

Limiting absorption principle for a degenerate problem modelling a plasma resonance

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Main subject: a problem from plasma physics

A limiting absorption principle for

$$-\operatorname{div}(\alpha \nabla u) + \lambda u = f \text{ in a bounded } \Omega$$

(equipped with 'well-chosen' BCs)

α is a **regular** function **changing its sign** inside the domain Ω .

The problem describing **plasma heating** due to the upper hybrid resonance

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Outline

- 1 the problem setting

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- 1 the problem setting
- 2 a self-adjoint operator associated to the problem

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- 1 the problem setting
- 2 a self-adjoint operator associated to the problem
- 3 a limiting absorption principle through a **viscosity limit** and the **definition of the limiting non-self adjoint operator**
- 4 some results on its spectrum (mostly in 1D)

Problem Setting

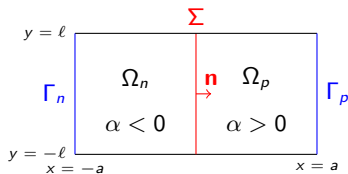


A boundary value problem

$$\operatorname{div}(\alpha(x, y) \nabla u) - \omega^2 u = f \text{ in } \Omega,$$

$$\gamma_0 u = 0 \text{ on } \Gamma_n \cup \Gamma_p,$$

periodic BCs on $y = |\ell|$.



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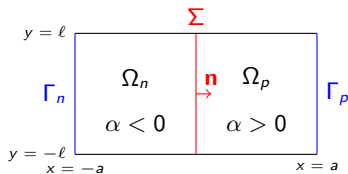
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Assumption

The function $\alpha \in C_{per,y}^\infty(\bar{\Omega}; \mathbb{R})$:

- 1 $\alpha(x, y) = 0$ on $\Sigma = \{x = 0\}$
- 2 $\partial_x \alpha(0, y) \neq 0$ (α is positive in Ω_p and negative in Ω_n).

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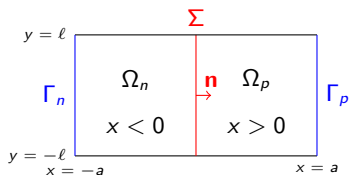
Brevity assumptions

For brevity:

$$\alpha(x, y) = x \text{ (not necessary), } \omega^2 = 0 \text{ (not necessary up to a discrete subset of } \mathbb{R})$$

BCs will be implicit till the end of the talk

An auxiliary space



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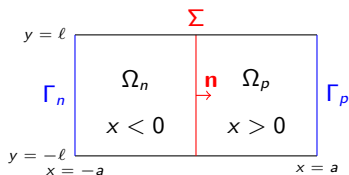
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Motivation

$$a(u, v) = \int_{\Omega_p} x \nabla u \nabla v + \int_{\Omega_n} x \nabla u \nabla v$$

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A space of 'regular' functions

$$\begin{aligned} \mathcal{V}_{reg} &= \mathcal{V}_{reg}(\Omega_n) \times \mathcal{V}_{reg}(\Omega_p), \\ \mathcal{V}_{reg}(\Omega_j) &= \{v \in L^2(\Omega_j) : \nabla v \in L^2_{loc}(\Omega_j), \\ &\quad |x|^{1/2} \nabla v \in L^2(\Omega_j) \text{ +BCs}\} \end{aligned}$$

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$$\mathcal{V}_{reg} = \mathcal{V}_{reg}(\Omega_p) \times \mathcal{V}_{reg}(\Omega_n),$$
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Properties

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- 2 $C^\infty(\overline{\Omega} \setminus \Sigma) \cap \mathcal{V}_{reg}$ are dense in \mathcal{V}_{reg} (Grisvard '63)
- 3 compact embedding into $L^2(\Omega)$ (cf. e.g. Hoai-Minh Nguyen '16)

The sign-changing operator on the space of regular functions

Those results are due to (Nicolopoulos, Desprès, Campos-Pinto, Ciarlet 2020)

Well-posedness result

$$\mathcal{A}_{reg} \mathbf{u} = \operatorname{div}(x \nabla \mathbf{u}), \quad D(\mathcal{A}_{reg}) = \{\mathbf{u} \in \mathcal{V}_{reg} : \operatorname{div}(x \nabla \mathbf{u}) \in L^2(\Omega) + BCs\}.$$

The operator $\mathcal{A}_{reg} : D(\mathcal{A}_{reg}) \rightarrow L^2(\Omega)$ is self-adjoint and invertible.

