First-order superintegrability with complex magnetic fields

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Introduction

- Superintegrable systems are distinguished by their symmetry, which often leads to exact, analytical solvability.
- Systems with linear terms in the Hamiltonian (in units where $m = 1, e = -1$),
$$H = \tfrac{1}{2}\left(\vec{p}^2 + A_j(\vec{x})p_j + p_jA_j(\vec{x}) + A_j(\vec{x})^2\right) + W(\vec{x}), \tag{1}$$
correspond to magnetic fields and are crucial models e.g. in particle physics or describe systems in uniform rotation e.g. the galaxies.
- Recent experimental observation of Lee-Yang zeros [5] increased the interest of mathematicians in complex magnetic fields.
- We connect these two fields by performing a classification of complex superintegrable systems in a restricted setting and by rigorously assessing their physical relevance.

Integrability and superintegrability

Integrability

A quantum Hamiltonian system with n degrees of freedom is integrable if it admits n algebraically independent integrals of motion in involution, i.e. no fully symmetrized (Jordan) polynomial of the integrals vanishes identically and the integrals commute.

Superintegrability

A quantum Hamiltonian system with n degrees of freedom is polynomially superintegrable if it admits $n + k$ algebraically independent integrals of motion (where $k \leq n - 1$), that are polynomial in the momenta and out of which n are in involution.

In our 3D case we need three independent integrals of motion X_1, X_2, Y plus the Hamiltonian H, two of them commuting:
$$[X_1, H] = [X_2, H] = [X_1, X_2] = [Y, H] = 0. \tag{2}$$

Fundamental problems with complex magnetic fields

There are at least two problems with rigorous physical interpretation even if we choose the fields W, B^j as complex functions of real variables $W, B^j : \mathbb{R}^3 \to \mathbb{C}$.

Lost gauge invariance

The time-independent gauge transformation $A'(\vec{x}) = A(\vec{x}) + \nabla\chi(\vec{x})$, $W'(\vec{x}) = W(\vec{x})$, in quantum mechanics corresponding to
$$U\psi(\vec{x}) = \exp\left(\frac{i}{\hbar}\chi(\vec{x})\right)\psi(\vec{x}), \tag{3}$$
is unbounded, not unitary, once $\chi(\vec{x})$ has non-vanishing imaginary part. The unboundedness of the transformation may change the spectral properties of the magnetic Hamiltonian (1) which is therefore not well defined by the complex-valued magnetic field only.

We are not able to address it, so we proceed formally and choose the simplest gauge.

Complex spectrum

Is usually not observable and its physical interpretation is unclear. (But see Pang et al. measuring complex Lee-Yang zeros [5].)

The mathematically correct approach to the problem entails pseudo-[4] or quasi-Hermiticity (or self-adjointness) [6, 1], i.e. the Hamiltonian H satisfies
$$H^\dagger = \Theta H \Theta^{-1}, \tag{4}$$
where the positive, respectively strongly positive operator Θ and its inverse Θ^{-1} are bounded (necessary to avoid spectral pathologies [2]).

Such pseudo- and quasi-Hermitian operator H is self-adjoint with respect to a modified "scalar" product $(\cdot, \Theta\cdot)$, which is indefinite, respectively positive definite. Only the positive definite case can be regarded as a nonstandard representation of quantum mechanics.

Search for a metric Θ is highly non-trivial and we only establish pseudo-Hermiticity trough parity $\mathcal{P}$ and/or time-reversal $\mathcal{T}$.

First-order cylindrical superintegrability

To avoid complications, we follow on the results in my Master's Thesis [3] and choose to classify cylindrical superintegrable systems with first-order additional integrals, i.e. with integrals of the form
$$\begin{aligned} X_1 &= (p_1^A)^2 + \frac{1}{2}\sum_j^3 (s_1^j(\vec{x})p_j + p_js_1^j(\vec{x})) + m_1(\vec{x}), \\ X_2 &= (L_3^A)^2 + \frac{1}{2}\sum_j^3 (s_2^j(\vec{x})p_j + p_js_2^j(\vec{x})) + m_2(\vec{x}), \\ Y &= k_1p_1^A + k_2p_2^A + k_3p_3^A + k_4L_1^A + k_5L_2^A + k_6L_3^A + m(\vec{x}), \end{aligned} \tag{5}$$
where we use the covariant linear and angular momenta w.r.t. a time-independent gauge transformation $A'(\boldsymbol{u}) = A(\vec{x}) + \mathrm{d}\chi(\vec{x})$, $W'(\vec{x}) = W(\vec{x})$
$$p_j^A = p_i + A_j = mv_j, \quad L_j^A = \epsilon_{jkl}x^kp_l^A. \tag{6}$$

Complex fields mean that all the functions B^j, W, s_μ^j, m_μ, m, with $\mu = 1, 2$, $j = x, y, z$ are complex functions of real variables and the constants $k_j \in \mathbb{C}$.

We impose (2), solve the equations obtained by collecting coefficients of derivatives such as ∂_{xy} coming from the momenta $\vec{p}_j = -i\hbar\partial_j$ that must separately vanish and obtain 1 new maximally superintegrable system and extend the 3 systems found in my Master Thesis [3] by allowing for complex coupling constants.

