

Unbounded damped waves: backward uniqueness and polynomial stability

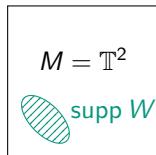
Perry Kleinhenz

Michigan State University

Joint work with Ruoyu P. T. Wang (University College London)

Damped Wave Equation

- (M, g) compact manifold, Δ Laplace-Beltrami
- $W(x) \in C^0(M)$ non-negative.



Let u solve

$$\begin{cases} \partial_t^2 u - \Delta u + W \partial_t u = 0 & \text{in } M \times \mathbb{R}^+, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in H^2(M) \times H^1(M). \end{cases}$$

$$E(u, t) = \int_M \underbrace{|\nabla u(x, t)|^2}_{\text{potential}} + \underbrace{|\partial_t u(x, t)|^2}_{\text{kinetic}} dx.$$

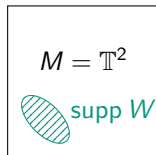
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$r(t) \rightarrow 0$ as $t \rightarrow \infty$, such that

$$E(u, t)^{1/2} \lesssim r(t)?$$

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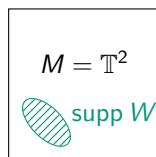
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Energy Decay Rates and Resolvent Estimates

$$\partial_t^2 u - \Delta u + W \partial_t u = 0,$$

$$\partial_t \mapsto i\lambda$$

$$(-\Delta u + i\lambda W - \lambda^2)u = f, \quad u := R(\lambda)f.$$

Resolvent estimates for $\lambda \in \mathbb{R}$, $|\lambda| \geq \lambda_0$, give stabilization rates (Gearhart, Prüss, Huang 1980s)

$$\|R(\lambda)\|_{L^2 \rightarrow L^2} \leq \frac{C}{|\lambda|} \iff E(u, t) \lesssim Ce^{-ct}.$$

(Borichev-Tomilov 2010)

$$\|R(\lambda)\|_{L^2 \rightarrow L^2} \leq C|\lambda|^{\frac{1}{\alpha}-1} \iff E(u, t) \lesssim \frac{1}{t^\alpha}.$$

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$$\|R(\lambda)\|_{L^2 \rightarrow L^2} \leq Ce^{c|\lambda|} \iff E(u, t) \lesssim \frac{1}{\log(2+t)}.$$

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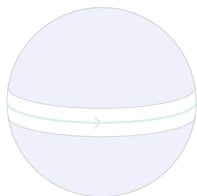
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Geometric Control Condition (GCC)

There exists $L > 0$, such that every unit speed geodesic γ of length at least L intersects $\{W > 0\}$.



$M = \mathbb{S}^2$, GCC



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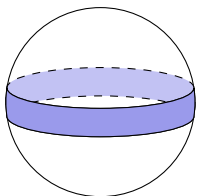
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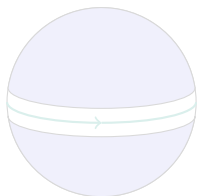
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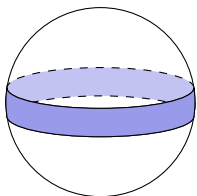
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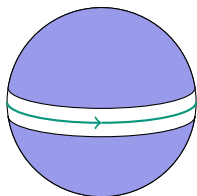
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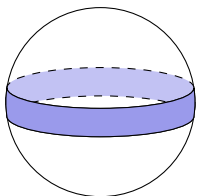
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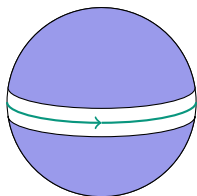
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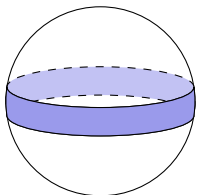
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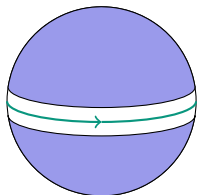
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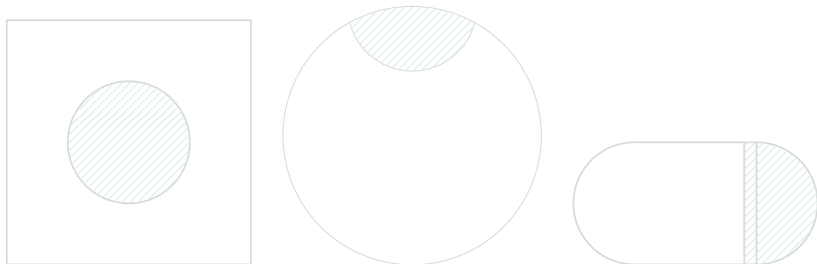
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K '23 generalizes to W time dependent

Schrödinger observability

The Schrödinger equation $(i\partial_t + \Delta)v(x, t) = 0$ is exactly observable from an open set $\Omega \subset M$, if there are $T, C_T > 0$ such that for all $v(x, t = 0) \in H^2$,

$$\|v(x, t = 0)\|_{L^2} \leq C_T \int_0^T \int_{\Omega} |v(x, t)|^2 dx dt.$$



Theorem (Anantharaman-Leautaud '14)

