Dislocated Dirac Operator

Duc Tho Nguyen

(Joint work with **D. Krejčiřík** and **L. Boulton**)

Mathematical aspects of the physics with non-self-adjoint operators
- The conference you can't refuse

Marseille, France, 3 - 7 June 2024

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Non-Self-Adjoint Op.  **Self-Adjoint Op.**

- Many tools are available to explore the **beauty** of the self-adjoint operator. But little theory exists to explain the **beast** of the non-self-adjoint operator.

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- Many tools are available to explore the **beauty** of the self-adjoint operator. But little theory exists to explain the **beast** of the non-self-adjoint operator.

If you want to learn how to live with the beast, you might be interested in

Elements of spectral theory without the spectral theorem
(**D. Krejčiřík** and **P. Siegl**, 2015)

This talk is about the following beast

$$\mathscr{L}_m = \underbrace{\begin{pmatrix} m & -\partial_x \\ \partial_x & -m \end{pmatrix}}_{\text{Free Dirac}} + i\,\mathrm{sign}(x)\mathrm{I}, \qquad m \geq 0, \qquad \mathrm{Dom}(\mathscr{L}_m) = H^1(\mathbb{R}, \mathbb{C}^2).$$

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- $\mathscr{L}_m$ is non-normal, but $\mathcal{T}$-self-adjoint, *i.e.*, $\mathscr{L}_m^* = \mathcal{T}\mathscr{L}_m\mathcal{T}$ with $\mathcal{T}u(x) = \overline{u(x)}$.

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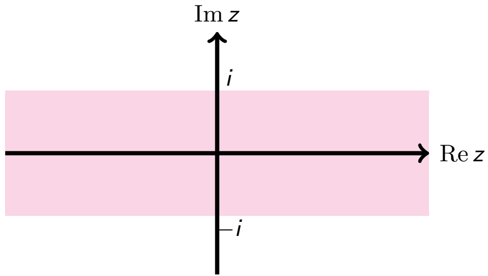
- The numerical range of $\mathscr{L}_m$, $\{\langle \mathscr{L}_m u, u\rangle : u \in \mathrm{Dom}(\mathscr{L}_m), \|u\| = 1\}$, is a strip:



Overview

1. Spectrum

2. Resolvent estimate

3. Location of eigenvalues of perturbations

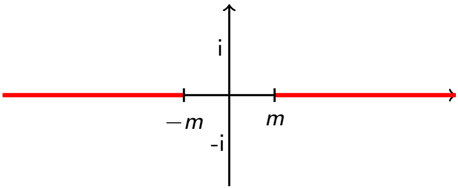
Table of Contents

The Spectrum

$$\mathscr{L}_m := \begin{pmatrix} m & -\partial_x \\ \partial_x & -m \end{pmatrix} + i\,\mathrm{sign}(x)\mathrm{I}, \qquad \mathrm{Dom}(\mathscr{L}_m) = H^1(\mathbb{R}, \mathbb{C}^2).$$

It is known that the spectrum of the free Dirac is

$$\sigma\left(\begin{pmatrix} m & -\partial_x \\ \partial_x & -m \end{pmatrix}\right) = (-\infty, -m] \cup [m, +\infty).$$



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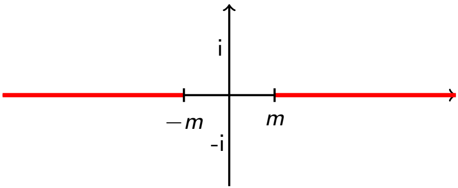
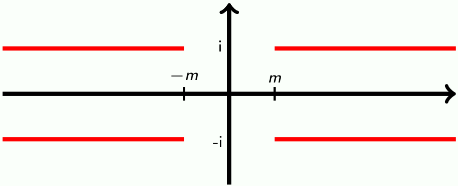


How does the spectrum change when we add $i\,\mathrm{sign}(x)\mathrm{I}$?