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A sketch of the proof

A property of the normal trace

If $\mathbf{u} \in D(\mathcal{A}_{reg})$, then

$$x \nabla \mathbf{u} \cdot \mathbf{n}|_{\Sigma} = 0.$$

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Decoupling

\mathbf{u} satisfies $\mathcal{A}_{reg} \mathbf{u} = f \iff \mathbf{u}_j = \mathbf{u}|_{\Omega_j}, j \in \{n, p\}$ solves :

$$\text{Find } \mathbf{u}_j \in \mathcal{V}_{reg}(\Omega_j) : \operatorname{div}(x \nabla \mathbf{u}_j) = f_j \text{ in } \Omega_j, \quad +BCs$$

(Lax-Milgram)

Extra regularity (Baouendi, Goulaouic '69)

$$\begin{aligned} D(\mathcal{A}_{reg}) &= \{v \in \mathcal{V}_{reg}(\Omega_n) \times \mathcal{V}_{reg}(\Omega_p) : \operatorname{div}(x\nabla v) \in L^2(\Omega) + BCs\} \\ &= \{v \in H^1(\Omega_n) \times H^1(\Omega_p) : \operatorname{div}(x\nabla v) \in L^2(\Omega) + BCs\}. \end{aligned}$$

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A 'problem' with \mathcal{A}_{reg}

For $\nu > 0$ (**dissipation in plasmas**), consider $u^\nu \in H^1(\Omega)$ solving

$$\operatorname{div}((x + i\nu)\nabla u^\nu) = f \quad \text{in } \Omega, +BCs.$$

Plasma heating: $\lim_{\nu \rightarrow 0^+} \nu \int_{\Omega \setminus \Sigma} |\nabla u^\nu|^2 dx \neq 0$

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Perhaps, $\lim_{\nu \rightarrow 0^+} u^\nu$ is not a solution of the **above** self-adjoint problem?

This discrepancy was first remarked by [Bruno Desprès](#) and his co-workers

A 1D illustration

For illustration: we consider 1D boundary-value problems

A boundary-value problem

$$\partial_x(x\partial_x u) = 0 \text{ on } (-1, 1), \quad u(-1) = c_1, \quad u(1) = c_2$$

Solution given by \mathcal{A}_{reg}^{-1}

$$u(x) = \begin{cases} c_1, & x < 0, \\ c_2, & x > 0. \end{cases}$$

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A boundary-value problem with absorption

$$\partial_x((x + i\nu)\partial_x u^\nu) = 0 \text{ on } (-1, 1), \quad u^\nu(-1) = c_1, \quad u^\nu(1) = c_2$$

Limiting solution

$$u^\nu = C^\nu + A^\nu \log(x + i\nu) \rightarrow C + A(\log|x| + i\pi\mathbb{1}_{x<0}), \quad C = c_2, \quad A = \frac{c_1 - c_2}{i\pi}.$$

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Two remarks

Plasma heating: $\int_{-1}^1 \nu |\partial_x u^\nu|^2 \rightarrow \pi A^2$
 $\log|x| \notin \mathcal{V}_{reg}(\Omega)$

The limiting absorption problem

Find $u^\nu \in H^1(\Omega)$, s.t.

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The goals

- 1 prove that $u^\nu \rightarrow u^+$ in $L^2(\Omega)$ as $\nu \rightarrow 0+$ and characterize the regularity of u^+
- 2 define the limiting operator \mathcal{A}^+ , s.t. $\mathcal{A}^+ u^+ = f$
- 3 end of the talk: some partial results on the spectrum of \mathcal{A}^+

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The first question: 1D/slab+constraints on α by [Bruno Desprès](#) and his co-workers ([L.-M. Imbert-Gerard](#), [R. Weder](#), [O. Lafitte](#), [M-C. Pinto](#), [A. Nicolopoulos](#)) using explicit/semi-explicit computations (no well-posedness etc.)

