# On the stability of non-autonomous Schrödinger equations and applications to Quantum Control

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# Outline

- Introduction and Motivation
- Existence of solutions and Stability
- Applications to Quantum Control

# What is Quantum Control?

- Bilinear Control Problem

  Non-autonomous Schrödinger Equation in a complex separable Hilbert space $\mathcal{H}$.

$$i\frac{d\Psi}{dt} = \left(H_0 + u(t)H_1\right)\Psi$$

  - $\{H(t) = H_0 + u(t)H_1\}$ is a family of self-adjoint operators acting on $\mathcal{H}$ with domains $\mathcal{D}(H(t))$ that might depend explicitly on time.
  - A space of controls $u \in \bigcup_{T\in\mathbb{R}^+} \mathcal{F}([0,T],\mathcal{C})$, where $\mathcal{F} = C^0, C^1, C^1_{\text{pw}}, \dots$.

- Approximate Controllability

  For any $\epsilon > 0$, $\Psi_0$, $\Psi_T \in \mathcal{H}$, does there exist $u : [0,T] \to \mathcal{C}$ such that:

$$\|\Psi_T - \Psi(T)\| < \epsilon$$

  with $\Psi(t)$ solution of the Schrödinger equation with initial condition $\Psi(0) = \Psi_0$.

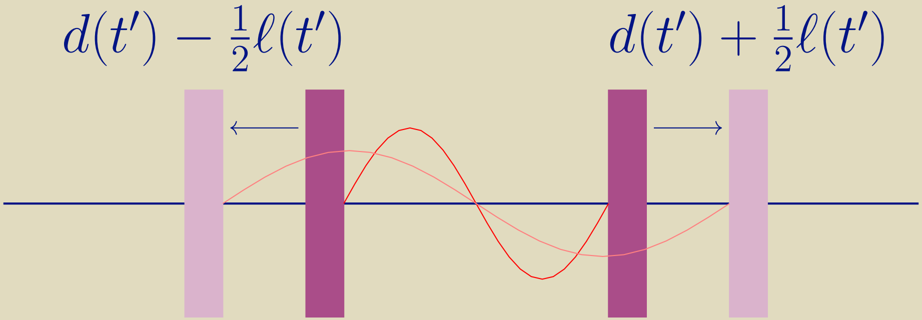
# Some considerations about time dependent domains

- In general domains $\mathcal{D}(H(t))$ depend on time.
- Solutions are given by Unitary Propagators:

$$U(t,s)U(s,r) = U(t,r)$$
$$U(t,s)\mathcal{D}(H(s)) = \mathcal{D}(H(t))$$
$$U^\dagger(t,s) = U(s,t)$$

- Case of Bilinear Control $H(t) = H_0 + u(t)H_1$
  - If $H_1$ is a bounded perturbation: $\mathcal{D}(H(t)) = \mathcal{D}(H_0)$.
  - If $H_1$ is relatively bounded and the values of the controls $\mathcal{C}$ are bounded (Kato-Rellich Theorem) we also have $\mathcal{D}(H(t)) = \mathcal{D}(H_0)$.
  - In general the situation can be very involved, even in this bilinear case.

- Free particle in a box with moving walls.




$$\Omega_{\ell,d} = \left[ d(t) - \tfrac{1}{2}\ell(t)\, , d(t) + \tfrac{1}{2}\ell(t) \right]$$

$$H(t) = -\frac{d^2}{dx^2}$$

$$\mathcal{D}(H(t)) = \mathcal{H}^2\left(\Omega_{\ell(t),d(t)}\right) \cap \mathcal{H}^1_0\left(\Omega_{\ell(t),d(t)}\right)$$

# Examples with time-dependent domains

- Free particle in a box with moving walls.
- Charged particle in the presence of a time-dependent magnetic field on a bounded region. In this case the boundary conditions depend naturally on the magnetic field.

