$



What about $\sigma_d(L_\xi)$?

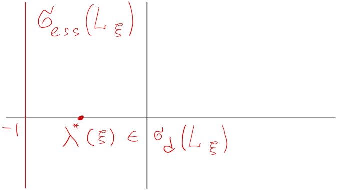
$(L_\xi - \lambda I) = (I + V(M_\xi - \lambda I)^{-1})(M_\xi - \lambda I)$.
Introduce the Birman-Schwinger operator
$K(\lambda, \xi) := I + V(M_\xi - \lambda I)^{-1}V = I + VR^0(\lambda, \xi)V \in \mathcal{B}(L^2_w(\mathbb{R}))$.

Discrete Spectrum:

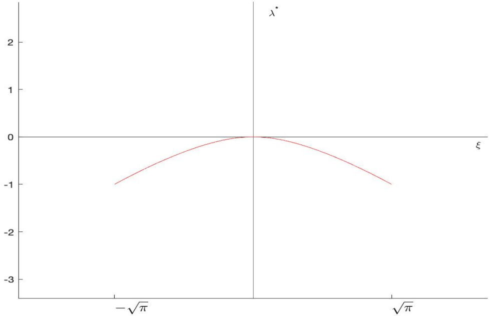
$$\omega(\lambda,\xi) = \det(K(\lambda,\xi)) = \det(1 + (R^0(\lambda,\xi)\mathbb{1}, \mathbb{1})_{L^2_w})$$
$$= 1 - \int_{\mathbb{R}} \frac{w(v)dv}{vi\xi + 1 + \lambda},\ \ w(v) = e^{-v^2}/\sqrt{\pi}.$$

Lemma

For any $\xi \in (-\sqrt{\pi}, \sqrt{\pi})$ there exists a unique $\lambda^*(\xi) \in (-1, 0]$ such that $\omega(\lambda^*(\xi), \xi) = 0$. Moreover, the multiplicity of such $\lambda^*(\xi)$ as a zero of $\omega(\cdot, \xi)$ is one. And if $\xi \in \mathbb{R} \setminus [-\sqrt{\pi}, \sqrt{\pi}]$, then $\omega(\cdot, \xi) \neq 0$.

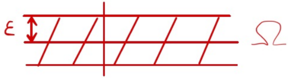


Moreover, $\lim_{\xi \to -\sqrt{\pi}^+} \lambda^*(\xi) = -1$ and $\lim_{\xi \to \sqrt{\pi}^-} \lambda^*(\xi) = -1$
The λ^* curve:



Spectral decomposition

Goal: spectral decomposition of $L_\xi = M_\xi + V$.
Challenge: generalized eigenfunctions cannot belong to $H = L^2_w(\mathbb{R})$.
Introduce a triple: $\Phi \subset H \subset \Phi^*$.



$\Omega = \Omega_\varepsilon = \{z \in \mathbb{C} : |\Im z| < \varepsilon\}, \ \varepsilon > 0.$

Definition (Φ = 'space of test functions')

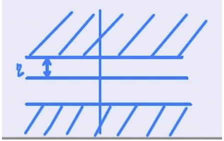
$\phi \in \Phi \Leftrightarrow z \to \phi(z)$ is holomorphic in Ω and for each $\varepsilon_1 \in [0, \varepsilon)$

$$\sup_{y \in [-\varepsilon_1, \varepsilon_1]} \int_{\mathbb{R}} |\phi(x+iy)|^2 w(x) dx < \infty.$$

We denote by Φ^* the space of semilinear continuous functionals over Φ.

Analytic representation of $\phi^* \in \Phi^*$

Each functional ϕ^* from Φ^* has a holomorphic representation $\phi^*(\cdot)$.



