# The space of Hardy-weights for quasilinear equations: Maz'ya-type characterization and sufficient conditions for existence of minimizers

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# The classical Hardy inequality

**Theorem (Hardy (1925))**

*For $1 < p < \infty$ the following inequality holds*

$$\left(\frac{p-1}{p}\right)^p \int_0^\infty \frac{|\varphi(x)|^p}{|x|^p}\,\mathrm{d}x \leq \int_0^\infty |\varphi'(x)|^p\,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\mathbb{R}_+).$$

*Moreover, $\left(\frac{p-1}{p}\right)^p$ is the best constant, and equality holds iff $\varphi = 0$.*

**Theorem (Multidimensional Hardy inequality)**

*For $1 < p < \infty$ the following inequality holds*

$$\left|\frac{p-N}{p}\right|^p \int_{\mathbb{R}^N\setminus\{0\}} \frac{|\varphi(x)|^p}{|x|^p}\,\mathrm{d}x \leq \int_{\mathbb{R}^N\setminus\{0\}} |\nabla\varphi|^p\,\mathrm{d}x \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^N\setminus\{0\}).$$

*Moreover, $\left|\frac{p-N}{p}\right|^p$ is the best constant, and equality holds iff $\varphi = 0$.*

# The setting

We consider a nonnegative energy functional of the form

$$Q_{p,A,V}[\varphi] \triangleq \int_\Omega \left(|\nabla\varphi|_A^p + V|\varphi|^p\right) \mathrm{d}x \geq 0 \qquad \forall \phi \in W^{1,p}(\Omega) \cap C_c(\Omega),$$

and its associated Euler-Lagrange equation

$$Q'_{p,A,V}(u) \triangleq -\operatorname{div}(|\nabla u|_A^{p-2} A\nabla u) + V|u|^{p-2}u = 0 \qquad \text{in } \Omega.$$

Here $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a domain, $1 < p < \infty$, $A \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^{N\times N})$ is a symmetric and locally uniformly positive definite matrix function,

$$|\xi|_A^2 \triangleq \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \qquad x \in \Omega,\ \xi \in \mathbb{R}^N,$$

and $V$ is a real valued potential in a certain local Morrey space $M^q_{\mathrm{loc}}(p;\Omega)$.

The operator $\Delta_{p,A}[u] \triangleq \operatorname{div}(|\nabla u|_A^{p-2} A\nabla u)$ is called the $(p, A)$-Laplacian.

# Agmon-Allegretto-Piepenbrink-type (AAP)-theorem

**Theorem (YP-Psaradakis (2016))**

*$Q_{p,A,V} \geq 0$ on $W^{1,p}(\Omega) \cap C_c(\Omega)$ iff the equation $Q'_{p,A,V}(u) = 0$ in $\Omega$ admits a positive weak solution (or positive supersolution) in $W^{1,p}_{\mathrm{loc}}(\Omega)$.*

# $\mathcal{H}(\Omega)$ the space of Hardy-weights for $Q_{p,A,V}$

**First aim:** Characterize the space $\mathcal{H}(\Omega) = \mathcal{H}(p, A, V; \Omega)$ of all Hardy-weights for $Q_{p,A,V}$, i.e., functions $g \in L^1_{\mathrm{loc}}(\Omega)$ s.t. the following Hardy-type inequality holds:

$$C \int_\Omega |g||\phi|^p \,\mathrm{d}x \leq Q_{p,A,V}(\phi) \qquad \forall \phi \in W^{1,p}(\Omega) \cap C_c(\Omega) \qquad \text{(HI)}$$

for some $C > 0$. We denote the best constant in (HI) by $\mathbb{S}_g(\Omega)$, i.e.,

$$\mathbb{S}_g(\Omega) = \inf\{Q_{p,A,V}(\phi) \mid \phi \in W^{1,p}(\Omega) \cap C_c(\Omega)\,, \int_\Omega |g||\phi|^p \,\mathrm{d}x = 1\}.$$

$\mathbb{S}_g(\Omega)$ is also called the generalized principal eigenvalue of $Q_{p,A,V}$ with respect to the weight $|g|$. Clearly, $\|g\|_{\mathcal{H}(\Omega)} \triangleq \mathbb{S}_g(\Omega)^{-1}$ is a norm on $\mathcal{H}(\Omega)$.

Suppose that $Q_{p,A,V} \geq 0$ in $\Omega$. If $\mathcal{H}(\Omega) = \{0\}$, then $Q_{p,A,V}$ is said to be critical in $\Omega$, otherwise, $Q_{p,A,V}$ is subcritical in $\Omega$. If $Q_{p,A,V} \not\geq 0$ in $\Omega$, then $Q_{p,A,V}$ is said to be supercritical in $\Omega$.

