SPECTRAL STABILITY OF QUANTUM SYSTEMS ON A HALF-LINE

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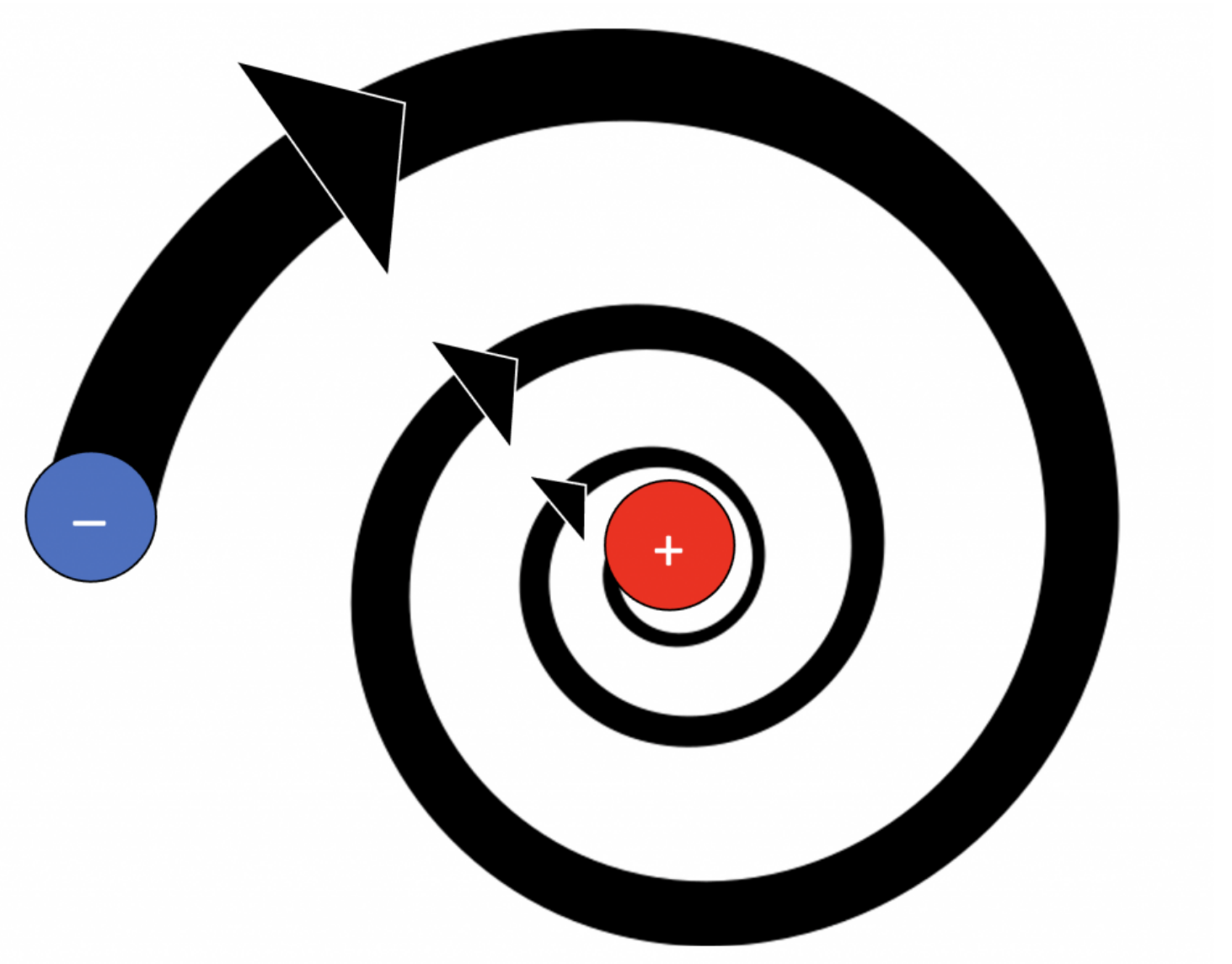
Motivation - Stability of matter

In 1911 Ernest Rutherford discovered that atoms consist of a centralized positive charge surrounded by an electron shell. According to the Rutherford model, electrons orbiting the center would lose their energy by radiation and eventually fall into it. However, classical physics predicted that this would have happened in a matter of nanoseconds and all the mass should have collapsed instantly, which is in direct contradiction to our experience. This paradox was one of the impeti that led to the development of quantum mechanics. The obvious stability of matter is consequnce of the stability of the spectrum of the Laplacian $-\Delta$ in $\mathbb{R}^3$ against additive perturbations $V:\mathbb{R}^3\to\mathbb{C}$ satisfying the subordination condition [4]

$$\forall\psi\in H^1(\mathbb{R}^3)\,, \qquad \int_{\mathbb{R}^3}|V|\,|\psi|^2\le a\int_{\mathbb{R}^3}|\nabla\psi|^2\,, \tag{1}$$

with $a<1$. Indeed, (1) together with the Hardy inequality implies that the spectrum of the Coulomb Hamiltonian is bounded from below. There are generalizations to higher dimensions $n>3$ [5, 4], but due to the absence of the Hardy inequality, there is none in dimensions $n=1,2$. Nevertheless, there exist two-dimensional analogues for the magnetic Laplacian [3]. To get a non-trivial result in dimension one, it is necessary to restrict the Laplacian to a half-line and subject to the Robin boundary condition [8]. We stress that half-line problems are especially relevant and important since they represent the radial part of a spherically symmetrical problem.



What - Stability of the spectrum

Consider the following formal problem. Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and B^*A possibly a non-self-adjoint but closed operator - potential, physically speaking. We say that the spectrum of $T+B^*A$ is **stable** (with respect to the free operator T) if

$$\sigma_\iota(T+B^*A)=\sigma_\iota(T),$$

where ι represent its individual components, *i.e.* discrete, essential and residual spectrum.