$\exists$ a complex-valued function $z \to \phi^*(z)$ which

- is holomorphic for $|\Im z| > \eta$ $(\varepsilon > \eta)$,
- satisfies the condition

$$\sup_{|y|>\eta} \int_{\mathbb{R}} |\phi^*(x+iy)|^2 w(x)dx < \infty,$$

- and

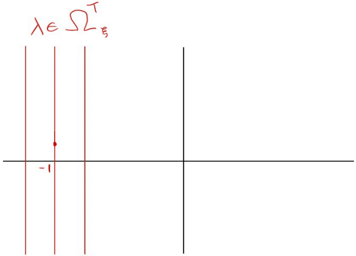
$$\langle \phi^*, \phi \rangle = \int_\gamma \phi^*(z)\overline{\phi(\overline{z})} w(z)dz,$$

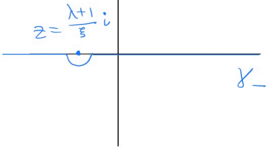
$$\int_\gamma := \int_{-\infty-i\gamma}^{\infty-i\gamma} - \int_{-\infty+i\gamma}^{\infty+i\gamma}$$

and γ is an arbitrary number in $[\eta, \varepsilon)$.

The resolvent extension of M_ξ through σ_{ess}

$$\langle R^0_\pm(\lambda,\xi)\phi,\psi\rangle = \int_{\gamma_\mp} \frac{\phi(z)\overline{\psi(\overline{z})}}{-i\xi z - 1 - \lambda} w(z)dz, \quad R^0_\pm(\lambda,\xi) : \Phi \to \Phi^*.$$





$\Omega^T_\xi = \{z \in \mathbb{C} : |\Re z + 1| < \varepsilon|\xi|\}, \quad \Pi_+ = \{z \in \mathbb{C} : \Re z > -1\}$
$\lambda \to (R^0(\lambda,\xi)\phi,\psi)_\pm$ from $\Pi_\pm$ to $\Pi_\pm \cup \Omega^T_\xi$.

The extensions of $\omega(\cdot,\xi)$ and $(L_\xi - \cdot)^{-1}$ through σ_{ess}

For any $\phi^* \in \Phi^*$ we define $V : \Phi^* \to L^2_w(\mathbb{R})$ as follows

$$V\phi^* = \langle \phi^*, \mathbb{1} \rangle \mathbb{1}.$$

The extensions of the the Birman-Schwinger operator

$$K_\pm(\lambda,\xi) = I + VR^0_\pm(\lambda,\xi)V$$

$$\omega_\pm(\lambda,\xi) = \det(K_\pm(\lambda,\xi)) = \det(1 + \langle R^0_\pm(\lambda,\xi)\mathbb{1}, \mathbb{1}\rangle) = 1 - \int_{\gamma_\mp} \frac{w(z)dz}{zi\xi + 1 + \lambda}.$$

The resolvent extension:

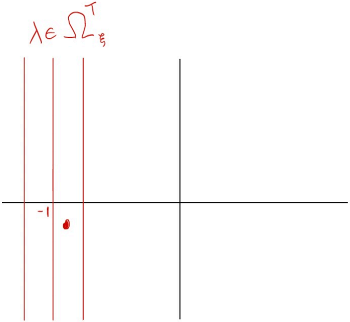
$$R_\pm(\lambda,\xi) = R^0_\pm(\lambda,\xi) - R^0_\pm(\lambda,\xi)VK^{-1}_\pm(\lambda,\xi)VR^0_\pm(\lambda,\xi)$$

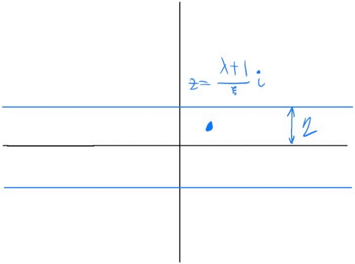
Generalized eigenfunctions of $M_\xi : \mathrm{dom}(M_\xi) \subset \Phi^* \to \Phi^*$

$\Omega_\xi^T = \{z \in \mathbb{C} : |\Re z + 1| < \varepsilon|\xi|\},\ \varepsilon > 0$. Let

$$\delta_\lambda^0(z) := \frac{1}{2\pi i} \frac{1}{z - \frac{i(\lambda+1)}{\xi}} w^{-1/2}(z),\ \lambda \in \Omega_\xi^T.$$

Note that $\delta_\lambda^0(\cdot) \in \Phi^*$ for $\eta > |\Re\lambda + 1|$.





$$\langle \delta^0_{-1-i\lambda\xi}, \phi\rangle = \overline{\phi(\overline{\lambda})}w^{1/2}(\lambda),\ \phi \in \Phi,\ \lambda \in \Omega.$$

Lemma

Let $\phi^* \in \mathrm{dom}(M_\xi)$ *and*

$$(M_\xi - \lambda)\phi^* = 0,\ \textit{where } M_\xi = -vi\xi - I.$$

If $|\Re\lambda + 1| < \varepsilon|\xi|$, *then* $\phi^* = C\delta^0_\lambda$, *and if* $|\Re\lambda + 1| \geq \varepsilon|\xi|$, *then* $\phi^* = 0$.