We attempt in particular at obtaining results without the above constraints

The limiting absorption problem

Find $u^\nu \in H^1(\Omega)$, s.t.

$$\operatorname{div}((x + i\nu)\nabla u^\nu) = f \quad \text{in } \Omega, \quad +BCs, \quad \nu > 0.$$

The principal observation

By integration by parts (take as a test function xu^ν) it follows:

$$\|u^\nu\|_{L^2(\Omega)} \leq C_1 \|f\| \implies \|xu^\nu\|_{L^2(\Omega)} \leq C_2 \|f\|.$$

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A space of 'singular' functions

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(cf. with $|x|^{1/2}\nabla v \in L^2$ for \mathcal{V}_{reg})

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(cf. with $|x|^{1/2}\nabla v \in L^2$ for \mathcal{V}_{reg})

An important difference with \mathcal{V}_{reg}

If $v \in \mathcal{V}_{sing}(\Omega_j)$ and $\operatorname{div}(x\nabla v) \in L^2(\Omega)$,

$$x\nabla v \cdot n|_\Sigma \neq 0.$$

E.g. $v(x, y) = \log|x|$, $x\nabla v \cdot n|_\Sigma = 1$.

The limiting absorption problem

Find $u^\nu \in H^1(\Omega)$, s.t.

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LAP

There exists $C > 0$, s.t. for all $\nu > 0$,

$$\|u^\nu\|_{\mathcal{V}_{\text{sing}}(\Omega)}^2 = \|u^\nu\|_{L^2}^2 + \int_{\Omega} x^2 |\nabla u^\nu|^2 \leq C \|f\|^2.$$

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Sketch of the proof: (multiplier-like techniques)

Test with

$v^\nu := x u^\nu + p^\nu$, $p^\nu \in H^1(\Omega \setminus \Sigma)$ to be chosen:

$$- \int_{\Omega \setminus \Sigma} (x + i\nu) \nabla u^\nu \overline{\nabla v^\nu} + \int_{\Sigma} (x + i\nu) \partial_x u^\nu \overline{[\gamma_0^\Sigma p^\nu]} = \int_{\Omega} f \overline{v^\nu}.$$

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$$- \int_{\Omega} |x \nabla u^\nu|^2 - \int_{\Omega \setminus \Sigma} x \partial_x u^\nu \overline{u^\nu} - \int_{\Omega \setminus \Sigma} x \nabla u^\nu \overline{\nabla p^\nu}$$

$$+ \int_{\Sigma} i\nu \partial_x u^\nu \overline{[\gamma_0^\Sigma p^\nu]} + \text{extra terms in } \nu = \int_{\Omega} f \overline{(x u^\nu + p^\nu)}.$$

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Key idea: take $p^\nu \in H^1(\Omega \setminus \Sigma)$ so that

$$- \int_{\Omega \setminus \Sigma} x \partial_x u^\nu \overline{u^\nu} - \underbrace{\int_{\Omega \setminus \Sigma} x \nabla u^\nu \overline{\nabla p^\nu}}_{a(u^\nu, p^\nu)} = - \int_{\Omega \setminus \Sigma} u^\nu \overline{u^\nu}$$

The LAP result

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There exists $C > 0$, s.t. for all $\nu > 0$,

$$\|u^\nu\|_{V_{\text{sing}}^2(\Omega)}^2 = \|u^\nu\|_{L^2}^2 + \int_{\Omega} x^2 |\nabla u^\nu|^2 \leq C \|f\|^2.$$

$$\begin{aligned} & - \int_{\Omega} |x \nabla u^\nu|^2 - \int_{\Omega \setminus \Sigma} x \partial_x u^\nu \overline{u^\nu} - \int_{\Omega \setminus \Sigma} x \nabla u^\nu \overline{\nabla p^\nu} \\ & + \int_{\Sigma} i\nu \partial_x u^\nu \overline{[\gamma_0^\Sigma p^\nu]} + \text{extra terms in } \nu = \int_{\Omega} \overline{f(xu^\nu + p^\nu)}. \end{aligned}$$