  For instance: $\Omega \subset \mathbb{R}^3$

$$H(t) = \Delta_{A(t)} = -\left(\vec{\nabla} - \vec{A}(t)\right)^2$$

$$\mathcal{D}\left(H(t)\right) = \left\{\Phi \in \mathcal{H}^2(\Omega) \;\middle|\; (\vec{\nabla}\Phi + i\vec{A}(t)) \cdot \vec{n}\big|_{\partial\Omega} = 0\right\}$$

# Examples with time-dependent domains

- Free particle in a box with moving walls.
- Charged particle in the presence of a time-dependent magnetic field on a bounded region. In this case the boundary conditions depend naturally on the magnetic field.
- Explicit time-dependent boundary conditions like time-dependent Robin parameter or a point-like interaction with varying strength.

$$\Omega = [0,1]$$

$$H(t) = -\frac{d^2}{dx^2}$$

$$\mathcal{D}(H(t)) = \left\{ \Phi \in \mathcal{H}^2(\Omega) \middle| \begin{array}{l} \phi'(0) = -\lambda(t)\Phi(0) \\ \Phi'(1) = \lambda(t)\Phi(1) \end{array} \right\}$$

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# Weak Form of the Schrödinger Equation

- There is a one-to-one correspondence between self-adjoint, semi-bounded operators and Hermitian, closed sesquilinear forms

$$\langle \Phi, H(t)\Psi \rangle = h_t(\Phi, \Psi)$$

  - Form domain: $\mathcal{D}(h_t)$
  - Operator domain is contained in the form domain: $\Psi \in \mathcal{D}(H(t)) \subset \mathcal{D}(h_t)$

- If the form domain is independent of $t$, $\mathcal{D}(h_t) = \mathcal{H}^+$, we say that the Hamiltonian has constant form domain.
  - Example for the time-dependent Robin parameter:

$$h_t(\Phi, \Psi) = \int_0^1 \partial_x \bar{\Phi} \partial_x \Psi dx + \lambda(t) \left( \bar{\Phi}(0)\Psi(0) + \bar{\Phi}(1)\Psi(1) \right)$$

$$\mathcal{D}(h_t) = \mathcal{H}^1([0,1])$$

- Weak form of the Schrödinger equation

$$i\frac{d}{dt}\langle \Phi, \Psi(t) \rangle = h_t(\Phi, \Psi(t))$$

# Scales of Hilbert spaces $\mathcal{H}^+ \subset \mathcal{H} \subset \mathcal{H}^-$

- If a Hilbert space $\mathcal{H}^+$ is continuously embedded into other $\mathcal{H}$, i.e. $\|\cdot\| \leq \|\cdot\|_+$ there is a canonical dual Hilbert space $\mathcal{H}^-$ that satisfies:
  - $\mathcal{H}^+ \subset \mathcal{H} \subset \mathcal{H}^-$, $\|\cdot\|_- \leq \|\cdot\|$.
  - The canonical pairing $(\cdot,\cdot)\colon \mathcal{H}^+ \times \mathcal{H}^- \to \mathbb{C}$ is the continuous extension of the scalar product

$$(\Phi, \Psi) = \langle \Phi, \Psi \rangle \quad \Phi \in \mathcal{H}^+,\ \Psi \in \mathcal{H}$$

- By Riesz's Theorem any form $h\colon \mathcal{H}^+ \times \mathcal{H}^+ \to \mathbb{C}$ can be represented by a bounded operator $H \in \mathcal{B}(\mathcal{H}^+, \mathcal{H}^-, )$

$$h(\Phi, \Psi) = (\Phi, H\Psi)$$

- Alternative Weak form of the Schrödinger equation with generator $h_t(\Phi, \Psi) = (\Phi, H(t)\Psi)$, $H(t) \in \mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$ as an evolution equation in $\mathcal{H}^-$.
  - Example for the time dependent Robin parameter. Evolution equation in $\mathcal{H}^-$:

$$i\frac{d}{dt}\Psi(t) = (H_0 + \lambda(t)H_1)\Psi \quad H_0,\ H_1 \in \mathcal{B}(\mathcal{H}^+, \mathcal{H}^-) \quad \Psi \in \mathcal{H}^+$$

Theorem [J. Kisyński, Sec. 8]:

Suppose that $t \mapsto h_t(\Phi, \Psi) \in C^1([0, T])$. Then there exists a unitary propagator $U(t, s)$, $t, s \in [0, T]$ such that:

(i) $U(t, s)\mathcal{H}^+ = \mathcal{H}^+$

(ii) $t \mapsto \langle \Phi, U(t, s)\Psi \rangle \in C^1([0, T])$

(iii) $U(t, 0)\Phi$ is a weak solution of the Schrödinger equation with initial value $\Phi \in \mathcal{H}^+$.

If $t \mapsto h_t(\Phi, \Psi) \in C^2([0, T])$

(iv) $U(t, s)\mathcal{D}(H(s)) = \mathcal{D}(H(t))$

(v) $U(t, 0)\Phi$ is a (strong) solution of the Schrödinger equation with initial value $\Phi \in \mathcal{D}(H(0))$.