The new system

The fields of the new system are
$$\begin{aligned} \vec{B}(\vec{x}) &= \left(-\frac{i(\alpha + i\beta)}{(x^1 - ix^2)^3}, -\frac{(\alpha + i\beta)}{(x^1 - ix^2)^3}, 0\right), \\ W(\vec{x}) &= \frac{w_1 + iw_2}{2(x^1 - ix^2)^2} - \frac{(\alpha + i\beta)^2}{8(x^1 - ix^2)^4}, \end{aligned} \tag{7}$$
where α, β, w_1, w_2 are all real.

We have 3 first order integrals,
$$Y_1 = ip_1^A + p_2^A,\ Y_2 = L_2^A + iL_1^A - \frac{\alpha + i\beta}{(x^1 - ix^2)},\ \tilde{X}_2 = p_3^A + \frac{\alpha + i\beta}{2(x^1 - ix^2)^2}, \tag{8}$$
i.e. one of the cylindrical integrals reduces to a first order one, $X_2 = \tilde{X}_2^2$, followed by the Hamiltonian and the second cylindrical integral
$$H = \frac{1}{2}((p_1^A)^2 + (p_2^A)^2 + (p_3^A)^2) + \frac{w_1 + iw_2}{2(x^1 - ix^2)^2} - \frac{(\alpha + i\beta)^2}{8(x^1 - ix^2)^4}, \tag{9}$$
$$X_1 = (L_3^A)^2 - (\alpha + i\beta)\frac{(x^1 + ix^2)}{x^1 - ix^2}p_3^A - (\alpha + i\beta)^2\frac{(x^1 + ix^2)}{2(x^1 - ix^2)^3} + (w_1 + iw_2)\frac{(x^1 + ix^2)}{x^1 - ix^2}. \tag{10}$$
The system is maximally superintegrable and separable in cylindrical as well as complex coordinates $z = x^1 + ix^2, \bar{z} = x^1 - ix^2$.

Formal gauge fixing and spectral analysis

We are analyzing the Hamiltonian (9). We no longer have gauge freedom, therefore the choice of the vector potential A is unclear. Another problem is the strong singularity of $(x^1 - ix^2)^{-4}$.

We proceed formally by choosing the gauge that eliminates the most singular term
$$\vec{A}(\vec{x}) = \left(0, 0, -\frac{\alpha + i\beta}{2(x^1 - ix^2)^2}\right), \tag{11}$$
so we have
$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) - \frac{\alpha + i\beta}{2(x^1 - ix^2)^2}p_3 + \frac{w_1 + iw_2}{2(x^1 - ix^2)^2}. \tag{12}$$
Such Hamiltonian is pseudo-Hermitian with respect to the indefinite partial parity $\Theta = \mathcal{P}_2$. It is also $\mathcal{PT}$ self-adjoint, but not self-adjoint with respect to $\mathcal{P}$ nor $\mathcal{T}$ alone.

We conjecture that its spectrum is purely essential, but are not able to prove it. In the complex coordinates the Schrödinger equation can be solved by separation of variables,
$$\Psi\left(z, \bar{z}, x^3\right) \sim N\exp\left(Cz + \frac{\lambda_3^2 - 2E}{4C\hbar^2}\bar{z} + \frac{w_1 + iw_2 - \lambda_3(\alpha + i\beta)}{4C\hbar^2}\frac{1}{\bar{z}}\right)\exp\left(\frac{i}{\hbar}\lambda_3x^3\right), \tag{13}$$
where C is the separation constant and $\lambda_3 \in \mathbb{R}$. The first exponential is not purely imaginary, so we cannot construct the Weyl sequence.

Known systems generalized to complex coupling constants

The other solutions are systems from [3] with complex coupling constants.

1. Purely constant magnetic field
$$\vec{B}(\vec{x}) = (0, 0, \alpha + i\beta), \quad W(\vec{x}) = 0, \tag{14}$$
2. Constant magnetic field with potential in the x^3 direction
$$\vec{B}(\vec{x}) = (0, 0, \alpha + i\beta), \quad W(\vec{x}) = W_3(x^3), \tag{15}$$
3. Non-constant magnetic field
$$\vec{B} = \left(\frac{4(\alpha + i\beta)}{(x^2)^3}, 0, 0\right), \quad W = -4\left(\frac{(\alpha + i\beta)^2}{2(x^2)^4} + \frac{w_1 + iw_2}{(x^2)^2}\right), \tag{16}$$

These are at most pseudo-Hermitian when we use only combinations of $\mathcal{P}$ and $\mathcal{T}$. We expect purely essential spectrum but, as above, we do not have a proof.

Conclusions

- We looked for the first order superintegrable systems of the cylindrical type that admit complex-valued fields, i.e. $B^j, W : \mathbb{R}^3 \to \mathbb{C}$.
- We have obtained four such systems, three of them known from [3] but generalized to complex coupling constants and one new system (7), with typically complex integrals.
- The new system separates in cylindrical and complex coordinates.
- The new system is maximally superintegrable with 3 first order integrals, the Hamiltonian and one quadratic cylindrical integral.
- We face 2 problems in rigorous analysis of these systems: lost gauge invariance and complex spectra.

Acknowledgment

Based on O. Kubů and L. Šnobl. Cylindrical first-order superintegrability with complex magnetic fields, *J. Math. Phys.*, **64** (2023) 062101

OK thanks David Krejčiřík for encouragement to investigate the topic of complex magnetic fields and discussions about the spectral analysis of the found systems.

OK was supported by the Grant Agency of the Czech Technical University in Prague, grant No. SGS22/178/OHK4/3T/14.

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