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Baouendi, Goulaouic '69: $p^\nu \in H^1(\Omega \setminus \Sigma)$, s.t.

$$\operatorname{div}(x \nabla p^\nu) = \overline{x \partial_x u^\nu - u^\nu} + BCs.$$

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There exists $C > 0$, s.t. for all $\nu > 0$,

$$\|u^\nu\|_{\mathcal{V}_{sing}(\Omega)}^2 = \|u^\nu\|_{L^2}^2 + \int_{\Omega} x^2 |\nabla u^\nu|^2 \leq C \|f\|^2.$$

Main problems with the above result

- $u^\nu \rightharpoonup u^+$ in $L^2(\Omega)$, but not better, since $\mathcal{V}_{sing}(\Omega)$ is **not compactly embedded** into $L^2(\Omega)$
- this space is a bad candidate for posing the limiting problem

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$u^+ \in \mathcal{V}_{sing}$ indeed satisfies

$$\mathcal{A}_{sing} u^+ = f, \quad \mathcal{A}_{sing} \cdot = \operatorname{div}(x \nabla \cdot), \quad D(\mathcal{A}_{sing}) = \{v \in \mathcal{V}_{sing}(\Omega) : \operatorname{div}(x \nabla v) \in L^2(\Omega), BCs\}.$$

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$$\mathcal{A}_{sing} u^+ = f, \quad \mathcal{A}_{sing} \cdot = \operatorname{div}(x \nabla \cdot), \quad D(\mathcal{A}_{sing}) = \{v \in \mathcal{V}_{sing}(\Omega) : \operatorname{div}(x \nabla v) \in L^2(\Omega), BCs\}.$$

Construction of the kernel of \mathcal{A}_{sing} of infinite dimension (Nicolopoulos et al 2020)

For any $g \in H_{per}^2(\Sigma)$, $(x, y) \mapsto s_g(x, y) = g(y) \log|x|$ in $D(\mathcal{A}_{sing})$

LAP

There exists $C > 0$, s.t. for all $\nu > 0$,

$$\|u^\nu\|_{\mathcal{V}_{sing}(\Omega)}^2 = \|u^\nu\|_{L^2}^2 + \int_{\Omega} x^2 |\nabla u^\nu|^2 \leq C \|f\|^2.$$

Main problems with the above result

- $u^\nu \rightharpoonup u^+$ in $L^2(\Omega)$, but not better, since $\mathcal{V}_{sing}(\Omega)$ is **not compactly embedded** into $L^2(\Omega)$
- this space is a bad candidate for posing the limiting problem

$u^+ \in \mathcal{V}_{sing}$ indeed satisfies

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Define $\Psi_g \in \mathcal{V}_{reg}(\Omega)$ satisfying the well-posed pb:

$$\operatorname{div}(x \nabla \Psi_g) = -\operatorname{div}(x \nabla s_g) + BCs.$$

Then $\Psi_g + s_g \in D(\mathcal{A}_{sing})$ and belongs to $\operatorname{Ker} \mathcal{A}_{sing}$.

Main idea

The situation is similar to formulating the limiting problem for $-\Delta u - (\omega + i\nu)^2 u = f$ in \mathbb{R}^d :
no existence in a smaller space (e.g. $H^1(\Omega)$), but **no uniqueness** in a larger space (e.g. $\mathcal{V}_{sing}(\Omega)$)

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Introduce a **radiation condition** to ensure the uniqueness

1D

Suppose that $u^\nu \in H_0^1(-1, 1)$ solves

$$\partial_x((x + i\nu)\partial_x u^\nu) = f \text{ in } (-1, 1).$$

Then $u^\nu \xrightarrow{L^2} u^+$, where

$$u^+ = u_{sing}^+ + u_{reg}^+, \quad u_{sing}^+(x) = A(\log|x| + i\pi\mathbb{1}_{x<0}), \quad u_{reg}^+ \in H^1(-1, 1).$$

