



UNIVERSITY OF  
LEICESTER

## Spectral analysis of the material-independent modes for the Helmholtz equation

Joint work with Anne-Sophie Bonnet-Ben Dhia and Christophe Hazard (ENSTA, Paris)

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Mathematical aspects of the physics with non-self-adjoint operators

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## Problem formulation

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We consider the operator  $\mathcal{L} : H^1(D) \rightarrow H^1(D)$  defined by

$$\mathcal{L}u = \nabla \cdot \int_D G(\cdot, y) \nabla u(y) dy,$$

- 
- $D \subset \mathbb{R}^d$ ,  $d = 2, 3$ , is bounded and not-necessarily simply connected.



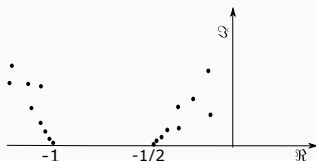
- $G$  is the Helmholtz Green function in free space, namely,  $\Delta G + \omega^2 G = \delta$  with Sommerfeld radiation conditions;  $\omega \in \mathbb{R}_+$  (frequency parameter).

$$G(x, y) = \begin{cases} -\frac{i}{4} H_0^{(1)}(\omega|x-y|) & \text{in } \mathbb{R}^2, \\ \frac{e^{i\omega|x-y|}}{4\pi|x-y|} & \text{in } \mathbb{R}^3. \end{cases}$$

$$\mathcal{L}u = \nabla \cdot \int_D G(\cdot, y) \nabla u(y) dy,$$

What is known [A. Kirsch, M. Costabel]:

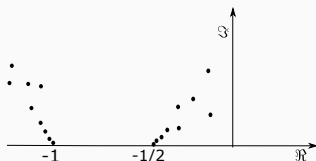
- $\mathcal{L} : H^1(D) \rightarrow H^1(D)$  is bounded, non-compact, **non-self-adjoint**, and dissipative.
- If  $D$  has boundary  $C^{1,\alpha}$ , ( $\alpha > 0$ ), then  $\sigma(\mathcal{L})$  is purely discrete and  $\sigma_{\text{ess}}(\mathcal{L}) = \{-1, -\frac{1}{2}\}$ :



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What we want to know:

- Are the root-vectors (generalised eigenfunctions) complete in  $H^1(D)$ ? i.e.

$$H^1(D) \stackrel{?}{=} \overline{\text{span}\{\text{root-vectors}\}}$$

- **Root-vectors** = the set of all  $v \in H^1(D)$  such that there exist non-negative  $m$  satisfying  $(\lambda - \mathcal{L})^m v = 0$ , where  $\lambda$  is an eigenvalue of  $\mathcal{L}$ .

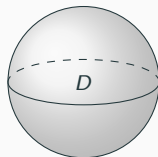
### Theorem (Bonnet-Ben Dhia, Hazard, R.)

The system of root vectors of  $\mathcal{L}$  is complete in  $H^1(D)$  provided that:

- $D \subset \mathbb{R}^2$  has a  $C^k$  boundary with  $k > 6$ .



- $D \subset \mathbb{R}^3$  has  $C^\infty$  boundary and is strictly convex.



### Remark

What we really need is that negative eigenvalues  $\lambda_k^-$  of the Neumann-Poincaré operator  $\mathcal{K}^*$  have an  $o(k^{-1})$  decay  $\Rightarrow$  conditions on the geometry of  $\partial D$  [Y.

Miyanishi, H. Kang, K. Ando, G. Rozenblum, and others]

$$\mathcal{K}^* \varphi(x) := \int_{\partial D} \frac{\partial G_0(x, y)}{\partial \nu(x)} \varphi(y) d\sigma(y), \quad x \in \partial D.$$

1. Motivation
2. The zero-frequency (self-adjoint) case
3. Proof of the main result (completeness)
4. Extensions
5. Towards a (Riesz) basis

## 1. Motivation

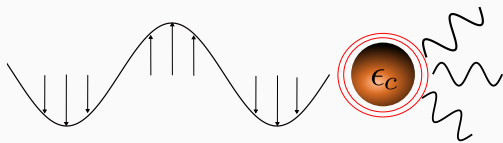
2. The zero-frequency (self-adjoint) case

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## Motivation: electromagnetic scattering problems in nanophotonics



### Plasmonic nanoparticles



$$\text{Re}(\epsilon_c) < 0$$

- Metallic particles, typically made of gold or silver

### High-index nanoparticles



$$|\epsilon_c| \gg 1$$

- Dielectric particles, typically made of silicon

## Motivation: scattering problem and volume integral equation



A diagram showing a grey, irregularly shaped region representing a scattering domain  $D$ . Inside the region, the text  $\varepsilon(x) = \varepsilon_c$  is written. To the right of the region, the text  $\varepsilon(x) = 1$  is written, indicating the permittivity of the surrounding medium.

