Non-self-adjoint scattering on graphs

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Scattering theory for self-adjoint operators

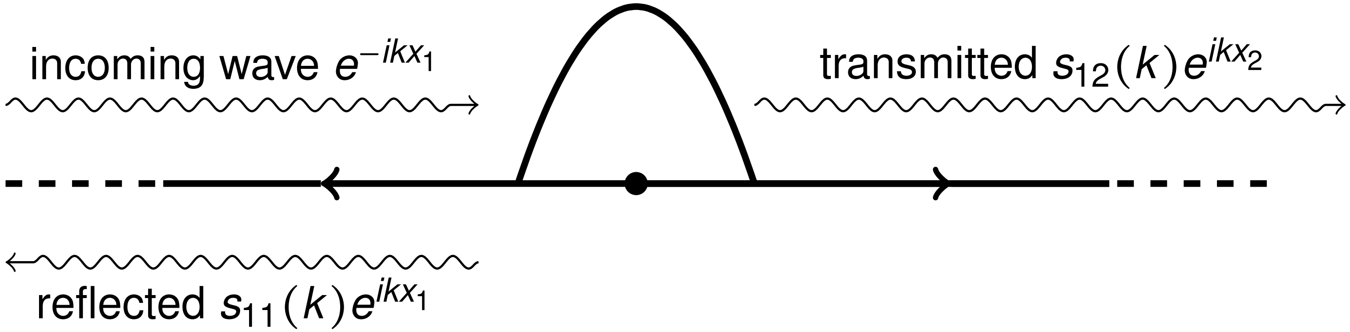
Schrödinger operators

$$\mathcal{H}_0 = -\frac{d^2}{dx^2} \quad \text{“free”}$$

$$\mathcal{H} = -\frac{d^2}{dx^2} + V \quad \text{“perturbed”}$$

Scattering matrix

$$S(k) = \begin{bmatrix} s_{11}(k) & s_{12}(k) \\ s_{21}(k) & s_{22}(k) \end{bmatrix}, \quad k > 0$$



- e.g. $V \in C_0^\infty(\mathbb{R};\mathbb{R})$
- $\mathcal{H} = \mathcal{H}^*$ in $L^2(\mathbb{R})$
- $(e^{it\mathcal{H}})_{t\in\mathbb{R}}$ unitary group

- $S(k)$ for $k > 0$ unitary
- $S(\cdot)$ describes scattering states of $\mathcal{H}$ by scattering states of $\mathcal{H}_0 = -\frac{d^2}{dx^2}$

Scattering theory for self-adjoint operators

Wave operators

$$W_+(\mathcal{H},\mathcal{H}_0) := s - \lim_{t\to+\infty} e^{it\mathcal{H}} e^{-it\mathcal{H}_0} P^{ac}_{\mathcal{H}_0}$$

$$W_-(\mathcal{H},\mathcal{H}_0) := s - \lim_{t\to-\infty} e^{it\mathcal{H}} e^{-it\mathcal{H}_0} P^{ac}_{\mathcal{H}_0}$$