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Observation

The singular part: $(\log|x| + i\pi\mathbb{1}_{x<0}) \left(\lim_{\nu \rightarrow 0^+} \log(x + i\nu) \right)$

The regular part: **globally in $H^1(\Omega)$** (i.e. **continuous** across the interface Σ)

One candidate (works in 1D)

$$u(x, y) = g(y)(\log |x| + i\pi \mathbb{1}_{x < 0}) + u_{reg}(x, y),$$

where $g \in H_{per}^1(\Sigma)$?, $u_{reg} \in H^1(\Omega)$?

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A 2D technical difficulty

$g \in H_{per}^{1/2}(\Sigma)$ and $u_{reg} \in H^{1/2}(\Omega)$ with $\partial_x u_{reg} \in L^2(\Omega)$ (**sharp**)

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$$u(x, y) = g(y)(\log |x| + i\pi \mathbb{1}_{x < 0}) + u_{\text{reg}}(x, y),$$

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A 2D technical difficulty

$g \in H_{\text{per}}^{1/2}(\Sigma)$ and $u_{\text{reg}} \in H^{1/2}(\Omega)$ with $\partial_x u_{\text{reg}} \in L^2(\Omega)$ (sharp)

$$\begin{array}{|c|c|} \hline \Delta U_g = 0 & \Delta U_g = 0 \\ \hline \end{array}$$

A construction of the 'radiation' condition

Instead of $g \in H_{\text{per}}^{1/2}(\Sigma)$, take its lifting $U_g(x, y) \in H^1(\Omega)$:

$$\Delta U_g = 0 \text{ in } \Omega \setminus \Sigma,$$

$$\gamma_0^\Sigma U_g = g + \text{BCs.}$$

A new radiation condition

We will say that $u \in D(\mathcal{A}_{sing})$ satisfies (RC) if it can be written as follows:

$$u(x, y) = u_{sing}(x, y) + u_{reg}(x, y),$$

with the **singular part**

$$u_{sing}(x, y) = U(x, y)(\log |x| + i\pi\mathbb{1}_{x < 0}),$$

where $U \in H^1(\Omega)$ is s.t. (this is a lifting of g above)

$$\Delta U = 0 \text{ in } \Omega \setminus \Sigma, +\text{BCs.}$$

and the **regular part** $u_{reg} \in H^{1-\varepsilon}(\Omega) \cap \mathcal{V}_{reg}(\Omega)$, for all $\varepsilon > 0$.

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Why $H^{1-\varepsilon}$ for the regular part

Not stable in $D(\mathcal{A}_{sing})$: $u_{sing} \in \mathcal{V}_{sing}$ but $\text{div}(x\nabla u_{sing}) \notin L^2(\Omega)$.

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If the above decomposition exists, it is **unique**.

Conormal derivative

$$x\nabla u \cdot n|_{\Sigma} = x\partial_x u|_{\Sigma} = \gamma_0^{\Sigma} U$$

Two questions

- 1 does the limiting solution $u^+(x, y)$ satisfy the new radiation condition?
- 2 definition of \mathcal{A}^+ with the above radiation condition

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The limiting problem

$\mathcal{A}^+ u^+ := \operatorname{div}(\alpha \nabla u^+) = f$, where

$$D(\mathcal{A}^+) = D(\mathcal{A}_{sing}) \cap \{v \in \mathcal{V}_{sing} : v \text{ satisfies the (RC)}\}.$$

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Theorem

$0 \in \rho(\mathcal{A}^+)$.

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Theorem

$0 \in \rho(\mathcal{A}^+)$.

For presentation purposes: the existence follows from the LAP. I will concentrate on the proof of uniqueness **only** (Rellich-type lemma)

An auxiliary result

Assume $u \in D(\mathcal{A}^+)$ and $\mathcal{A}^+ u = 0$. If $x \nabla u \cdot n|_{\Sigma} = 0$, then $u = 0$.

