L^p estimates for degenerate problems in the half-space

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Introduction

Let

$$\mathcal{L} = y^{\alpha_1}\Delta_x + y^{\alpha_2}\left(D_{yy} + \frac{c}{y}D_y - \frac{b}{y^2}\right) \quad \text{and} \quad D_t - \mathcal{L}$$

in the half-space $\mathbb{R}^{N+1}_+ = \{(x,y) : x \in \mathbb{R}^N, y > 0\}$ or $(0,\infty) \times \mathbb{R}^{N+1}_+$.

Here $b,\ c$ are constant real coefficients and we use

$$L_y = D_{yy} + \frac{c}{y}D_y - \frac{b}{y^2}.$$

When $b = 0$, then L_y is a Bessel operator (we shall denote it by B_y) The real numbers α_1, α_2 satisfy $\alpha_2 < 2$ and $\alpha_2 - \alpha_1 < 2$ but are not assumed to be nonnegative. The reasons for these restrictions will be explained later.

$\mathcal{L}$ is the sum of a degenerate diffusion $y^{\alpha_1}\Delta_x$, tangential to $\partial\mathbb{R}^{N+1}_+$, and of a 1d degenerate normal diffusion $y^{\alpha_2}L_y$ which commute only when $\alpha_1 = 0$. It satisfies the scaling property

$$I_s^{-1}\mathcal{L}I_s = s^{2-\alpha_2}\mathcal{L}, \quad I_s u(x,y) = u(s^{1-\frac{(\alpha_2-\alpha_1)}{2}}x, sy).$$

Here we study unique solvability of the problems $\lambda u - \mathcal{L}u = f$ and $D_t v - \mathcal{L}v = g$ in L^p spaces under appropriate boundary conditions, and initial conditions in the parabolic case, together with the regularity of u, v.

When $\alpha_1 = \alpha_2 = 0$,

$$\mathcal{L} = \Delta_x + \left(D_{yy} + \frac{c}{y} D_y - \frac{b}{y^2} \right)$$

reduces to the so-called Caffarelli-Silvestre extension operators. Note that in this case $\mathcal{L}$ is the sum of two commuting operators.

The case $\alpha_1 = \alpha_2 = 1$ and $b = 0$, namely

$$\mathcal{L} = y \left(\Delta_x + D_{yy} \right) + c D_y,$$

is also widely treated in the literature on degenerate problems.

When $\alpha_2 = 0$ our operators generalize the class of Baouendi-Grushin operators

$$\mathcal{L} = y^{\alpha} \Delta_x + D_{yy},$$

to which they reduce when $c = b = 0$.

The aim of this talk is to provide a unified approach which allows to prove elliptic and parabolic L^p estimates and solvability of the associated problems.

In the language of semigroup theory, we prove that $\mathcal{L}$ generates an analytic semigroup, characterize its domain as a weighted Sobolev space and show that it has maximal regularity, which means that both $D_t v$ and $\mathcal{L}v$ have the same regularity as $(D_t - \mathcal{L})v$.

Surprisingly enough, the case $\alpha_1 = \alpha_2$ implies all other cases by a change of variables, as described below.

However this change of variables modifies the underlying measure and the procedure works if one is able to deal with the simpler case

$$\mathcal{L} = y^{\alpha}(\Delta_x + L_y)$$

in the whole scale of L^p_m spaces, where $L^p_m = L^p(\mathbb{R}^{N+1}_+; y^m dxdy)$.

A similar simplification holds also for the 1d operator L_y: it is sufficient to deal in full generality with the case where $b = 0$, that is when L_y is a Bessel operator.

Finally, it is sufficient to deal only with Neumann boundary conditions, since the case of Dirichlet boundary conditions is again deduced by a change of variables.