Scattering operators

$$S := W_+^*(\mathcal{H},\mathcal{H}_0) W_-(\mathcal{H},\mathcal{H}_0)$$

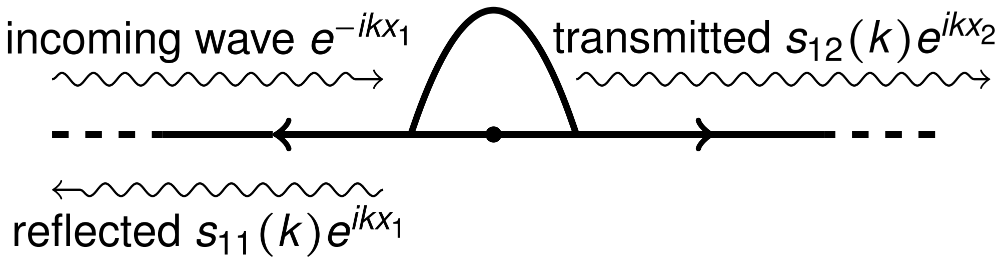
Absolutely continuous spectrum

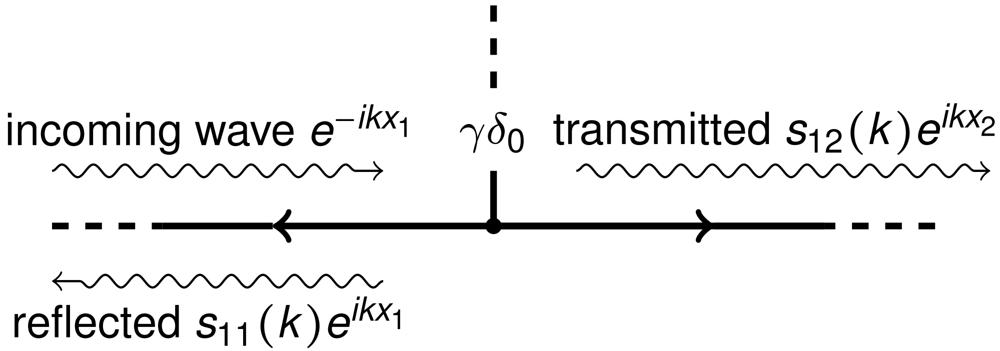
- For $\mathcal{H}_0$, $\mathcal{H}$ self-adjoint absolutely continuous spectrum defined via spectral measure
- Absolutely continuous part defined via image of spectral projection

Existence and Completness of $W_\pm$

- If $W_\pm$ exist and surjective, then absolutely continuous part of $\mathcal{H}_0$ and $\mathcal{H}$ unitarily equivalent
- Scattering theory $\approx$ perturbation theory for absolutely continuous spectrum

Scattering theory for self-adjoint operators: point interaction





Point interactions = boundary conditions

$$D(\Delta(A,B)) := \{\psi \in H^2(0,\infty;\mathbb{C})^2 \colon A\psi(0) + B\partial_x\psi(0) = 0\},$$
$$\Delta(A,B)\psi := \partial_x^2\psi$$

Generalized eigenfunctions

Find $S(k)$ such that

$$\psi(x,k) = e^{-ikx} + S(k)e^{ikx}$$

satisfies

$$A\psi(0,k) + B\partial_x\psi(0,k) = 0$$

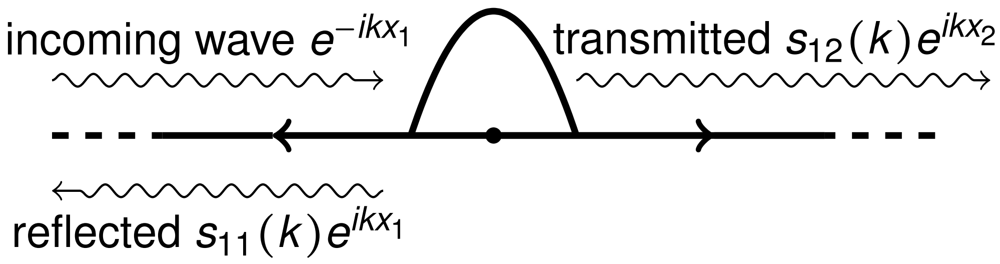
then

$$S(k) = -(A + ikB)^{-1}(A - ikB)$$

Scattering matrix

- $S(k)$ for $k > 0$ unitary if and only if $-\Delta(A,B)$ self-adjoint
- $S(.)$ defines scattering operator $S = W_+^* W_-$
- $\psi(\cdot,k)$ span absolutely continuous part of $-\Delta(A,B)$