It remains to show that for $u \in D(\mathcal{A}^+)$

$$\operatorname{div}(x \nabla u) = 0 \implies x \nabla u \cdot n|_{\Sigma} = 0 \quad (\iff U|_{\Sigma} = 0)$$

The Green formula (Ciarlet, MK, Peillon '23, subm.)

Let $v, q \in D(\mathcal{A}^+)$, and let additionally:

$$v = V(\log |x| + i\pi \mathbb{1}_{x < 0}) + v_{reg},$$

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Then

$$(\mathcal{A}^+ v, q) - (v, \mathcal{A}^+ q) = \int_{\Sigma} x \nabla v \cdot n [\gamma_0 \bar{q}] - \int_{\Sigma} \overline{x \nabla q \cdot n} [\gamma_0 v]$$

Here $[\gamma_0 \cdot] = \gamma_{0,n}^{\Sigma} - \gamma_{0,p}^{\Sigma}$.

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An application (proof of uniqueness)

Let $u \in D(\mathcal{A}^+)$, $\operatorname{div}(x \nabla u) = 0$, take $v = q = u$ in the above:

$$0 = -2i\pi \int_{\Sigma} |U|^2 d\Sigma.$$

Thus $U|_{\Sigma} = 0 \implies u = 0$.

What we have

- LAP
- \mathcal{A}^+ with the radiation condition

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The remaining part of the talk

What can be said about $\sigma(\mathcal{A}^+)$?

On a spectrum of the operator \mathcal{A}^+

A compact embedding

With the radiation condition: $D(\mathcal{A}^+) \subset H^{1/2-\varepsilon}(\Omega)$

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If the spectrum of \mathcal{A}^+ is non-empty, it is a discrete set.

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A property of the eigenvalues

From the Green's formula it follows that the solutions of $\mathcal{A}^+ u = \lambda u$ satisfy

$$\operatorname{Im} \lambda = -\pi \frac{\|x \nabla u \cdot n\|_{L^2(\Sigma)}^2}{\|u\|^2}.$$

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Therefore, $\sigma(\mathcal{A}^+) \subset \{z \in \mathbb{C} : \operatorname{Im} z \leq 0\}$

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Conjecture: $\sigma(\mathcal{A}^+)$ is non-empty, has infinitely many eigenvalues and belongs to $\mathbb{C}^- := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$.

A toy problem

Find $u \in D(\mathcal{A}_{1D}^+) = \{u \in \mathcal{V}_{sing} : u(x) = U(x)(\log|x| + i\pi\mathbb{1}_{x<0}) + u_{H_1}(x)\}$ and λ , s.t.

$$\partial_x(x\partial_x u) = \lambda u, \text{ on } (-1, 1), \quad u(-1) = u(1) = 0.$$

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Explicit solution (princ. def. of $\sqrt{\cdot}$)

$$u(x) = a \underbrace{I_0(2\sqrt{\lambda x})}_{\text{entire, even}} + b \underbrace{K_0(2\sqrt{\lambda x})}_{\sim -\log(2\sqrt{\lambda x}), |x| \rightarrow 0}.$$

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The spectrum

From the reflection symmetry arguments, it follows that

$$\sigma(\mathcal{A}_{1D}^+) \subset i\mathbb{R}_-.$$

Results for a toy 1D problem

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From the reflection symmetry arguments, it follows that

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A more precise information on the spectrum can be obtained by solving, for $\lambda \in i\mathbb{R}^-$:

$$\det \begin{pmatrix} I_0(2\sqrt{\lambda}) & K_0(2\sqrt{\lambda}) \\ I_0(2\sqrt{-\lambda}) & K_0(2\sqrt{-\lambda}) \end{pmatrix} = 0.$$

The spectrum of \mathcal{A}^+ (asymptotics of special functions)