The change of variables is (for $\beta \neq -1$)

$$Tu(x,y) := |\beta+1|^{\frac{1}{p}} y^k u(x, y^{\beta+1}), \quad (x,y) \in \mathbb{R}^{N+1}_+.$$

Then T maps isometrically $L^p_{\tilde{m}}$ onto L^p_m where

$$\tilde{m} = \frac{m+kp-\beta}{\beta+1} \quad \text{and}$$

$$T^{-1}\left(y^{\alpha_1}\Delta_x + y^{\alpha_2}L_y\right)Tu = \left(y^{\frac{\alpha_1}{\beta+1}}\Delta_x + (\beta+1)^2 y^{\frac{\alpha_2+2\beta}{\beta+1}}\tilde{L}_y\right)u$$

where $\tilde{L}$ is as $\mathcal{L}$ with

$$\tilde{b} = \frac{b-k\,(c-1+k)}{(\beta+1)^2},$$

$$\tilde{c} = \frac{c+2k+\beta\,(c+1+2k+\beta)}{(\beta+1)^2}.$$

We use β to to have $\alpha_1 = \alpha_2$ and k to get $b = 0$. But then the density of the underlying measure changes from y^m to $y^{\tilde{m}}$ and we use this strategy also when $m = c - \alpha_2$, that is in the case of the symmetrizing measure $y^{c-\alpha_2} dx\, dy$.

Let me explain the meaning of the restrictions $\alpha_2 < 2$, $\alpha_2 - \alpha_1 < 2$. assuming first that $\alpha_1 = \alpha_2 = \alpha$, so that the unique requirement is $\alpha < 2$.
It turns out that when $\alpha \geq 2$ the problem is easily treated in the strip $\mathbb{R}^N \times [0, 1]$ in the case of the Lebesgue measure and all problems are due to the strong diffusion at infinity.
The case $\alpha \geq 2$ in the strip $\mathbb{R}^N \times [1, \infty[$ requires therefore new investigation even though the 1d operator $y^\alpha L_y$ alone can be treated for any $\alpha \in \mathbb{R}$, by the above tranformation T.

When $\alpha_1 \neq \alpha_2$, the change of variables T with $k = 0$ and $\beta = \frac{\alpha_1 - \alpha_2}{2}$, transforms $y^{\alpha_1}\Delta_x + y^{\alpha_2}B_y$ into $y^{\alpha}(\Delta_x + \tilde{B}_y)$, $\alpha = \frac{2\alpha_1}{\alpha_1 - \alpha_2 + 2}$. However, the strip $\mathbb{R}^N \times [0,1]$ is mapped into itself only when $\alpha_2 - \alpha_1 < 2$.

Under this condition it is possible, though not done, that the restriction $\alpha_2 < 2$ can be removed, at least when the operator is studied in $\mathbb{R}^N \times [0,1]$ rather than in $\mathbb{R}^{N+1}_+$. But dealing with the case $\alpha_2 - \alpha_1 \geq 2$ requires further investigation.

Assuming, in addition, that $\alpha_1 \geq 0$, the range of parameters for which we prove solvability is optimal, since it coincides with that of L_y. However, when $\alpha_1 < 0$ it can happen that $\mathcal{L}$ generates in a range of parameters for which the domain is less regular.

The operators $\mathcal{L}$, $D_t - \mathcal{L}$ are studied through estimates like

$$\|y^\alpha \Delta_x u\|_{p,m} + \|y^\alpha B_y u\|_{p,m} \le C\|\mathcal{L}u\|_{p,m}, \tag{1}$$

and

$$\|D_t u\|_{p,m} + \|\mathcal{L}u\|_{p,m} \le C\|(D_t - \mathcal{L})u\|_{p,m}, \tag{2}$$

where the L^p norms are taken over $\mathbb{R}^{N+1}_+$ and on $(0,\infty) \times \mathbb{R}^{N+1}_+$.

This kind of estimates are quite natural in this context but not easy to prove. Of course they imply $\|y^\alpha D_{x_i x_j} u\|_{p,m} \le C\|\mathcal{L}u\|_{p,m}$, by the Calderón-Zygmund inequalities in the x-variables, and can be restated by saying that $\mathcal{L}$ is closed on $D(y^\alpha \Delta_x) \cap D(y^\alpha B_y)$ or that $y^\alpha \Delta_x \mathcal{L}^{-1}$ is bounded.