### Scattering problem (T.M polarization)

Find  $u \in H_{\text{loc}}^1(\mathbb{R}^d)$  such that, for  $\varepsilon = \varepsilon_c \mathbb{1}(D) + \mathbb{1}(\mathbb{R}^d \setminus D)$ ,

$$\nabla \left( \frac{1}{\varepsilon} \nabla u \right) + \omega^2 u = f,$$

$$\lim_{|x| \rightarrow \infty} \sqrt{|x|} \left( \frac{\partial u}{\partial |x|} - i\omega^2 u \right) = 0,$$

is equivalent to (A. Kirsch)

### Lippmann-Schwinger type equation

Find  $u \in H^1(D)$  such that, for  $\varepsilon_c$  given

$$u + \frac{1 - \varepsilon_c}{\varepsilon_c} \mathcal{L}u = \hat{f}.$$

$\Rightarrow$  Study of Fredholm properties of  $\mathcal{L}$  and **numerical solutions**

$\Rightarrow$  **modal expansion approach**:  $u = \sum \alpha_k u_k$  (D. Bergman, Y. Sivan, C.H. Chen, C. Forestiere, V. A. Markel)

## Motivation: eigenvalue problems


$$\varepsilon(x) = \varepsilon \quad \varepsilon(x) = 1$$

### PDE eigenvalue problem

Find  $u \in H_{\text{loc}}^1(\mathbb{R}^d)$  and  $\varepsilon = \varepsilon \mathbf{1}(D) + \mathbf{1}(\mathbb{R}^d \setminus D)$ ,

$$\nabla \left( \frac{1}{\varepsilon} \nabla u \right) + \omega^2 u = 0,$$

$$\lim_{|x| \rightarrow \infty} \sqrt{|x|} \left( \frac{\partial u}{\partial |x|} - i\omega^2 u \right) = 0,$$

is equivalent to

### Integral operator eigenvalue problem

Find non-trivial  $u \in H^1(D)$  and  $\varepsilon$  given

$$\mathcal{L}u = \frac{\varepsilon}{1-\varepsilon} u$$

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## Zero-frequency case: spectral properties

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Consider the zero-frequency version of  $\mathcal{L}$ ,

$$\mathcal{L}_0 u = \nabla \cdot \int_D G_0(\cdot, y) \nabla u(y) dy,$$

where  $G_0$  is the Green function for the Laplacian.

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### Remark

$\mathcal{L}$  is a compact perturbation of  $\mathcal{L}_0$ .

Integration by parts:

$$\mathcal{L}_0 u = -u + \underbrace{\int_{\partial D} \frac{\partial G_0(\cdot, y)}{\partial \nu} u d\sigma(y)}_{\mathcal{D}_0 \text{ double layer potential}}$$

$\sigma(\mathcal{D}_0) = \{\text{harmonic functions with real eigenvalues accumulating at } \frac{1}{2}\} \cup \{\text{functions that vanish on } \partial D \text{ (Dirichlet functions) with eigenvalue } 0\}$ .

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Let  $H_0^1 := \{\text{Dirichlet functions}\}$  and  $U := \{\text{harmonic functions}\}$ . We have

$$H^1(D) = \underbrace{H_0^1 \oplus \tilde{U}}_{:=\tilde{H}^1} \oplus \{c\}; \quad \text{for } \langle u, v \rangle_{\tilde{H}^1} := \int_D \nabla u \cdot \nabla v dy,$$

$\mathcal{L}_0$  is complete and invariant in each of these subspaces

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## Proof of the main result

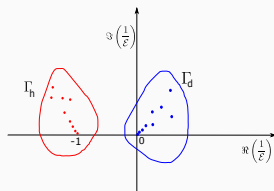
We can do similarly for  $\mathcal{L}$ :

$$H^1(D) = \underbrace{H_h + H_d}_{\tilde{H}^1} + \{c\}$$

**Riesz projection.** Consider three contours  $\Gamma_h, \Gamma_d$  lying on the resolvent set of  $\mathcal{L}$  and enclosing disjointly the totality of its spectrum. Then,

$$\Pi_h = -\frac{1}{2\pi i} \int_{\Gamma_h} (\lambda - \mathcal{L})^{-1} d\lambda,$$

(resp.  $\Pi_d$ ) projects  $H^1(D)$  into a subspace  $H_h$  (resp.  $H_d$ ) wherein  $\mathcal{L}$  is invariant.