The generalized eigenfunctions of the operator M^*_ξ

$$\delta^{a0}_\lambda(z) := \frac{1}{2\pi i}\frac{1}{z + \frac{i(\lambda+1)}{\xi}}w^{-1/2}(z),\ \lambda \in \Omega^T_\xi$$

$$\langle \delta^{a0}_{-1+i\bar{\lambda}\xi}, \phi\rangle = \overline{\phi(\lambda)}w^{1/2}(\bar{\lambda}),\ \phi \in \Phi,\ \lambda \in \Omega.$$

Generalized eigenfunctions of $L_\xi : \mathrm{dom}(L_\xi) \subset \Phi^* \to \Phi^*$

We now determine the generalized eigenfunctions of the extended operator L_ξ.

Lemma

Fix $\xi \neq 0$. Let $\lambda \in \Omega_\xi^T$ and let $K_\pm^{-1}(\lambda, \xi)$ exist. If

$$(L_\xi - \lambda)\phi^* = 0, \;\; u^* \in \mathrm{dom}(L_\xi),$$

then $\phi^* = C\delta_\lambda^\pm$, where

$$\delta_\lambda^\pm = (1 - R_\pm^0(\lambda, \xi) V K_\pm^{-1}(\lambda, \xi) V)\delta_\lambda^0.$$

One can also show that if $\lambda \in \Omega_\xi^T$ and $K_\pm(\lambda, -\xi)$ are invertible, then

$$\delta_\lambda^{a\pm} = (1 - R_\pm^0(\lambda, -\xi) V K_\pm^{-1}(\lambda, -\xi) V)\delta_\lambda^{a0}$$

are the generalized eigenfunctions of the extended operator L_ξ^*.

Generalized Fourier transforms

Let $\xi \neq 0$, $\lambda \in \Omega$ and $\phi \in \Phi$. We also introduce the following transforms (the generalized Fourier transforms)

$$(\mathcal{U}_\xi \phi)(\lambda) := \frac{1}{w^{1/2}(\bar{\lambda})} \langle \phi, \delta^{+}_{-1-i\lambda\xi} \rangle,$$
$$(\mathcal{B}_\xi \phi)(\lambda) := \frac{1}{w^{1/2}(\lambda)} \langle \phi, \delta^{a-}_{-1+i\bar{\lambda}\xi} \rangle.$$

$$(\mathcal{U}_\xi L_\xi^* f)(\lambda) = (i\xi\lambda - 1)(\mathcal{U}_\xi f)(\lambda), \; (\mathcal{B}_\xi L_\xi f)(\lambda) = (-i\xi\lambda - 1)(\mathcal{B}_\xi f)(\lambda).$$

Theorem (S.,Zumbrun)

- *if* $\xi \in (-\sqrt{\pi}, 0) \cup (0, \sqrt{\pi})$, *then for any* $f, g \in H = L^2_w(\mathbb{R})$ *the following generalized Parseval's identity holds*

  $$(f,g)_{L^2_w} = \int_{\mathbb{R}} (\mathcal{B}_\xi f)(\lambda)\overline{(\mathcal{U}_\xi g)(\lambda)} w(\lambda) d\lambda + (P_{\lambda^*}(\xi) f, g)_{L^2_w},$$

  where $\lambda^*(\xi) \in (-1, 0)$ *is an isolated eigenvalue of* L_ξ *and* $P_{\lambda^*}(\xi)$ *is the Riesz projection corresponding to the simple eigenvalue* $\lambda^*(\xi)$ *of* L_ξ.
- *if* $\xi = \pm\sqrt{\pi}$, *then for any* $f, g \in H^1_w(\mathbb{R})$

  $$(f,g)_{L^2_w} = \int_{\mp} (\mathcal{B}_\xi f)(\lambda)\overline{(\mathcal{U}_\xi g)(\lambda)} w(\lambda) d\lambda + \langle P_{-1^\pm} f, g\rangle.$$