Let me explain how to obtain (1).
Assuming that $y^\alpha(\Delta_x u + B_y u) = f$ and taking the Fourier transform with respect to x we obtain

$$-|\xi|^2 \hat{u}(\xi, y) + B_y \hat{u}(\xi, y) = y^{-\alpha} \hat{f}(\xi, y)$$

and then $y^\alpha |\xi|^2 \hat{u}(\xi, y) = -y^\alpha |\xi|^2 (|\xi|^2 - B_y)^{-1} y^{-\alpha} \hat{f}(\xi, y)$. This means

$$y^\alpha \Delta_x \mathcal{L}^{-1} = \mathcal{F}^{-1} \left(y^\alpha |\xi|^2 (|\xi|^2 - B_y)^{-1} y^{-\alpha} \right) \mathcal{F}$$

and the boundedness of $y^\alpha \Delta_x \mathcal{L}^{-1}$ is equivalent to that of the operator valued multiplier

$$\xi \to M(\xi) = y^\alpha |\xi|^2 (|\xi|^2 - B_y)^{-1} y^{-\alpha}$$

in $L^p(\mathbb{R}^N; L^p_m(0, \infty)) = L^p_m(\mathbb{R}^{N+1}_+)$.

We prove this by a vector valued Mikhlin multiplier theorem which rests on square function estimates for the family $M(\xi)$ and its derivatives. The strategy for proving (2) is similar after taking the Fourier transform with respect to t.

However the a-priori estimates (1) and (2) are not sufficient for the solvability of the equation $\lambda u - \mathcal{L}u = f$. In fact, $\mathcal{L}$ is not dissipative unless additional restrictions on the parameters and on the underlying measure are assumed.
In order to prove existence results, or generation results in the language of semigroups, we use that the operator valued map

$$\xi \in \mathbb{R}^N \to N(\xi) = (\lambda + y^\alpha |\xi|^2 - y^\alpha B_y)^{-1} \quad \lambda \in \mathbb{C}_+$$

is a Fourier multiplier in $L^p(\mathbb{R}^N; L^p_m(0,\infty)) = L^p_m(\mathbb{R}^{N+1}_+)$.

The operator $\mathcal{L} = y^\alpha \Delta_x + y^\alpha B^n_y$ in $L^2_{c-\alpha}$

We use the Sobolev space $H^1_{\alpha,c} := \{u \in L^2_{c-\alpha} : y^{\frac{\alpha}{2}} \nabla u \in L^2_{c-\alpha}\}$ equipped with the inner product

$$\langle u, v \rangle_{H^1_{\alpha,c}} := \langle u, v \rangle_{L^2_{c-\alpha}} + \left\langle y^{\frac{\alpha}{2}} \nabla u, y^{\frac{\alpha}{2}} \nabla v \right\rangle_{L^2_{c-\alpha}}.$$

Then

$$\mathcal{L} = y^\alpha \Delta + c y^{\alpha-1} D_y = y^{-c+\alpha} \operatorname{div}\Big(y^c \nabla u\Big).$$

is associated to the form in $L^2_{c-\alpha}(\mathbb{R}^{N+1}_+)$

$$\mathfrak{a}(u, v) := \int_{\mathbb{R}^{N+1}_+} \langle \nabla u, \nabla \overline{v} \rangle \, y^c dx\, dy, \quad D(\mathfrak{a}) = H^1_{\alpha,c}.$$

By construction $\mathcal{L}$ is a non-positive self-adjoint operator and generates a contractive analytic semigroup $\left\{e^{z\mathcal{L}} : \ z \in \mathbb{C}_+\right\}$ in $L^2_{c-\alpha}$.