**Lemma:** The kernel of  $\mathcal{L}$  is the constant functions

**Main tool of our proof:** Spectral theory for Schatten class operators.

**Theorem (M. G. Krein (1960))**

*The system of root vectors of a compact operator  $\mathcal{A} : H \rightarrow H$ , where  $\text{Im } \mathcal{A}$  is definite and 1-Schatten, is complete in  $H$  provided that*

$$\lim_{\rho \rightarrow \infty} \frac{n_+(\rho; \text{Re } \mathcal{A})}{\rho} = 0 \quad \text{or} \quad \lim_{\rho \rightarrow \infty} \frac{n_-(\rho; \text{Re } \mathcal{A})}{\rho} = 0, \quad (1)$$

*where  $n_{\pm}(\rho; \text{Re } \mathcal{A})$  denotes the characteristic numbers of  $\mathcal{A}$  (inverse of eigenvalues) in the interval  $[0, \rho]$  and  $[-\rho, 0]$ , respectively.*

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I. C. Gohberg and M. G. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators (Translations of Mathematical Monographs)-American Mathematical Society (1969)

### Lemma

There exist an operator  $\tilde{\mathcal{L}} : \tilde{H}^1 \rightarrow \tilde{H}^1$  with eigenvalues  $1/\mathcal{E}$  and the same root-vectors as  $\mathcal{L}$  (excepting constant functions). Furthermore,  $\tilde{\mathcal{L}} = \mathcal{R}' + \mathcal{R}'' + i\mathcal{H}$ , with

$$\langle \mathcal{R}' u, v \rangle_{\tilde{H}^1} = \omega^2 \int_D u \bar{v} \, dy,$$

$$\langle \mathcal{R}'' u, v \rangle_{\tilde{H}^1} = \int_{\partial D} \operatorname{Re} DtN^+ u \bar{v} \, d\sigma(y),$$

$$\langle \mathcal{H} u, v \rangle_{\tilde{H}^1} = \int_{\partial D} \operatorname{Im} DtN^+ u \bar{v} \, d\sigma(y),$$

where  $DtN^+$  is the exterior Dirichlet-to-Neumann map, and  $\operatorname{Re} DtN^+$  and  $\operatorname{Im} DtN^+$  denote the real and imaginary parts of  $DtN^+$  in  $L^2(\partial D)$ .

### Lemma

The operators  $\mathcal{R}'$ ,  $\mathcal{R}''$ , and  $\mathcal{H}$ , defined above have the following properties:

1.  $\mathcal{R}'$  is Hilbert-Schmidt;
2.  $\mathcal{R}''$  is invariant under  $\tilde{H}_H$  and  $\mathcal{R}''(H_D) = 0$ . Moreover, the restriction of  $\mathcal{R}'' + I$  to  $\tilde{H}_H$  is trace class;
3.  $\mathcal{H}$  is positive and trace class.

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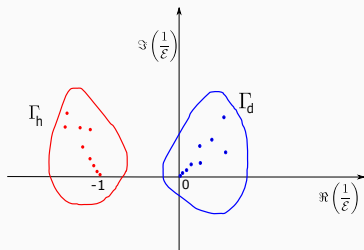
Let  $u = \mathcal{S}\varphi$  and  $v = \mathcal{S}\psi$ , where  $\mathcal{S}$  denotes the Helmholtz single layer potential. From the properties of layer potentials we have

$$\operatorname{Re} DtN^+(\mathcal{S}(\varphi)) = \left(\frac{1}{2}\mathcal{I} + \mathcal{K}^*\right)\varphi - \mathcal{K}^{(1)}\varphi$$

where  $\mathcal{K}^*$  is the Neumann-Poincaré operator and  $\mathcal{K}_D^{(1)}$  is trace-class.