# $Q_{p,A,V}$-capacity

## Definition

Let $\mathbf{u}$ be a positive solution of $Q'_{p,A,V}(u) = 0$ in $\Omega$. For a compact set $F \Subset \Omega$, the $Q_{p,A,V}$-capacity of $F$ with respect to $(\mathbf{u}, \Omega)$ is defined by

$$\mathrm{Cap}_{\mathbf{u}}(F,\Omega) \triangleq \inf\{Q_{p,A,V}(\phi) \mid \phi \in W^{1,p}(\Omega) \cap C_c(\Omega), \phi \geq \mathbf{u} \text{ on } F\}.$$

We introduce a norm on the subspace $H(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$ of all functions $g$ satisfying:

$$\|g\|_{\mathcal{H}^{\mathbf{u}}(\Omega)} \triangleq \sup\left\{\frac{\int_F |g||\mathbf{u}|^p \,\mathrm{d}x}{\mathrm{Cap}_{\mathbf{u}}(F,\Omega)} \mid F \Subset \Omega \text{ compact s.t. } \mathrm{Cap}_{\mathbf{u}}(F,\Omega) \neq 0\right\} < \infty.$$

In fact, the above space is a Banach function space.

# Characterization of Hardy-weights: $\mathcal{H}(\Omega) = H(\Omega)$

$$\|g\|_{\mathcal{H}^{\mathbf{u}}(\Omega)} = \sup\left\{ \frac{\int_F |g||\mathbf{u}|^p \, \mathrm{d}x}{\mathrm{Cap}_{\mathbf{u}}(F,\Omega)} \mid F \Subset \Omega \text{ compact s.t. } \mathrm{Cap}_{\mathbf{u}}(F,\Omega) \neq 0 \right\}.$$

## Theorem (YP - Das (2022))

*Let $p \in (1,\infty)$, and $\mathbf{u}$ be a positive solution of $Q'_{p,A,V}(v) = 0$ in $\Omega$. Let $g \in L^1_{\mathrm{loc}}(\Omega)$, then $\|g\|_{\mathcal{H}^{\mathbf{u}}(\Omega)} < \infty$ iff the Hardy-type inequality*

$$C \int_\Omega |g||\phi|^p \, \mathrm{d}x \leq Q_{p,A,V}(\phi) \qquad \forall \phi \in W^{1,p}(\Omega) \cap C_c(\Omega) \qquad \text{(HI)}$$

*holds with $C > 0$. Moreover,*

$$\|g\|_{\mathcal{H}^{\mathbf{u}}(\Omega)} \leq \|g\|_{\mathcal{H}(\Omega)} = \mathbb{S}_g(\Omega)^{-1} \leq C_p \|g\|_{\mathcal{H}^{\mathbf{u}}(\Omega)} \quad \forall g \in \mathcal{H}(\Omega),$$

*where $C_p$ depends only on $p$. Thus, up to the equivalence relation of norms, the norm $\|\cdot\|_{\mathcal{H}^{\mathbf{u}}(\Omega)}$ is independent of the positive solution $\mathbf{u}$ of $Q'_{p,A,V}(v) = 0$ in $\Omega$.*

### Definition (Beppo Levi space)

The generalized Beppo Levi space $\mathcal{D}^{1,p}_{A,V^+}(\Omega)$ for $Q_{p,A,V}$ is the completion of $W^{1,p}(\Omega) \cap C_c(\Omega)$ with respect to the norm

$$\|\phi\|_{\mathcal{D}^{1,p}_{A,V^+}(\Omega)} \triangleq \left( \| \, |\nabla \phi|_A \, \|^p_{L^p(\Omega)} + \|\phi\|^p_{L^p(\Omega, V^+ \, \mathrm{d}x)} \right)^{1/p} .$$

For $g \in \mathcal{H}(\Omega)$, consider the generalized principal eigenvalue $\mathbb{S}_g(\Omega)$. In fact,

$$\mathbb{S}_g(\Omega) = \inf\{ Q_{p,A,V}(\phi) \mid \phi \in \mathcal{D}^{1,p}_{A,V^+}(\Omega) \, , \int_\Omega |g||\phi|^p \, \mathrm{d}x = 1 \}$$

(instead of $\phi \in W^{1,p}(\Omega) \cap C_c(\Omega)$).
We say that the generalized principal eigenvalue $\mathbb{S}_g(\Omega)$ is attained if

$$\mathbb{S}_g(\Omega) \text{ is attained in } \mathcal{D}^{1,p}_{A,V^+}(\Omega).$$

# Sufficient condition for attainment of the best constant

Let $\mathcal{H}_0(\Omega) \triangleq \overline{\mathcal{H}(\Omega) \cap L^\infty_c(\Omega)}^{\|\cdot\|_{\mathcal{H}(\Omega)}}$.

**Theorem (For the case $V = V^+$)**

*If $V = V^+$ and $g \in \mathcal{H}_0(\Omega)$, then the functional $T_g : \mathcal{D}^{1,p}_{A,V^+}(\Omega) \to \mathbb{R}$:*

$$T_g(\phi) \triangleq \int_\Omega |g||\phi|^p \,\mathrm{d}x$$

*is compact on $\mathcal{D}^{1,p}_{A,V^+}(\Omega)$. In particular, $\mathbb{S}_g(\Omega, V^+)$ is attained in $\mathcal{D}^{1,p}_{A,V^+}(\Omega)$.*

Since $\mathcal{H}_0(p, A, V; \Omega) \subset \mathcal{H}_0(p, A, V^+; \Omega)$, it follows that if $g \in \mathcal{H}_0(p, A, V; \Omega)$, then $T_g$ is compact in $\mathcal{D}^{1,p}_{A,V^+}(\Omega)$.