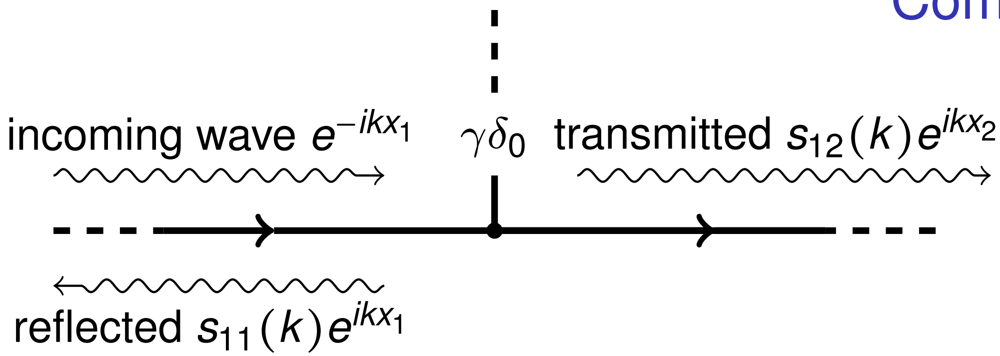
Scattering theory for dissipative operators



Optical model

$$\mathcal{H} = \mathcal{H}_0 + V + iC^*C,$$
$$V = V^* \text{ and } C \in \mathcal{L}(X)$$

- Davies (1978, . . .)
- Faupin, Fröhlich et al. (2018, . . .)

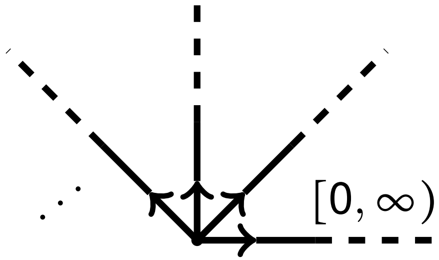


Complex δ-coupling

$$\mathcal{H} = \mathcal{H}_0 + \alpha\delta_0,$$
$$\alpha \in \{\operatorname{Re}\alpha \le 0, \operatorname{Im}\alpha < 0\}$$

- Kadowaki et al. (2003, . . .)
- Behrndt, Malamud, Neidhardt (2007, . . .)

Scattering theory for non-self-adjoint operators on star graphs



Spectral singularities

$$\det(A + ikB) = 0$$

- $\operatorname{Im} k > 0$, then $-k^2$ eigenvalue
- otherwise k spectral singularity

Generalized eigenfunctions

Find $S(k)$ such that

$$\psi(x,k) = e^{-ikx} + S(k)e^{ikx}$$

satisfies

$$A\psi(0,k) + B\partial_x\psi(0,k) = 0$$

then

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Resolvent

Integral kernel for $\operatorname{Im} k > 0$

$$r_{A,B}(x,y;-k^2) = \frac{i}{2k}\operatorname{diag}\{e^{ik|x_j - y_j|}\}_{j\in\mathcal{E}} + \frac{i}{2k}\operatorname{diag}\{e^{ikx_j}\}_{j\in\mathcal{E}} S(k,A,B)\operatorname{diag}\{e^{iky_j}\}_{j\in\mathcal{E}}$$

Model case on the half line

Operators and forms

Consider in $L^2(0,\infty;\mathbb{C})$ the operator

$$\Delta_\gamma u = \partial_x^2 u \quad \text{with} \quad D(\Delta_\gamma) := \{u \in H^2(0,\infty;\mathbb{C}) \colon \gamma u(0) + u'(0) = 0\}, \quad \gamma \in \mathbb{C} \cup \{\infty\}$$

The operator $-\Delta_\gamma$ is associated with the quadratic form

$$\delta_\gamma[u] := \int_0^\infty |\partial_x u(x)|^2 dx - \gamma |u(0)|^2, \quad u \in H^1(0,\infty;\mathbb{C}) \quad \text{if} \quad \gamma \in \mathbb{C},$$
$$\delta_\infty[u] := \int_0^\infty |\partial_x u(x)|^2 dx, \quad u \in H_0^1(0,\infty;\mathbb{C}) \quad \text{if} \quad \gamma = \infty.$$