**Ref:** J. Kisyński, Studia Mathematica 3 (23), 1964

Corollary:

Suppose that $t \mapsto h_t(\Phi, \Psi) \in C^1_{\text{pw}}([0, T])$. Then there exists a unitary propagator $U(t, s)$, $t, s \in [0, T]$ such that:

(i) $U(t, s)\mathcal{H}^+ = \mathcal{H}^+$

(ii) $t \mapsto \langle \Phi, U(t, s)\Psi \rangle \in C^1_{\text{pw}}([0, T])$

(iii) $U(t, 0)\Phi$ is a piecewise weak solution of the Schrödinger equation with initial value $\Phi \in \mathcal{H}^+$.

Piecewise weak solution:

$$U(t, s) := U_1(t, t_1)U_2(t,_1 t_2)U_3(t_2, t_3) \cdots U_{n+1}(t_n, s)$$

Assumptions:

Let $\{H_n(t)\}_{n\in\mathbb{N}}$ be a family of Hamiltonians with constant form domain

A1) Uniformly semibounded from below $h_t(\Phi,\Phi) > -m\|\Phi\|^2$

A2) The mapping $t \mapsto h_t(\Phi,\Psi) \in C^2_{\text{pw}}$

A3) There is $c>0$ such that $c^{-1}\|\cdot\|_{+,n,t} \leq \|\cdot\|_{+,n_0,t_0} \leq c\|\cdot\|_{+,n,t}$

A4) $\sup_n \sum_j \|C_n(t)\|_{L^1(I_j)} < M$ where $C_n(t) := \left\|\frac{d}{dt}H_n(t)\right\|_{\mathcal{B}(\mathcal{H}^+,\mathcal{H}^-)}$.

Theorem [Balmaseda, Lonigro, PP]:

$$\|U_j(t,s) - U_k(t,s)\|_{\mathcal{B}(\mathcal{H}^+,\mathcal{H}^-)} \leq L(c,M)\int_s^t \|H_j(\tau) - H_k(\tau)\|_{\mathcal{B}(\mathcal{H}^+,\mathcal{H}^-)} d\tau$$

**Ref:** Balmaseda, Lonigro, PP, arXiv: 2306.10203.

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# Form Bilinear Control Systems

- Time dependent Hamiltonian with constant form domain represented in $\mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$ by

$$H(t) = H_0 + u(t)H_1 \qquad H_0, H_1 \in \mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$$

- Properties Related with well-posedness:
  - Uniformly bounded from below
  - sup $|u(t)| < C$
  - $u$ is piecewise differentiable and $\sum_i \|u'\|_{L^1(I_i)} < M$.
- Properties Related with Controllability:
  - The operator $H_0 \colon \mathcal{D}(H_0) \to \mathcal{H}$ has compact resolvent
  - The operator $H_1$ is compact in the sense of $\mathcal{B}(\mathcal{H}^+, \mathcal{H}^-)$
  - There exists $\{P_n\}$ a family of finite rank projections such that:

$$\lim_{n\to\infty} \|P_n\Phi - \Phi\|_+ = 0$$

# How to prove Approximate Controllability

- We will assume that the solution of the finite dimensional control problem exists.

$$i\frac{\partial\Psi}{\partial t} = P_n(\mathcal{H}_0 + u_n(t)H_1)P_n\Psi \qquad i\frac{\partial\Psi}{\partial t} = (\mathcal{H}_0 + u_n(t)H_1)\Psi$$

Solution $U_n(t,s)$ such that $\|\Psi_T - U_n(T_n,0)\Psi_0\| < \epsilon$

- It is easy to prove that

$$\|U_n(t,s) - U(t,s)\|_{\mathcal{B}(\mathcal{H}^+,\mathcal{H}^-)} \to 0$$
$$\Downarrow$$
$$\|U_n(t,s)\Phi - U(t,s)\Phi\| \to 0$$

- Using Stability

$$\|U_n(T_n,0) - U(T_n,0)\|_{\mathcal{B}(\mathcal{H}^+,\mathcal{H}^-)} \leq L\int_0^{T_n} \|P_nH_0P_n - H_0\|_{\mathcal{B}(\mathcal{H}^+,\mathcal{H}^-)} + |u(\tau)|\|P_nH_1P_n - H_1\|_{\mathcal{B}(\mathcal{H}^+,\mathcal{H}^-)}d\tau$$

# How to prove Approximate Controllability

- We will assume that the solution of the finite dimensional control problem exists.