To characterize the domain let me introduce the Sobolev space

$$W^{2,p}(\alpha_1,\alpha_2,m)=\left\{u\in W^{2,p}_{loc}(\mathbb{R}^{N+1}_+):\ u,\ y^{\alpha_1}D_{x_ix_j}u,,\ y^{\alpha_2}D_{yy}u\in L^p_m\right\}$$

(estimates for first order derivatives follow by interpolation) and add a Neumann boundary condition for $y=0$ setting

$$W^{2,p}_{\mathcal{N}}(\alpha_1,\alpha_2,m)=\{u\in W^{2,p}(\alpha_1,\alpha_2,m):\ y^{\alpha_2-1}D_yu\ \in L^p_m\}.$$

To work in a dense set of smooth functions I introduce

$$\mathcal{D}=\left\{u\in C_c^\infty([0,\infty)),\ D_yu(y)=0 \text{ for } y\le\delta \text{ and some } \delta>0\right\}$$

and finally (finite sums below)

$$C_c^\infty(\mathbb{R}^N)\otimes\mathcal{D}=\left\{u(x,y)=\sum_i u_i(x)v_i(y),\ u_i\in C_c^\infty(\mathbb{R}^N),\ v_i\in\mathcal{D}\right\}.$$

Proposition

If $\frac{m+1}{p} > \alpha_1^-$ *then* $C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D}$ *is dense in* $W^{2,p}_{\mathcal{N}}(\alpha_1, \alpha_2, m)$*.*

Note that the condition $(m+1)/p > \alpha_1^-$ is necessary for the inclusion $C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D} \subset W^{2,p}_{\mathcal{N}}(\alpha_1, \alpha_2, m)$.
The domain if $\mathcal{L}$ in $L^2_{c-\alpha}$ is now given.

Theorem

If $c + 1 > |\alpha|$ *then*

$$D(\mathcal{L}) = W^{2,2}_{\mathcal{N}}(\alpha, \alpha, c - \alpha)$$

The proof uses the boundedness of the multiplier

$$\xi \to M(\xi) = y^\alpha |\xi|^2 (|\xi|^2 - B_y)^{-1} y^{-\alpha}$$

in $L^2(\mathbb{R}^N; L^2_{c-\alpha}(0, \infty)) = L^2_{c-\alpha}$.

The operator $\mathcal{L} = y^\alpha \Delta_x + y^\alpha B_y^n$ in L^p_m

Here the situation is more complicated since it is not easy to construct a semigroup, first, due to the lack of dissipativity. We have to prove analyticity, directly.
The main point is the following lemma whose proof relies on the boundedness of the multipliers M, N introduced before.

Lemma

Let $\alpha^- < \frac{m+1}{p} < c + 1 - \alpha$. *Then for any* $\lambda \in \mathbb{C}^+$ *the operators*

$$(\lambda - \mathcal{L}_{c-\alpha,2})^{-1}, \quad y^\alpha \Delta_x (\lambda - \mathcal{L}_{c-\alpha,2})^{-1}, \quad y^\alpha B_y^n (\lambda - \mathcal{L}_{c-\alpha,2})^{-1}$$

initially defined on $L^p_m \cap L^2_{c-\alpha}$ *extend to bounded operators on* L^p_m *which we denote respectively by* $\mathcal{R}(\lambda)$, $y^\alpha \Delta_x \mathcal{R}(\lambda)$, $y^\alpha B_y^n \mathcal{R}(\lambda)$.
Moreover the family $\{\lambda \mathcal{R}(\lambda) : \lambda \in \mathbb{C}^+\}$ *is* $\mathcal{R}$*-bounded on* L^p_m.

At this point "pseudo resolvents" arguments yield an analytic semigroup in L^p_m and the L^2 argument extends to L^p_m and characterizes the domain.

Theorem

If $\alpha^- < \frac{m+1}{p} < c + 1 - \alpha$, then $\mathcal{L}_{m,p}$ has maximal regularity and

$$D(\mathcal{L}_{m,p}) = W^{2,p}_{\mathcal{N}}(\alpha, \alpha, m).$$

In particular $C_c^\infty(\mathbb{R}^N) \otimes \mathcal{D}$ is a core for $\mathcal{L}_{m,p}$.