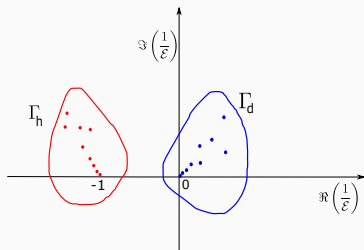
$$\begin{aligned}\langle \mathcal{R}''u, v \rangle_{\tilde{H}^1} &= \int_{\partial D} \left(\frac{1}{2}\mathcal{I} + \mathcal{K}^*\right)\varphi \mathcal{S}\bar{\psi} \, d\sigma(y) - \int_{\partial D} \mathcal{K}^{(1)}\varphi \mathcal{S}\bar{\psi} \, d\sigma(y) \\ &= \left\langle -\left(\frac{1}{2}\mathcal{I} + \mathcal{K}^*\right)\varphi, \psi \right\rangle_{\hat{H}^{-\frac{1}{2}}} + \left\langle \mathcal{K}^{(1)}\varphi, \psi \right\rangle_{\hat{H}^{-\frac{1}{2}}}\end{aligned}$$

## Proof of the main result



Define  $\tilde{\mathcal{L}}_h : H_h \rightarrow H_h$  and  $\tilde{\mathcal{L}}_d : H_d \rightarrow H_d$ , the restrictions of  $\tilde{\mathcal{L}}$  to  $H_h$  and  $H_d$ , respectively  $\Rightarrow$  **we need to show that both  $\tilde{\mathcal{L}}_h$  and  $\tilde{\mathcal{L}}_d$  are complete**

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What are  $H_h$  and  $H_d$ ?

$$H_h = \Pi_h H^1(D); \quad \Pi_h = -\frac{1}{2\pi i} \int_{\Gamma_h} (\lambda - \tilde{\mathcal{L}})^{-1} d\lambda,$$

$$H_d = \Pi_d H^1(D); \quad \Pi_d = -\frac{1}{2\pi i} \int_{\Gamma_d} (\lambda - \tilde{\mathcal{L}})^{-1} d\lambda,$$

Useful identity

$$(A + B - \lambda)^{-1} = (A - \lambda)^{-1} + (A + B - \lambda)^{-1} B (A - \lambda)^{-1}$$

## Proof of the main result

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$$(A + B - \lambda)^{-1} = (A - \lambda)^{-1} + (A + B - \lambda)^{-1} B (A - \lambda)^{-1}$$

$$\Pi_h = \underbrace{\int_{\Gamma_h} (\lambda - \mathcal{R}'')^{-1} d\lambda}_{\Pi_{\tilde{U}}} + \underbrace{\int_{\Gamma_h} (\lambda - \tilde{\mathcal{L}})^{-1} (\mathcal{R}' + i\mathcal{H}) (\lambda - \mathcal{R}'')^{-1} d\lambda}_{=:\mathcal{C}_h}$$

$$\Pi_d = \underbrace{\int_{\Gamma_d} (\lambda - \mathcal{R}'')^{-1} d\lambda}_{\Pi_{H_0^1}} + \underbrace{\int_{\Gamma_d} (\lambda - \tilde{\mathcal{L}})^{-1} (\mathcal{R}' + i\mathcal{H}) (\lambda - \mathcal{R}'')^{-1} d\lambda}_{=:\mathcal{C}_d}$$

## Proof of the main result: completeness of $\tilde{\mathcal{L}}_h$ and $\tilde{\mathcal{L}}_d$

### Lemma

The system of root vectors of  $\tilde{\mathcal{L}}_h$  is complete in  $H_h$ .

### Lemma

The system of root vectors of  $\tilde{\mathcal{L}}_d$  is complete in  $H_d$ .

