Lieb-Thirring type inequalities for multidimensional Schrödinger operators with complex-valued potentials

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Abstract

The purpose of this research is to investigate a conjecture that was stated by Demuth, Hansmann and Katriel in 2013. We study a possible generalization of Lieb-Thirring type inequalities for eigenvalues of non-selfadjoint Schrödinger operators with complex-valued potentials, acting on $L^2(\mathbb{R}^d)$ where $d \geq 2$. In particular, we find the asymptotic behavior for the discrete spectra of Schrödinger operators with a one-parameter family of rapidly decaying complex-valued potentials and present a disproof of this conjecture. This is a joint work with S. Bögli (Durham) and F. Štampach (Prague).

Introduction

A Schrödinger operator $H := -\Delta + V$ in $L^2(\mathbb{R}^d)$, where $d \geq 1$, with a potential V acts like $Hf = -\Delta f + Vf$, for $f \in \mathcal{D}(H) \subset L^2(\mathbb{R}^d)$.

$$\gamma \geq 1/2 \text{ if } d = 1, \gamma > 0 \text{ if } d = 2 \text{ and } \gamma \geq 0 \text{ if } d \geq 3. \qquad (1)$$

Note: If $V \in L^{\gamma+d/2}(\mathbb{R}^d)$ satisfies (1), then $\sigma_{ess}(H) = [0,\infty)$. Define $\sigma_d(H) := \sigma(H) \backslash \sigma_{ess}(H)$. In [5], the Lieb-Thirring (L-T) type inequality states that there exists $(\gamma, d$-dependent) $C_{\gamma,d} > 0$ such that for all real-valued $V \in L^{\gamma+d/2}(\mathbb{R}^d)$,

$$\sum_{\lambda\in\sigma_d(H)} |\lambda|^\gamma \leq C_{\gamma,d} ||V||^{\gamma+d/2}_{L^{\gamma+d/2}}.$$

For $\gamma > 1/2$, the L-T type inequality no longer holds for non-real V as $\sigma_d(H)$ may have an accumulation point in $(0,\infty)$, see [1, 2]. This motivated Demuth, Hansmann and Katriel to suggest a weaker form of an L-T type inequality.

Open problem: In [4], assuming (1), does there exist $(\gamma, d$-dependent) $C_{\gamma,d} > 0$ such that for all complex-valued $V \in L^{\gamma+d/2}(\mathbb{R}^d)$,

$$\sum_{\lambda\in\sigma_d(H)} \frac{\mathrm{dist}(\lambda,[0,\infty))^{\gamma+d/2}}{|\lambda|^{d/2}} \leq C_{\gamma,d} ||V||^{\gamma+d/2}_{L^{\gamma+d/2}}? \qquad (2)$$

For $d = 1$, (2) is not true and a counter-example can be found in [3]. This has led us to generalize the example from [3] to dimensions $d \geq 2$.

Results

Let $h > 1,\ 0 < \tau < 2\alpha < 2\delta < 2\beta < 2/5$. Focus on $h^{\alpha+1/2} \leq \ell \leq h^{\delta+1/2}$. Then we solve (3) for m in the restrictions

$$-\beta \log(h) \leq \mathrm{Im}(m) - \mathrm{Im}\big(\theta_\nu(m)\big) \leq -\alpha \log(h) \text{ and } h^\tau|\mathrm{Im}(m)| \leq \mathrm{Re}(m).$$

Here $\theta_\nu(z) = z - \nu\pi/2 - \pi/4 - \arctan\left(\frac{Y_\nu(z)}{J_\nu(z)}\right)$ where $\nu \geq 0$, $|\arg(z)| < \pi/2$ and Y_ν is the Bessel function of the second kind, see [6]. Let $0 < \varepsilon < 1$. For large h and every $j \in \mathbb{Z}$ with $\ell(\log(\ell))^{(1-\varepsilon)/(d-1)} \leq j \leq (8\pi)^{-1}h^{\beta+1/2}$, there is a solution

$$m_j = 2\pi j + \frac{\nu\pi}{2} + \frac{\pi}{2} + \theta_\nu(m_j) + \mathrm{i}\log\left(\frac{\sqrt{h}}{4\pi j}\right) + O\left((\log(h))^{(\varepsilon-1)/(d-1)}\right).$$

This implies $\mathrm{Im}(\lambda_j) > h/2$ and $|\lambda_j| \leq 4\pi^2 j^2$. Consequently,

$$\frac{1}{h^{\gamma+d/2}}\sum_j \frac{\mathrm{Im}(\lambda_j)^{\gamma+d/2}}{|\lambda_j|^{d/2}} \geq \mathrm{const}_{d,\alpha,\gamma,\delta,\varepsilon}\overset{\text{unbounded in } h}{(\log(h))^\varepsilon}(1 + o(1)).$$

Methodology

1 Take $H_h = -\Delta + \mathrm{i}h\chi_{B_1(0)}$, $h > 0$. Then $\left\|\mathrm{i}h\chi_{B_1(0)}\right\|^{\gamma+d/2}_{L^{\gamma+d/2}} = \mathrm{const}_d h^{\gamma+d/2}$ and $\mathrm{dist}(\lambda,[0,\infty)) = \mathrm{Im}(\lambda)$.

Aim to show

$$\frac{1}{h^{\gamma+d/2}}\sum_{\lambda\in\sigma_d(H_h)} \frac{\mathrm{Im}(\lambda)^{\gamma+d/2}}{|\lambda|^{d/2}} \to \infty, \quad \text{as } h \to \infty.$$

2 Use $\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{d-1}}$, $x \in \mathbb{R}^d$, $r = |x|$. Eigenvalues of $-\Delta_{S^{d-1}}$ are $\lambda = \ell(\ell + d - 2)$, $\ell \in \mathbb{N}_0$ where eigenfunctions are spherical harmonics of degree ℓ, denoted by $Y^{(\ell)}$.

- ☐ $\dim\left(\mathcal{L}_\lambda\left(-\Delta_{S^{d-1}}\right)\right) \leq 2$ if $d = 2$.
- ☐ $\dim\left(\mathcal{L}_\lambda\left(-\Delta_{S^{d-1}}\right)\right) = \frac{(2\ell+d-2)(\ell+d-3)!}{(d-2)!\ell!}$ if $d \geq 3$.

Choose eigenfunctions of H_h in the form $\psi(r)Y^{(\ell)}(x/r)$.

- ☐ $\lambda \in \sigma_d(H_h) \Rightarrow \dim(\mathcal{L}_\lambda(H_h)) \geq \mathrm{const}_d \ell^{d-2}$.

3 Using this separation ansatz gives an eigenvalue equation in ψ for every ℓ. Then we solve this eigenvalue equation, asymptotically for large ℓ and h. **Setting:**

- ☐ $\lambda = k^2 (k \in \mathbb{C})$.
- ☐ $k = (ih + m^2)^{1/2}$ $(m \in \mathbb{C})$ with $\mathrm{Im}(k) > 0$.

Aim to solve the characteristic equation for m:

$$\mathrm{i}h = m^2\left[\left(\frac{J'_\nu(m)\, H_\nu^{(1)}(k)}{J_\nu(m)(H_\nu^{(1)})'(k)}\right)^2 - 1\right], \qquad (3)$$

where $\nu = \ell + d/2 - 1$, J_ν and $H_\nu^{(1)}$ are Bessel and Hankel functions of the first kind, respectively.

Conclusion

We have found a family of potentials depending on a parameter h such that, for large h, we can find the asymptotics of finitely many eigenvalues in an h–dependent region of the complex plane. The eigenvalue asymptotics imply that (2) does not hold.

References

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