The condition $\alpha^- < \frac{m+1}{p} < c + 1 - \alpha$ is "quite" precise. In fact the $1d$ operator $y^\alpha B_y$ (with Neumann boundary conditions) is a generator if and only if $0 < \frac{m+1}{p} < c + 1 - \alpha$.

The mixed derivatives

I never inserted the mixed derivatives $y^\alpha D_{xy} u$ in the definition of Sobolev spaces, since this simplifies some argument. However

Theorem

Let $\alpha^- < \frac{m+1}{p} < c + 1 - \alpha$*. Then there exists* $C > 0$ *such that for every* $u \in D(\mathcal{L}_{m,p})$

$$\|y^\alpha D_y \nabla_x u\|_{L^p_m} \le C \|\mathcal{L} u\|_{L^p_m}.$$

This is actually a property of the Sobolev space $W^{2,p}_{\mathcal{N}}(\alpha, \alpha, m)$ in such a range of parameters, saying that the mixed derivatives can be controlled by the pure ones. However we do not have a Sobolev space proof of this fact and we do not know if this is true without restrictions or without boundary conditions.

The operator $y^{\alpha_1}\Delta_x + y^{\alpha_2}B_y^n$

Let now

$$\mathcal{L}_{m,p}^{\alpha_1,\alpha_2} = y^{\alpha_1}\Delta_x + y^{\alpha_2}B_y^n$$

in the space L^p_m. The generation and domain properties for $\mathcal{L}^{\alpha_1,\alpha_2}$ are deduced from the case $\alpha_1 = \alpha_2$ by using the isometry

$$Tu(x,y) := |\beta+1|^{\frac{1}{p}} y^k u(x, y^{\beta+1}), \quad (x,y) \in \mathbb{R}^{N+1}_+.$$

Theorem

Let $\alpha_2 - \alpha_1 < 2$ and

$$\alpha_1^- < \frac{m+1}{p} < c + 1 - \alpha_2.$$

Then $\mathcal{L}^{\alpha_1,\alpha_2}$ with domain $D(\mathcal{L}_{m,p}^{\alpha_1,\alpha_2}) = W^{2,p}_{\mathcal{N}}(\alpha_1,\alpha_2,m)$ generates a bounded analytic semigroup in L^p_m which has maximal regularity.

Dirichlet boundary conditions

It is possible to add a potential term to B and study the operator

$$\mathcal{L}^{\alpha_1,\alpha_2} = y^{\alpha_1}\Delta_x + y^{\alpha_2}\left(D_{yy} + \frac{c}{y}D_y - \frac{b}{y^2}\right), \quad \alpha_2 < 2$$

in L^p_m, under Dirichlet boundary conditions at $y = 0$. One uses again the isometry T above with $\beta = 0$ and a suitable k to cancel the potential term.
I prefer not to give a list of the results can be obtained in this way but I point out that the domain is, in general, less regular than for Neumann boundary conditions.

Dissipativity

I mentioned several times that these kind of operators are dissipative only for a certain range of parameters.
It is not difficult to see that $y^{\alpha_1}\Delta_x + y^{\alpha_2}B_y^n$ is dissipative in L^p_m if and only if $y^{\alpha_2}B_y^n$ is dissipative. This last case is easily reduced to $\alpha_2 = 0$, by a change of variable. Let us formulate the result, therefore, only for B^n.

Proposition

Assume $0 < \frac{m+1}{p} < c+1$. The operator B_p^n is dissipative in $L^p_m(\mathbb{R}_+)$ if and only if

(i) *$m = c$ or*

(ii) *$m \geq 1$ and $\frac{m-1}{p} \leq c-1$.*

Baouendi-Grushin operator

Our results apply to generalized Baouendi-Grushin operators

$$\mathcal{L} = y^\alpha \Delta_x + L_y, \quad \alpha > -2$$

in the half space $\mathbb{R}^{N+1}_+$ both with Neumann and Dirichlet boundary conditions, but we restrict ourselves to the classical case $\mathcal{L} = y^\alpha \Delta_x + D_{yy}$ in the whole space $\mathbb{R}^{N+1}$ with the Lebesgue measure.