Adjoint in $L^2(0,\infty;\mathbb{C})$

$$(\Delta_\gamma)^* = \Delta_{\overline{\gamma}}, \quad \gamma \in \mathbb{C} \cup \{\infty\}.$$

Spectrum and spectral singularities

$$\sigma_r(-\Delta_\gamma) = \varnothing, \quad \sigma_c(-\Delta_\gamma) = [0,\infty) \quad \text{and} \quad \sigma_p(-\Delta_\gamma) = \sigma_{pp}(-\Delta_\gamma) = \begin{cases} \varnothing & \text{if } \operatorname{Re}\gamma \le 0, \\ -\gamma^2 & \text{if } \operatorname{Re}\gamma > 0, \end{cases}$$

$$\text{spectral singularity} = \begin{cases} -\gamma^2 & \text{if } \gamma \in \mathbb{R}, \\ \varnothing & \text{else}, \end{cases}$$

Model case on the half line

C_0-contraction semigroups

(a) The operator $+i\Delta_\gamma$ is m-dissipative if and only if $\mathrm{Re}\,(-i\gamma) \leq 0$, that is, $\mathrm{Im}\,(\gamma) \leq 0$;

(b) The operator $-i\Delta_\gamma$ is m-dissipative if and only if $\mathrm{Re}\,(+i\gamma) \leq 0$, that is, $\mathrm{Im}\,(\gamma) \geq 0$;

(c) The operator $i\Delta_\gamma$ is skew-self-adjoint if and only if $\mathrm{Im}\,(\gamma) = 0$;

(d) All $\pm i\Delta_\gamma$ generate (not necessarily) bounded C_0-groups.

Pre-wave operators

$$W(-\Delta_\gamma, -\Delta_0, t) := e^{it\Delta_\gamma} e^{-it\Delta_0} \quad \text{for} \quad t \in \mathbb{R},$$

$$W(-\Delta_0, -\Delta_\gamma, t) := e^{it\Delta_0} e^{-it\Delta_\gamma} P_p^\perp \quad \text{for} \quad t \in \mathbb{R},$$

Absolutely continuous subspace

If $-i\mathcal{H}$ dissipative[1], then

$$H_{ac}(\mathcal{H}) := \overline{M(\mathcal{H})}$$

where

$$M(\mathcal{H}) := \left\{ u \in H, \exists c_u > 0, \forall v \in H, \int_0^\infty |< \mathrm{e}^{-it\mathcal{H}} u, v > |^2 dt \leqslant c_u \|v\|^2 \right\}.$$

[1] compare Davies (1978)

Model case on the half line: Existence of wave operators

Resolvent difference

$$r_{A,B}(x,y;-k^2) = \frac{i}{2k} e^{ik|x-y|} + \frac{i}{2k} e^{ikx} S(k,A,B) e^{iky}$$

Therefore,

$$r_{0,1}(x,y;-k^2) - r_{A,B}(x,y;-k^2) = \frac{i}{2k} e^{ikx} [1 - S(k,A,B)] e^{iky}$$

and

$$\frac{i}{2k} e^{ikx} e^{iky} = \frac{1}{2} (r_{0,1}(x,y;-k^2) - r_{1,0}(x,y;-k^2))$$

Enß method

Splitting into “Ingoing” and “outgoing” spaces using dilation operator

$$\frac{1}{2i}(x\partial_x + \partial_x x)$$

$\ldots \Rightarrow$ Existence wave operators

Cook-Kuroda method

Difference of resolvents trace class
$\ldots \Rightarrow$ Existence wave operators