# Attainment of best constant I (spectral gap condition I)

**Theorem**

*Assume that* $A \in C^\gamma$, $g \in \mathcal{H}(\Omega) \cap \mathcal{M}^q_{\mathrm{loc}}(p;\Omega)$, *and*

$$\mathbb{S}_g(\Omega) < \mathbb{S}_g^{\overline{\infty}}(\Omega) \triangleq \sup\{K \Subset \Omega \mid \mathbb{S}_g(\Omega \setminus K)\}.$$

*Assume that* $\int_{\Omega\setminus K_1} V^- G^p \,\mathrm{d}x < \infty$, *where* $G$ *is a positive solution of the equation* $Q'_{p,A,V-\mathbb{S}_g(\Omega)|g|}[u] = 0$ *in* $\Omega \setminus K$ *of minimal growth in a neighborhood of infinity in* $\Omega$, *where* $K \Subset K_1 \Subset \Omega$.
*Then,* $\mathbb{S}_g(\Omega)$ *is attained in* $\mathcal{D}^{1,p}_{A,V^+}(\Omega)$.

Proof's outline: If $\mathbb{S}_g(\Omega) < \mathbb{S}_g^{\overline{\infty}}(\Omega)$, then $Q_{p,A,V-\mathbb{S}_g(\Omega)|g|}$ is critical in $\Omega$, with a ground state $\Phi$ and a null-sequence $0 \le \phi_n \le \Phi$. By assumption and the nnecessary condition for being Hardy-weight[KP], $\int_\Omega (V^- + |g|)\Phi^p \,\mathrm{d}x < \infty$, $\hat{\phi}_n \rightharpoonup \Phi$ in $\mathcal{D}^{1,p}_{A,V^+}(\Omega)$, and we may pass to the limit in the identity:

$$\int_\Omega |\nabla \phi_n|_A^p \mathrm{d}x + \int_\Omega V_+ |\phi_n|_A^p \mathrm{d}x = Q_{p,A,V-\mathbb{S}_g(\Omega)|g|}(\phi_n) + \int_\Omega V_- \phi_n^p \mathrm{d}x + \mathbb{S}_g(\Omega)\int_\Omega |g|\phi_n^p \mathrm{d}x.$$

# Attainment of best constant II (spectral gap condition II)

For $x \in \overline{\Omega}$ and $g \in \mathcal{H}(\Omega)$, define the Hardy constant of $g$ at $x$ by

$$\mathbb{S}_g(x,\Omega) \triangleq \liminf_{r\to 0}\left\{Q_{p,A,V}(\phi) \mid \phi \in \mathcal{D}^{1,p}_{A,V^+}(\Omega\cap B_r(x)), \int_{\Omega\cap B_r(x)} |g||\phi|^p \mathrm{d}x = 1\right\},$$

and let $\Sigma_g \triangleq \{x \in \overline{\Omega} \mid \mathbb{S}_g(x,\Omega) < \infty\}$, and $\mathbb{S}^*_g(\Omega) \triangleq \inf_{x\in\overline{\Omega}} \mathbb{S}_g(x,\Omega)$.

**Theorem**

*Let $\Omega$ be a bounded domain, $V \in \mathcal{M}^q_{\mathrm{loc}}(p;\Omega)$ s.t. $V^- \in \mathcal{H}_0(p,A,V^+;\Omega)$.*
*Suppose that $g \in \mathcal{H}(p,A,V;\Omega)$ s.t. $\left|\overline{\Sigma_g}\right| = 0$.*
*If $\mathbb{S}_g(\Omega) < \mathbb{S}^*_g(\Omega)$, then $\mathbb{S}_g(\Omega)$ is attained in $\mathcal{D}^{1,p}_{A,V^+}(\Omega)$.*

The proof relies on concentration compactness arguments.

# Conclusion remark

All the above results presented here have been recently extended to the case of the Finsler Laplacian plus potential

$$Q'_{p,\mathcal{A},V}(u) \triangleq -\operatorname{div}(\mathcal{A}(x,\nabla u)) + V|u|^{p-2}u = 0 \qquad \text{in } \Omega.$$

and

$$Q_{p,\mathcal{A},V}[\varphi] \triangleq \int_\Omega \Big(\mathcal{A}(x,\nabla\varphi)\cdot\nabla\varphi) + V|\varphi|^p\Big)\,\mathrm{d}x \geq 0 \ \ \forall\phi \in W^{1,p}(\Omega)\cap C_c(\Omega),$$

by my student Yongjun Hou under some restrictions.

Thank you for your attention!