The spectrum

When $m > 0$

The spectrum of $\mathscr{L}_m$ is $\sigma(\mathscr{L}_m) = (-\infty, -m] \cup [m, +\infty) + i\{-1, 1\}$.



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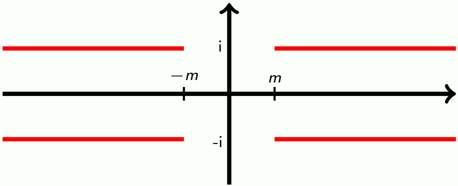

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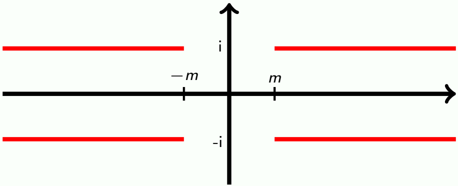

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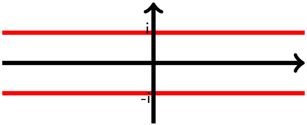

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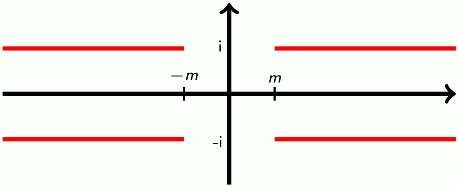

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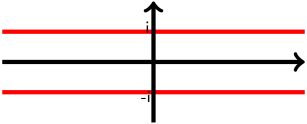

Unfortunately, It is not enough!

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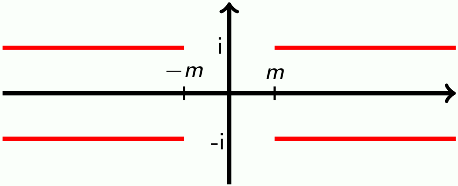
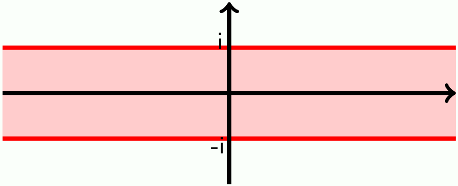


When $m = 0$

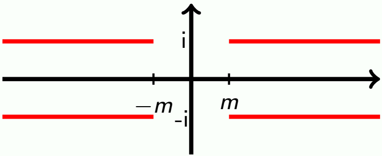
The spectrum of $\mathscr{L}_0$ is $\sigma(\mathscr{L}_0) = \{z \in \mathbb{C} : |\mathrm{Im}\, z| \leq 1\}$.



Characterization of the spectrum: $\sigma(\mathscr{L}_m) = \sigma_r(\mathscr{L}_m) \cup \sigma_c(\mathscr{L}_m) \cup \sigma_p(\mathscr{L}_m)$

When $m > 0$

$$\sigma_r(\mathscr{L}_m) = \emptyset, \quad \sigma_c(\mathscr{L}_m) = \{z \in \mathbb{C} : |\mathrm{Re}\, z| \geq m,\ |\mathrm{Im}\, z| = 1\}, \quad \sigma_p(\mathscr{L}_m) = \emptyset.$$



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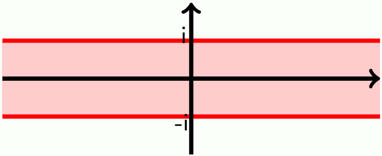
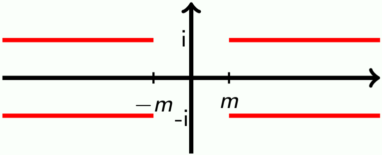


Essential spectra

When $m > 0$

$$\sigma_{ess,k}(\mathscr{L}_m) = \{z \in \mathbb{C} : |\mathrm{Re}\, z| \geq m, |\mathrm{Im}\, z| = 1\} \qquad \text{for all } k \in \{1,2,3,4,5\}.$$



When $m = 0$

$$\sigma_{ess,k}(\mathscr{L}_0) = \{z \in \mathbb{C} : |\mathrm{Im}\, z| = 1\} \qquad \text{for } k \in \{1,2,3,4\},$$

$$\sigma_{ess,5}(\mathscr{L}_0) = \{z \in \mathbb{C} : |\mathrm{Im}\, z| \leq 1\}.$$