$$i\frac{\partial\Psi}{\partial t} = P_n(\mathcal{H}_0 + u_n(t)H_1)P_n\Psi \qquad i\frac{\partial\Psi}{\partial t} = (\mathcal{H}_0 + u_n(t)H_1)\Psi$$

Solution $U_n(t,s)$ such that $\|\Psi_T - U_n(T_n,0)\Psi_0\| < \epsilon$

- It is easy to prove that

$$\|U_n(t,s) - U(t,s)\|_{\mathcal{B}(\mathcal{H}^+,\mathcal{H}^-)} \to 0$$
$$\Downarrow$$
$$\|U_n(t,s)\Phi - U(t,s)\Phi\| \to 0$$

- Using Stability

$$\|U_n(T_n,0) - U(T_n,0)\|_{\mathcal{B}(\mathcal{H}^+,\mathcal{H}^-)} \leq L\int_0^{T_n} \|P_nH_0P_n - H_0\|_{\mathcal{B}(\mathcal{H}^+,\mathcal{H}^-)} + |u(\tau)|\|P_nH_1P_n - H_1\|_{\mathcal{B}(\mathcal{H}^+,\mathcal{H}^-)}d\tau$$

# Approximate Controllability II: Using Interaction Picture

- $P_n$ is the orthogonal projection onto span$\{\Phi_j\}_{j=1}^n$, O.N.B. of $H_0$.

- $\tilde{U}(t,s) = e^{itH_0}U(t,s)e^{-isH_0}$ and $\tilde{U}_n(t,s) = e^{itH_0}U_n(t,s)e^{-isH_0}$

$$\begin{aligned}
\|U_n - U\|_{\mathcal{B}(\mathcal{H}^+,\mathcal{H}^-)} &\leq \|e^{-itH_0}\|_- \|\tilde{U}_n - \tilde{U}\|_{\mathcal{B}(\mathcal{H}^+,\mathcal{H}^-)} \|e^{isH_0}\|_+ \\
&\leq L \int_0^{T_n} |u_n(\tau)| \|P_n \tilde{H}^1 P_n - \tilde{H}^1\|_{\mathcal{B}(\mathcal{H}^+,\mathcal{H}^-)} d\tau \\
&\leq L \|P_n H^1 P_n - H^1\|_{\mathcal{B}(\mathcal{H}^+,\mathcal{H}^-)} \int_0^{T_n} |u_n(\tau)| d\tau
\end{aligned}$$

- If the finite dimensional control problem can be solved with controls *a priori* bounded in $\mathcal{L}^1$ then this proves approximate controllability. This is indeed the case.

# Limitations of the Assumptions

- Compactness of $H^1 \colon \mathcal{H}^+ \to \mathcal{H}^-$

  If $\mathcal{H}^+$ is compactly embedded in $\mathcal{H}$, which is guaranteed in the case that $H_0$ has compact resolvent, then

  $$H^1 \in \mathcal{B}(\mathcal{H}^+, \mathcal{H}) \Rightarrow H^1 \in \mathcal{K}(\mathcal{H}^+, \mathcal{H}^-)$$

  Moreover there are the trivial inclusions $\mathcal{B}(\mathcal{H}^+) \subset \mathcal{B}(\mathcal{H}^+, \mathcal{H})$ and $\mathcal{B}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H}^+, \mathcal{H})$.

- Restrictions to the projections onto the orthonormal base of $H_0$.

  To obtain the proof of existence of solutions of the control problem is fine. But, for the applications and specially when one wants to look for the actual controls, this restriction can be limiting. It would be very desirable to lift this restriction and to be able to consider more general compressions.

# Applications and some references

- Main reference for this presentation:

  A. Balmaseda, D. Lonigro, PP. (2023) arXiv: 2306.10203

- Applications to prove Approximate Controllability in several cases:
  - Free Quantum Particle in a one-dimensional domain is approximately controllable by moving walls.

    A. Balmaseda, D. Lonigro, PP. SIAM Journal on Control and Optimization 62 2, 826-852. (2024). doi: 10.1137/22M1518980
  - Charged particles are approximately controllable under the action of time-dependent magnetic fields in graph-like regions of $\mathbb{R}^n$.

    A. Balmaseda, D. Lonigro, PP. J. Phys. A: Math. Theor 56 32, 325201. (2023). doi: 10.1088/1751-8121/ace505
  - Particles under the effect of point-like interactions with time-dependent strength, including time-dependent Robin boundary conditions, are approximately controllable.

    A. Balmaseda, D. Lonigro, PP. (2024). arXiv: 2402.02955

THANKS!