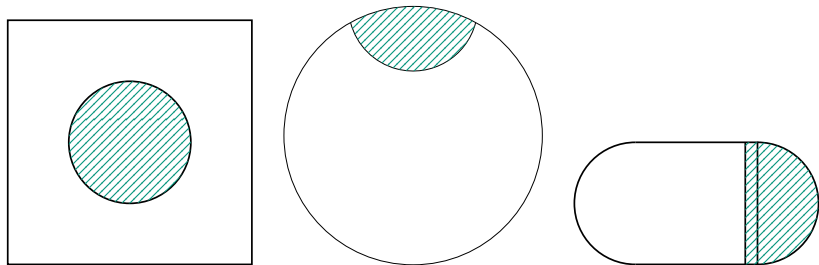
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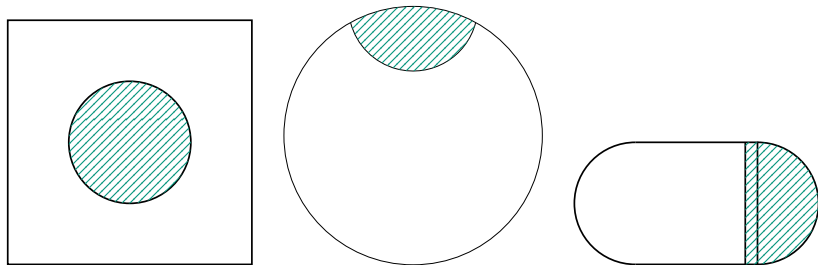
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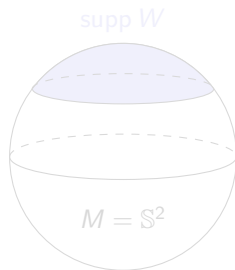
$\{W > 0\}$ Open

Theorem (Lebeau '93, Burq '96)

If $\{W > 0\}$ is open and nonempty, then

$$E(u, t)^{1/2} \lesssim \frac{1}{\log(2+t)} \quad \text{and} \quad \|R(\lambda)\|_{L^2 \rightarrow L^2} \leq C e^{C|\lambda|}.$$

This logarithmic decay is sharp.



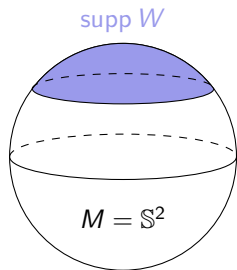
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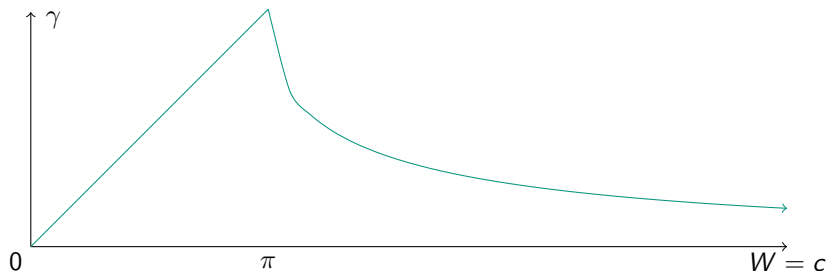
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Theorem (Cox-Zuazua '03)

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Decay Speed



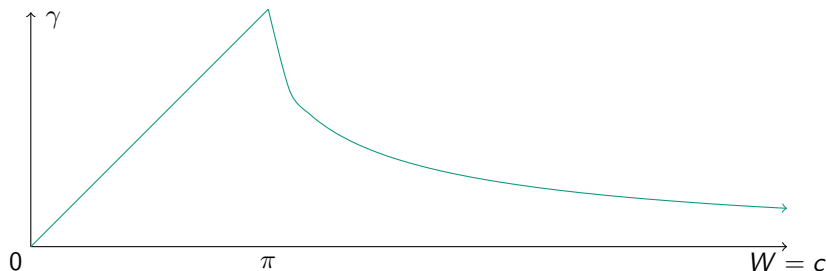
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Key observation: energy decay is **not monotonic** in the size of the damping.

Abstract damped wave equation

- The initial value problem for damped wave equation:

$$\begin{cases} (\partial_t^2 - \Delta + W\partial_t) u(t) = 0 \\ u(0) = u_0 \in H^2, \partial_t u(0) = u_1 \in H^1. \end{cases}$$

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Example

- Unbounded damped waves: $P = -\Delta$, $Q = \sqrt{W}$, where $W \in L^p(M)$ for $p \in (d, \infty)$, $d = \dim(M)$. Here $Q : H^{\frac{d}{2p}} \rightarrow L^2$, $\gamma = \frac{d}{4p}$.