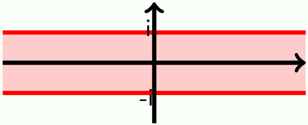


Table of Contents

When $m = 0$

$$\|(\mathscr{L}_0 - z)^{-1}\| = \frac{1}{|\operatorname{Im} z| - 1}.$$

Pseudospectrum: $\sigma_\varepsilon(\mathscr{L}_0) = \{z \in \mathbb{C} : |\operatorname{Im} z| < 1 + \varepsilon\}$.

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Non-normality $\not\Leftrightarrow$ Non-triviality of pseudospectrum

When $m > 0$

$$\|(\mathscr{L}_m - z)^{-1}\| = \frac{|\operatorname{Re} z|^2}{m(1-|\operatorname{Im} z|^2)} \left(1 + \frac{1}{|\operatorname{Re} z|^2}\right)$$

as $|\operatorname{Re} z| \to +\infty$ and uniformly for all $|\operatorname{Im} z| < 1$.

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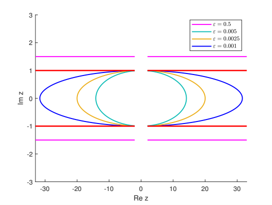

Figure: Illustration of (asymptotic) level set $\|(\mathscr{L}_m - z)^{-1}\| = \frac{1}{\varepsilon}$.

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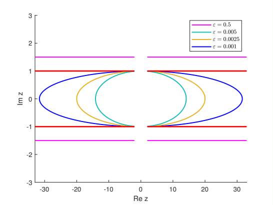

Figure: Illustration of (asymptotic) level set $\|(\mathscr{L}_m - z)^{-1}\| = \frac{1}{\varepsilon}$.

This is an improvement of the following result:

Schrödinger (**Henry-Krejčiřík**-17)

$$(1-o(1))\frac{\mathrm{Re}\ z}{\sqrt{1-(\mathrm{Im}\, z)^2}} \leq \left\|\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + i\mathrm{sign}(x) - z\right)^{-1}\right\| \leq \frac{4\mathrm{Re}\ z}{1-|\mathrm{Im}\, z|}(1+o(1)).$$

as $\mathrm{Re}\, z \to +\infty$.

Table of Contents

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Let $V : \mathbb{R} \to \mathbb{C}^{2\times 2}$. It is known that the eigenvalues of the "Free Dirac$+V$" lies in the union of two disks. [**J.-C. Cuenin, A. Laptev, and C. Tretter**, 2014]

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Where can we find the eigenvalues of the operator "$\mathscr{L}_m + V$" ?

Definition of the operator

Let $V : \mathbb{R} \to \mathbb{C}^{2\times 2}$. We apply Kato's construction to define the operator

$$\mathscr{L}_{m,\varepsilon V} = \text{“}\mathscr{L}_m + V\text{”} \quad .$$

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Proposition

Assume that one of the following conditions holds:

- **H1**: $\|V\|_{L^1} < 1$.
- **H2**: $V \in L^1(\mathbb{R}, \mathbb{C}^{2\times 2}) \cap L^p(\mathbb{R}, \mathbb{C}^{2\times 2})$ *for some* $p \in (1, \infty]$.

Then, there exists a closed densely defined extension $\mathscr{L}_{m,V}$ *of* $\mathscr{L}_m + V$ *with a non empty resolvent.*

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Then, there exists a closed densely defined extension $\mathscr{L}_{m,V}$ *of* $\mathscr{L}_m + V$ *with a non empty resolvent.Furthermore, we have*

- $m > 0$: $\sigma_{\text{ess,k}}(\mathscr{L}_{m,V}) = \sigma_{\text{ess,k}}(\mathscr{L}_m)$, $\forall k = 1, 2, 3, 4, 5.$
- $m = 0$: $\sigma_{\text{ess,k}}(\mathscr{L}_{0,V}) = \sigma_{\text{ess,k}}(\mathscr{L}_0)$, $\forall k = 1, 2, 3, 4.$