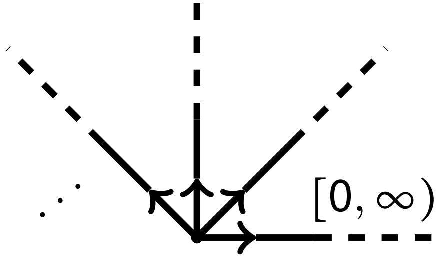
Extension to self-adjoint system

Extension theory for symmetric operators with finite deficiency indices
$\ldots \Rightarrow$ Existence wave operators

Absolutely continuous subspace

Replace semigroup by resolvents
$\ldots \Rightarrow H_{ac}(i\Delta_\gamma) = H_p((i\Delta_\gamma)^*)^\perp$

Scattering theory for non-self-adjoint operators on star graphs



Assumption

For orthogonal projection P,

$$A = L + P \quad \text{and} \quad B = P^{\perp}, \quad \text{where}$$
$$L = P^{\perp} L P^{\perp} \quad \text{and} \quad L \quad \text{diagonaliziable}$$
$$\mathrm{Im}\,(L) \leq 0$$

Generalized eigenfunctions

Find $S(k)$ such that

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Wave operator

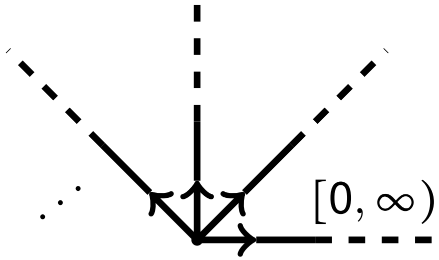
Then one has the existence of

$$s - \lim_{t\to-\infty} W(-\Delta(0,1), -\Delta(A,B), t)$$
$$= s - \lim_{t\to-\infty} e^{it\Delta(0,1)} e^{-it\Delta(A,B)} P_p^{\perp}$$
$$s - \lim_{t\to+\infty} W(-\Delta(A,B), -\Delta(0,1), t)$$
$$= s - \lim_{t\to+\infty} e^{it\Delta(A,B)} e^{-it\Delta(0,1)}$$

and on the half line

$$s - \lim_{t\to+\infty} W(-\Delta_\gamma, -\Delta_0, t) f = \int_0^\infty \psi(\cdot,k)\hat{f}(k)\,\tfrac{dk}{2},$$
$$\hat{f}(k) = \tfrac{2}{\pi}\int_0^\infty \cos(xk) f(x)\,dx$$

Scattering theory for non-self-adjoint operators on star graphs



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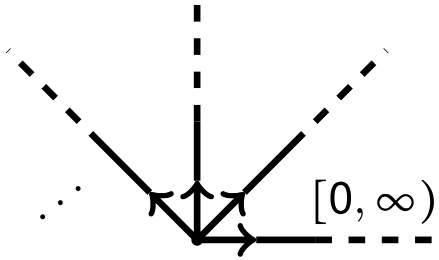
Conjecture

Wave operators

$$s - \lim_{t \to -\infty} W(-\Delta(0,1), -\Delta(A,B), t)$$
$$= s - \lim_{t \to -\infty} e^{it\Delta(0,1)} e^{-it\Delta(A,B)} P_p^{\perp}$$
$$s - \lim_{t \to +\infty} W(-\Delta(A,B), -\Delta(0,1), t)$$
$$= s - \lim_{t \to +\infty} e^{it\Delta(A,B)} e^{-it\Delta(0,1)}$$

(asymptotically) complete if and only if there is **no** spectral singularity

Scattering theory for non-self-adjoint operators on star graphs



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$$\operatorname{Im}(L) \leq 0$$

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Conjecture

Wave operators

$$s - \lim_{t \to -\infty} W(-\Delta(0,1), -\Delta(A,B), t)$$
$$= s - \lim_{t \to -\infty} e^{it\Delta(0,1)} e^{-it\Delta(A,B)} P_p^{\perp}$$
$$s - \lim_{t \to +\infty} W(-\Delta(A,B), -\Delta(0,1), t)$$
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Thanks for your attention!