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- Water waves: $P = |D| = \sqrt{-\Delta}$, $Q = \sqrt{W}|D|^s$, $s \in [0, 1/2]$

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- Damped beam/plate: $P = \Delta^2$, $Q = \sqrt{W} \nabla_x$.

$$(\partial_t^2 + \Delta^2 + \nabla_x^* W(x) \nabla_x \partial_t) u(t) = 0$$

Here $Q : H^1(M) \rightarrow L^2(T^*M)$, $\gamma = \frac{1}{4}$.

Stabilization Rates

Theorem (K-Wang '23)

u solves DWE on compact manifold M with $d = \dim(M)$.

$$E(u, t)^{1/2} \lesssim r(t).$$

$\{W \geq \varepsilon\}$	$P = -\Delta$ $W \in L^\infty$	$P = -\Delta$ $W \in L^p$	$P = D $ $ D ^s W D ^s$	$P = \Delta^2$ $\nabla_x^* W \nabla_x$
Geometric control	e^{-t}			
Schrödinger obs.	$1/\sqrt{t}$			
Open	$\frac{C}{\log \langle t \rangle}$			

Sharp

Stabilization Rates

Theorem (K-Wang '23)

u solves DWE on compact manifold M with $d = \dim(M)$.

$$E(u, t)^{1/2} \lesssim r(t).$$

$\{W \geq \varepsilon\}$	$P = -\Delta$ $W \in L^\infty$	$P = -\Delta$ $W \in L^p$	$P = D $ $ D ^s W D ^s$	$P = \Delta^2$ $\nabla_x^* W \nabla_x$
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- Schrödinger observability: there is $C_T, T > 0$, uniformly for $u \in H_1$,

$$\underbrace{\|u\|_H}_{\text{initial data}} \leq C_T \underbrace{\int_0^T \|Qe^{itP}u\|_Y dt}_{\text{observation in time } [0, T]}.$$

- Control estimates: for all $\lambda \in \mathbb{R}_{\geq 0}$, $u \in H_1$:

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$$\|u\|_H \leq C\langle \lambda \rangle^\beta (\langle \lambda \rangle^{-1} \|(P - \lambda^2)\Lambda^{\mu-\gamma} u\|_H + \|Q\Lambda^\mu u\|_Y) + C\|\Lambda^{-N} u\|_H.$$

Here $N > 0$, $\mu \geq 0$ is a flexible parameter.

- Unique Continuation Principle (UCP): $(P - \lambda^2)u = 0 \Rightarrow Qu \neq 0$ for all $\lambda > 0$.

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Let $Q : H_\gamma \rightarrow Y$, and assume UCP holds. Fix $\mu \in [0, \gamma]$, $\beta \geq 0$. If modified control estimate holds uniformly for $u \in H_1$, for $\lambda \geq \lambda_0 > 0$, Then

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Use P^α with $\alpha \geq 2\gamma$, with loss (or gain!)

Thank you for your attention!

Finite Time Extinction

- Finite time extinction: $u(t) \equiv 0$ for all $t \geq T$, while initial data non-trivial.
- Nonlinear phenomenon. But happens with singular damped waves.
- Castro–Cox '01, Freitas–Hefti–Siegl '20: $M = [0, 1]$ with Dirichlet boundary condition, $W = 2x^{-1}$, then $u \equiv 0$ for all $t \geq 2$ for all initial data. There is finite time extinction.
- $\sqrt{W} = Cx^{-\frac{1}{2}}$ maps $H_{1/4+\epsilon}(= H_0^{\frac{1}{2}+\epsilon})$ to L^2 , but is unbounded on $H_{1/4} = H^{\frac{1}{2}}$.
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Theorem (Kleinhenz-W. '23 (I): optimal backward uniqueness)

Let $Q \in \mathcal{L}(H_{1/4}, Y)$. Then $u(t) = 0$ for some $t > 0$ implies $u(0) = u'(0) = 0$.
 Index $\gamma = 1/4$ is optimal.

Corollary (Weak monotonicity)

Let $Q, \bar{Q} \in \mathcal{L}(H^\gamma : Y)$ such that $\|Qu\|_Y \leq C\|\bar{Q}u\|_Y$ for all $u \in H^1$. Then

$$E_Q^{1/2} \leq C\langle t \rangle^{-\frac{1}{\alpha}} \Rightarrow E_{\bar{Q}}^{1/2} \leq C\langle t \rangle^{-\frac{1}{2\alpha+4\gamma}}.$$

- Larger damping does not guarantee faster/equal decay, but does guarantee a decay at the square root of the original speed.
- In other words, the energy decay is **monotonic at a slower scale**.

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Future directions

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- Know: control estimates (monotonic) \Leftrightarrow suboptimal decay (monotonic)
- Control estimates + commutator condition on $[P, Q^*Q]$ (not monotonic) \Leftrightarrow optimal decay (not monotonic)?
- When $Q = Q_0 P^\gamma$ for $Q_0 : H \rightarrow Y$ bounded, can we remove the loss 4γ induced by the unboundedness? In this case the commutator

$$[P, Q^*Q] = P^\gamma [P, Q_0^*Q_0] P^\gamma,$$

has a better structure.

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