Proposition

Let $\alpha > -\frac{1}{p}$. Then $\mathcal{L} = |y|^\alpha \Delta_x + D_{yy}$ with domain

$$D(\mathcal{L}) = \left\{ u,\ y^\alpha D_{x_i x_j} u,\ D_{yy} u,\ y^{\frac{\alpha}{2}} D_{x_i y} u \in L^p(\mathbb{R}^{N+1}) \right\}$$

generates a bounded analytic semigroup in $L^p(\mathbb{R}^{N+1})$ which has maximal regularity.

The operator $y\Delta_x + yB_y$

Another popular example appears when $\alpha_1 = \alpha_2 = 1$, that is for $\mathcal{L} = y\Delta_x + yB_y = y\Delta_x + yD_{yy} + cD_y$. If $0 < \frac{m+1}{p} < c$ the domain is

$$W^{2,p}_{\mathcal{N}}(1,1,m) = \{u,\ \nabla_x u,\ D_y u,\ yD_{x_i x_j} u,\ yD_{yy} u,\ yD_{x_i y} u \in L^p_m\},$$

Note that, when $m = 0$, then

$$W^{2,p}_{\mathcal{N}}(1,1,0) = \{u \in W^{1,p}(\mathbb{R}^{N+1}_+) : yD_{x_i x_j} u,\ yD_{yy} u,\ yD_{x_i y} u \in L^p(\mathbb{R}^{N+1}_+)\}$$

and the associated elliptic and parabolic problems seem to have no boundary condition. In our approach, the Neumann boundary condition is indeed imposed to $yD_y u$, by requiring that $\frac{1}{y}(yD_y u) \in L^p(\mathbb{R}^{N+1}_+)$.

All the admissible ranges of m, c follow from our results, both in the Dirichlet and in the Neumann case.

The main tool

A few words on the boundedness of the operator valued multipliers on which this talk is based, namely

$$M(\lambda,\xi)=y^{\alpha}|\xi|^2(\lambda+|\xi|^2-B_y)^{-1}y^{-\alpha}$$

$$N(\lambda,\xi)=(\lambda+y^{\alpha}|\xi|^2-y^{\alpha}B_y)^{-1}\quad \lambda\in\mathbb{C}_+.$$

We need $\mathcal{R}$-boundedness of M and λN (and of a certain number of ξ-derivatives) with respect to ξ and λ which we deduce from estimates of the heat kernels of B_y-V and $y^{\alpha}B_y-V$, where V is a suitable potential. The role of $\mu=|\xi|^2$ and λ is quite symmetric since

$$\left(\lambda-y^{\alpha}B^n+\mu y^{\alpha}\right)^{-1}f=\left(\mu-B^n+\frac{\lambda}{y^{\alpha}}\right)^{-1}\left(\frac{f}{y^{\alpha}}\right).$$

Even though this formula is formally obvious, its proof requires some care.

The main tool in all proofs consists in the following R-boundedness result which we apply to kernel estimates of the Bessel semigroup (with $M = 1$ below). We consider the family of operators

$$S_t^{\alpha,\beta} f(y) = t^{-\frac{M}{2}} \left(\frac{|y|}{\sqrt{t}} \wedge 1 \right)^{-\alpha} \int_{\mathbb{R}^M} \left(\frac{|z|}{\sqrt{t}} \wedge 1 \right)^{-\beta} \exp\left(-\frac{|y-z|^2}{\kappa t} \right) f(z)\, dz$$

where κ is a positive constant.

Theorem

Let $1 < p < \infty$ and let us suppose that $\alpha < \frac{M}{p} < M - \beta$. Then the family $\left(S_t^{\alpha,\beta} \right)_{t \geq 0}$ is $\mathcal{R}$-bounded on $L^p(\mathbb{R}^M)$.