- *if* $\xi \notin [-\sqrt{\pi}, \sqrt{\pi}]$, *then for any* $f, g \in H = L^2_w(\mathbb{R})$

  $$(f,g)_{L^2_w} = \int_{\mathbb{R}} (\mathcal{B}_\xi f)(\lambda)\overline{(\mathcal{U}_\xi g)(\lambda)} w(\lambda) d\lambda.$$

We introduce

$$P(\xi) := \begin{cases} P_{\lambda^*}(\xi),\ \xi \in (-\sqrt{\pi}, \sqrt{\pi}), \\ P_{-1^\pm},\ \xi = \pm\sqrt{\pi}, \\ 0,\ \xi \notin [-\sqrt{\pi}, \sqrt{\pi}], \end{cases}$$

Theorem (S.,Zumbrun)

Let $f \in L^2(\mathbb{R}, H^1_w)$. Then the following eigenfunction expansion formula is valid:

$$f = \mathcal{U}^*_\xi \mathcal{B}_\xi f + P(\xi) f,$$

where the equality is understood in the weak sense (the H^{-1}_w-sense).

$$\partial_t \hat{f}(t,\xi,v) = (L_\xi \hat{f})(t,\xi,v).$$

$$\hat{f} = \mathcal{U}_\xi^* \mathcal{B}_\xi \hat{f} + P(\xi)\hat{f}.$$

Then the solution can be found in the form:

$$\hat{f}(t,\xi,v) = \mathcal{U}_\xi^* \big((\mathcal{B}_\xi \hat{f})(\xi,\lambda) \big) + P(\xi)\hat{f}.$$

Theorem (S.,Zumbrun)

Let $\hat{f}_0(\xi,v) := \hat{f}(0,\xi,v)$ *represent the Fourier transform of the initial distribution function of the gas and assume that* $\hat{f}_0 \in L^2(\mathbb{R}, H_w^1(\mathbb{R}))$*. Then the Cauchy problem has a unique solution and its Fourier transform is*

$$\hat{f}(t,\xi,v) = e^{-t}\mathcal{U}_\xi^* \big(e^{-i\xi\lambda t} (\mathcal{B}_\xi \hat{f}_0)(\xi,\lambda) \big) + e^{\lambda^*(\xi)t} P(\xi)\hat{f}_0,$$

where $\hat{f}(t,\cdot,\cdot)$ *belongs to the space* $L^2(\mathbb{R}, H_w^{-1}(\mathbb{R}))$*.*

Decay to Grossly Determined Solutions

Theorem (S.,Zumbrun)

Let f be the general solution with the initial distribution function f_0 such that $\hat{f}_0 \in L^2(\mathbb{R}, H^1_w(\mathbb{R}))$ and let $g(t,x,v) := \mathfrak{F}^{-1}(P(\xi)\hat{f})$. Then

$$\hat{g}(t,\xi,v) = \hat{\mu}(t,\xi)e_1 = e^{\lambda^*(\xi)t}\hat{\mu}_0(\xi)e_1,$$

where $\hat{\mu}(t,\xi) := \langle \hat{g}(t,\xi,\cdot), \mathbb{1}\rangle$ is the density function. Moreover,

$$\|f-g\|_{L^2(\mathbb{R},H^{-1}_w(\mathbb{R}))} \leq Ce^{-t}\|f_0\|_{L^2(\mathbb{R},H^1_w(\mathbb{R}))}.$$

If $\hat{f}_0 \in L^2(\mathbb{R}, L^2_w)$ and is compactly supported on $(-\sqrt{\pi}, \sqrt{\pi})$, then

$$\|f-g\|_{L^2(\mathbb{R},L^2_w(\mathbb{R}))} \leq Ce^{-t}\|f_0\|_{L^2(\mathbb{R},L^2_w(\mathbb{R}))}.$$