### Proof.

- We check the requirements of the completeness theorem for Schatten operators and show that

$$\lim_{\rho \rightarrow \infty} \frac{n_-(\rho; \operatorname{Re} \tilde{\mathcal{L}}_d)}{\rho} = 0,$$

- $\operatorname{Re} \tilde{\mathcal{L}}_d = \Pi_d \mathcal{R}' \Pi_d - \Pi_d \mathcal{R}'' \Pi_d$
- $|\eta_k^-(\operatorname{Re} \tilde{\mathcal{L}}_d)| \leq \langle |\Pi_d \mathcal{R}'' \Pi_d \phi_k, \phi_k\rangle_{\tilde{H}^1} |$ .
- Using  $\Pi_d = \Pi_{H_0^1} + \mathcal{C}_d$  we have

$$\begin{aligned} \langle \mathcal{R}'' \Pi_d \phi_k, \Pi_d \phi_k \rangle_{\tilde{H}^1} &= \langle \mathcal{R}'' \mathcal{C}_h \phi_k, \mathcal{C}_h \phi_k \rangle_{\tilde{H}^1} \\ &\leq \|\mathcal{R}''\| \|\mathcal{C}_h \phi_k\|_{\tilde{H}^1}^2 \\ &\leq \|\mathcal{R}''\| \max_{\lambda \in \mathbb{C}} \left\| \int_{\Gamma_d} (\lambda - \tilde{\mathcal{L}})^{-1} (\mathcal{R}' + i\mathcal{H}) (\lambda - \mathcal{R}'')^{-1} \phi_k d\lambda \right\|_{\tilde{H}^1}^2 \\ &\leq C \|\mathcal{R}' + i\mathcal{H}\| \|\mathcal{Q} \phi_k\|_{\tilde{H}^1}^2 \end{aligned}$$

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## Extensions: Perturbations to the Green function

1. Many problems where the Green function can be written as

$$G_p = G + P$$

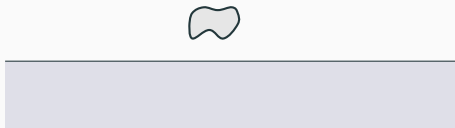
where  $P$  is smooth.

2. One can define  $\tilde{\mathcal{L}}_p = \mathcal{R}' + \mathcal{R}''_p + i\mathcal{H}_p$  with  $\mathcal{R}''_p = \mathcal{R}'' + \mathcal{P}_1$  and  $\mathcal{H}_p = \mathcal{H} + \mathcal{P}_2$ .  
If  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are both trace class, then all the previous holds.

### Periodic gratings



### Half penetrable space



## The classical Lippmann-Schwinger operator

Consider  $\mathcal{M} : L^2(D) \rightarrow L^2(D)$ , defined by

$$\mathcal{M}u = \omega^2 \int_D u(y) G(\cdot, y) dy$$

Find  $u \in L^2_{\text{loc}}(\mathbb{R}^d)$  such that, for  $\hat{\mu} = \mu_c \mathbf{1}(D) + \mathbf{1}(\mathbb{R}^d \setminus D)$ ,

$$\Delta u + \omega^2 \hat{\mu} u = f,$$

$$\lim_{|x| \rightarrow \infty} \sqrt{|x|} \left( \frac{\partial u}{\partial |x|} - i\omega^2 u \right) = 0$$

is equivalent to

Find  $u \in H^1(D)$  such that, for  $\mu_c$  given

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Find  $u \in H^1(D)$  such that, for  $\mu_c$  given

$$u + \frac{1 - \mu_c}{\mu_c} \mathcal{M}u = \hat{f}.$$

**Completeness:** we can write

$$\mathcal{M}u = \mathcal{M}_1 u + \mathcal{M}_0 u.$$

where  $\mathcal{M}_1$  is trace class and  $\mathcal{M}_0$  (the Newtonian potential) is positive self-adjoint .

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**Riesz basis:**

- (i) Root-vectors are **complete**
- (ii) for every sequence  $\alpha_n \in l^2$

$$c \sum_n |\alpha_n|^2 \leq \left\| \sum_n \alpha_n u_n \right\|^2 \leq C \sum_n |\alpha_n|^2$$

**Theorem (I. M. Glazmann (1958))**

*A set of eigenvectors  $u_k$ , with eigenvalues  $\lambda_k$ , of a dissipative operator form a Riesz basis of its linear hull if*

$$\sum_{k \neq m} \frac{\operatorname{Im} \{ \lambda_m \} \operatorname{Im} \{ \lambda_k \}}{|\lambda_m - \lambda_k|} < \infty.$$

## Towards a Riesz basis: the disk

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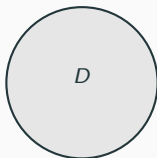
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The disk:



Explicit asymptotic behaviour of eigenvalues for the disk  $\Rightarrow$  can prove Riesz basis.

Thank you for your attention