How - Birman-Schwinger principle

Let T be a self-adjoint operator acting in $\mathcal{H}$ and $A,B^*:\mathcal{H}\to\mathcal{H}$ possibly non-self-adjoint but closed linear maps with $\operatorname{dom}(T)\subset\operatorname{dom}(A)\cap\operatorname{dom}(B)$. Let us introduce the *Birman-Schwinger operator* $K^T(z)$ as

$$K^T(z):=A(T-z)^{-1}B^*,$$

for every $z\in\rho(T)$. Here we recall famous Kato's result [6] which, in terms of $K^T(z)$, guarantees preservation of the spectrum of T under perturbation B^*A.

Theorem [6, Kato, 1966]

Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and suppose that B^*,A are closed operators in $\mathcal{H}$ with $\operatorname{dom}(T)\subset\operatorname{dom}(A)\cap\operatorname{dom}(B)$ such that

$$\sup_{z\in\rho(T)}\|A(T-z)^{-1}B^*\|<1\,.$$

Then there exists a closed extension T_V of $T+B^*A$ which is similar to T satisfying for all $z\in\rho(T)$

$$(T-z)^{-1}-(T_V-z)^{-1}=\overline{(T-z)^{-1}B^*}A(T_V-z)^{-1}.$$

Consequently, the possibly non-self-adjoint operator T_V is **quasi-self-adjoint** [9].

Why - Half-line Robin Laplacians

Our main motivation is to find relativistic analogue to the following result and investigate the compatibility in the non-relativistic limit. Given by non-negative parameter $\beta\in(0,+\infty)$ and mass $m>0$ consider the half-line Robin Laplacian

$$H_\beta:=-\frac{1}{2m}\frac{d^2}{dx^2},$$
$$\operatorname{dom}(H_\beta):=\left\{\psi\in H^2((0,+\infty))\mid\psi'(0)=\beta\psi(0)\right\}.$$

Theorem [8, Krejčiřík, Laptev, Štampach, 2022]

Let $V\in L^1\big((0,+\infty),\mathbb{C};(1+x)\,dx\big)$ satisfy

$$\int_0^{+\infty}\int_0^{+\infty}|V(x)|\left(\frac{2m}{\beta}+2m\min(x,y)\right)^2|V(y)|\,dx\,dy<1\,. \tag{2}$$

Then $\sigma(H_\beta+V)=\sigma_{\mathrm{c}}(H_\beta+V)=\sigma(H_\beta)=\sigma_{\mathrm{c}}(H_\beta)$.

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Half-line Dirac operators

For $\alpha\in(0,+\infty)$ and $m,c>0$ we define family of Dirac operators on a half-line as follows

$$D_\alpha:=\begin{pmatrix} mc^2 & -c\dfrac{d}{dx}\\ c\dfrac{d}{dx} & -mc^2\end{pmatrix}.$$

Contrary to the recent approaches in [2, 1], however, we also make the boundary condition c-dependent (and in fact also m-dependent):

$$\operatorname{dom}(D_\alpha):=\left\{\psi\in H^1((0,+\infty),\mathbb{C}^2)\mid\psi_1(0)\frac{\alpha}{mc}=\psi_2(0)\right\}.$$

– Our goal is twofold. First, we present the sufficient condition for the preservation of the spectrum of D_α.

Theorem [7, Kramár, Krejčiřík, 2024]

Let $V\in L^1\big((0,+\infty),\mathbb{C}^{2,2};(1+x)\,dx\big)$ satisfty

$$\frac{1}{c^2}\int_0^{+\infty}\int_0^{+\infty}|V(x)|\left[1+\big(q_c+2mc\min(x,y)\big)^2\right]|V(y)|\,dx\,dy<1\,, \tag{3}$$

where $q_c:=\max\left(\frac{\alpha}{mc},\frac{mc}{\alpha}\right)$. Then $D_\alpha+V$ is similar to D_α.

Note that similarity of the perturbed and original operators yields a coincidence of spectra and all of their components individually.

Non-relativistic limit

Second step is to relate the sufficient condition (3) to the non-relativistic analogue given by (2). Sending c in (3) to infinity, we formally (here V is a matrix-valued function) arrive at the sufficient condition (2), which guarantees the stability of the spectrum of the non-relativistic operator $H_\beta+V$ under the identification $\beta=2\alpha$. In addition, not only that the sufficient conditions are compatible, but the operators do so as well.

Theorem [7, Kramár, Krejčiřík, 2024]

Let $\alpha\in(0,+\infty)$ and $m>0$. If $z\in\rho(H_{2\alpha})$, then, for all sufficiently large c, $z\in\rho(D_\alpha-mc^2)$ and

$$\lim_{c\to+\infty}\left\|(D_\alpha-mc^2-z)^{-1}-\begin{pmatrix}(H_{2\alpha}-z)^{-1} & 0\\ 0 & 0\end{pmatrix}\right\|=0\,.$$

Compatibility of the non-relativistic limit

In other words, our results can be summarized in the following commutative diagram, which holds for all $\alpha\in(0,+\infty)$. The bottom row symbolically represents the sufficient conditions for the corresponding operators.

$$\begin{tikzcd}
D_\alpha \arrow[r, "c\to+\infty"] \arrow[d] & H_{2\alpha} \arrow[d] \\
\|K^{D_\alpha}(z)\|<1 \arrow[r, "c\to+\infty"] & \|K^{H_{2\alpha}}(z)\|<1
\end{tikzcd}$$

In the special setting when $V_{12}=V_{21}=V_{22}=0$ with $V:(0,+\infty)\to\mathbb{C}^{2,2}$, the correspondence is truly one to one.

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