$|\lambda_0| \leq |\lambda_1| \leq \dots$, and we have, as $k \rightarrow +\infty$:

$$\lambda_k = -i \frac{\pi^2 k^2}{16 \sin^2 \frac{\pi}{4}} + o(k^2).$$

The spectrum of \mathcal{A}^+ (asymptotics of special functions)

$|\lambda_0| \leq |\lambda_1| \leq \dots$, and we have, as $k \rightarrow +\infty$:

$$\lambda_k = -i \frac{\pi^2 k^2}{16 \sin^2 \frac{\pi}{4}} + o(k^2).$$

Why is this somewhat surprising

For the eigenfunction

$$u = a_\lambda I_0(2\sqrt{\lambda x}) + K_0(2\sqrt{\lambda x}),$$

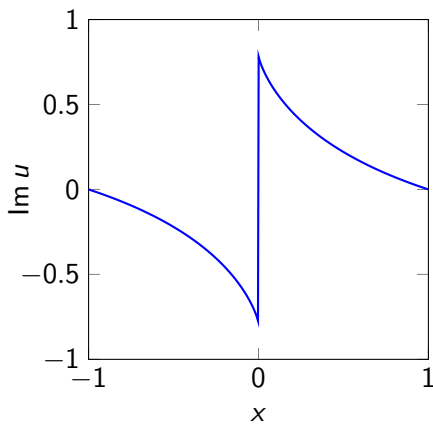
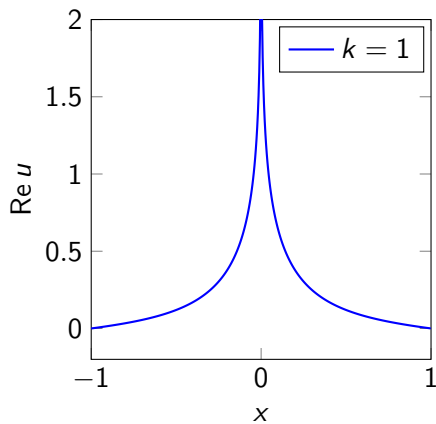
recall that (in 1D):

$$\operatorname{Im} \lambda = -\frac{\pi}{4} \frac{1}{\|\alpha_\lambda I_0(2\sqrt{\lambda x}) + K_0(2\sqrt{\lambda x})\|_{L^2}^2},$$

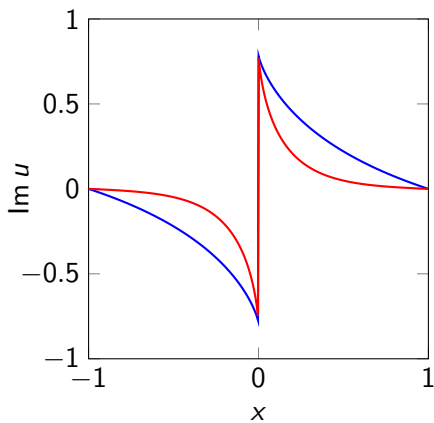
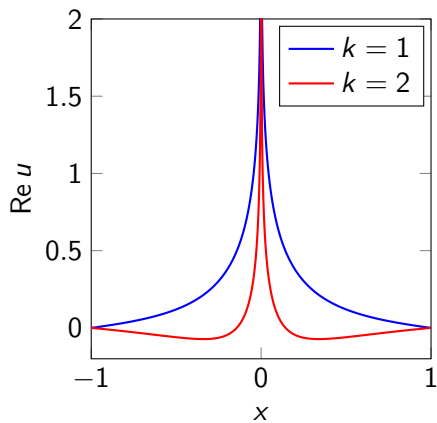
As $\lambda \rightarrow \infty$,

$$a_\lambda I_0(2\sqrt{\lambda x}) + K_0(2\sqrt{\lambda x}) \sim K_0(2\sqrt{\lambda x}) \sim \begin{cases} -\log(2\sqrt{\lambda x}), & |x| \ll |\lambda|^{-1}, \\ e^{-\sqrt{2}|\lambda x|}, & |x| \gg |\lambda|^{-1}. \end{cases}$$