Theorem

- *For* $m > 0$, *assume that there exists a positive constant* C *such that*

$$\int_{\mathbb{R}} \sqrt{1+x^2}|V(x)|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}x \leq \frac{1}{C},$$

then

$$\sigma_p(\mathscr{L}_{m,V}) \subseteq \left\{z \in \mathbb{C} : |\mathrm{Im}\, z| \leq 1,\ |\mathrm{Re}\, z| \geq C\sqrt{\frac{m}{\|V\|_{L^1}}}\right\}.$$

- *For* $m = 0$, *assume that* $V \in L^1(\mathbb{R})$ *and* $\|V\|_{L^1} < 1$*. Then,*

$$\sigma(\mathscr{L}_{0,V}) \subset \{z \in \mathbb{C} : |\mathrm{Im}\, z| \leq 1\}.$$

Example: Step potential

Let $a > 0$ and $b \in \mathbb{R}$: $V_{a,b}(x) = (-i\text{sign}(x) - b)\chi_{[-a,a]}(x)I$. We consider

$$\mathscr{L}_{m,V_{a,b}} = \mathscr{L}_m + V_{a,b}(x), \qquad \text{Dom}(\mathscr{L}_{m,V_{a,b}}) = H^1(\mathbb{R}, \mathbb{C}^2).$$

When $m > 0$

- $\sigma_{\text{ess,k}}(\mathscr{L}_{m,V_{a,b}}) = \sigma_{\text{ess,k}}(\mathscr{L}_m), \qquad \forall k \in \{1,2,3,4,5\}.$
- $\mathscr{L}_{m,V_{a,b}}$ possesses infinitely many large positive and negative isolated eigenvalues of finite multiplicity.

When $m = 0$

- $\sigma_{\text{ess,k}}(\mathscr{L}_{0,V_{a,b}}) = \sigma_{\text{ess,k}}(\mathscr{L}_0), \qquad \forall k \in \{1,2,3,4\}.$
- $\sigma_{\text{p}}(\mathscr{L}_{0,V_{a,b}}) = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < 1\}.$

Weakly couple

Let us consider the perturbed operator $\mathscr{L}_{m,\varepsilon V} = \text{“}\mathscr{L}_m + \varepsilon V\text{”}$ as $\varepsilon \to 0$.

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Let $m > 0$ and let $V \in L^1(\mathbb{R}, (1+x^2)\mathrm{d}x)$. Then, as $\varepsilon \to 0$,

$$\sigma_{\mathrm{dis}}\left(\mathscr{L}_{m,\varepsilon V}\right) \subset \left\{ z \in \mathbb{C} : |\mathrm{Im}\, z| < 1,\ \frac{4m}{4|z|^2+m^2} = \varepsilon \int_{\mathbb{R}} e^{2\eta_z(x)} \langle V(x), \overline{\Upsilon_z} \rangle_F \, \mathrm{d}x + o(\varepsilon^2) \right\}$$

where

$$\eta_z(x) = -i\left(z - \frac{m^2}{2\mathrm{Re}\, z}\right)x - |x|, \qquad \Upsilon_z = \frac{1}{2(4|z|^2+m^2)} \begin{pmatrix} -(2z+m)^2 & i(4z^2-m^2) \\ i(4z^2-m^2) & (2z-m)^2 \end{pmatrix}.$$

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If we assume further that $V \in W^{2,1}(\mathbb{R})$, then there exists $C > 0$ such that

$$\sigma_{\mathrm{dis}}\left(\mathscr{L}_{m,\varepsilon V}\right) \subset \left\{ z \in \mathbb{C} : |\mathrm{Im}\, z| < 1,\ |\mathrm{Re}\, z| \geq \frac{C}{\varepsilon} \right\} \quad \text{as } \varepsilon \to 0.$$

Thank you very much for your attention!