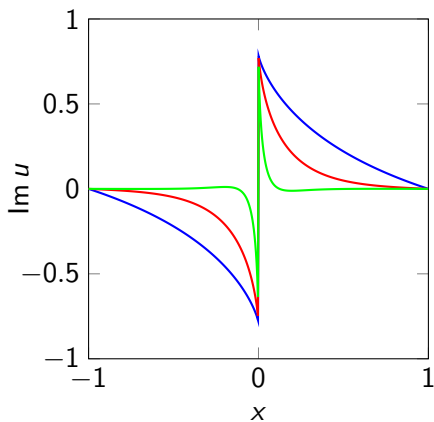
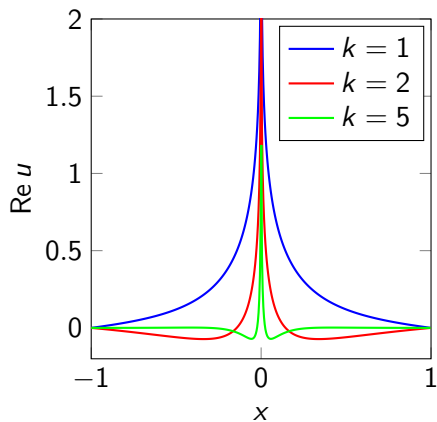
Eigenfunctions



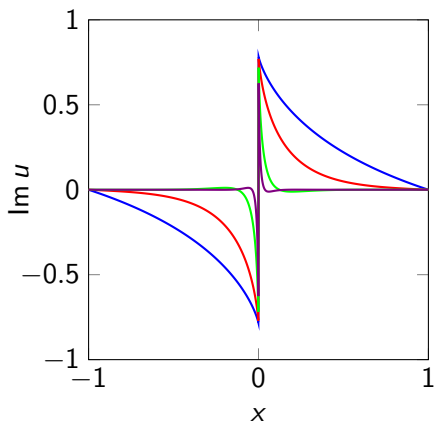
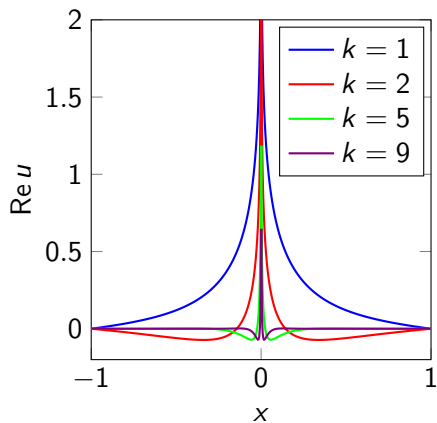
Eigenfunctions

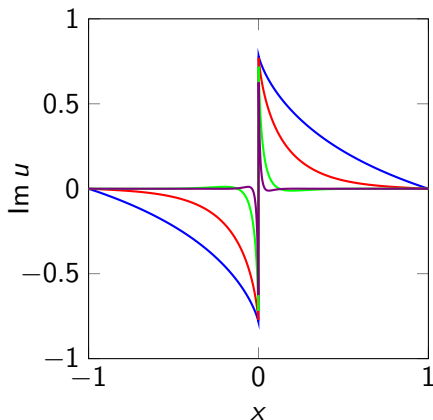
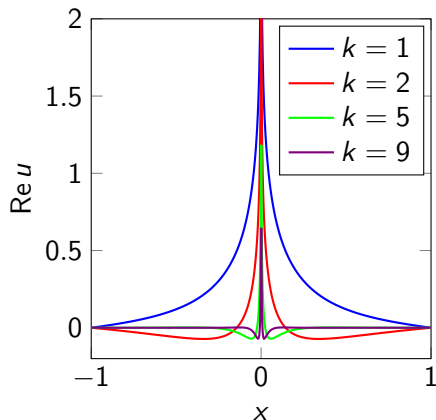


Eigenfunctions



Eigenfunctions





Many open questions (convergence of eigenvalues when $\nu \rightarrow 0$, spectral asymptotics, eigenfunction 'localization' etc etc)

Thank you for your attention !