

Powers of discrete Laplacians and Hardy-type inequalities

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Criticality transition for positive powers of the discrete Laplacian on the half line



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[Isr. J. Math. (to appear)]

An improved discrete Rellich inequality on the half line

- 1 Motivation
- 2 Criticality transition and Hardy type inequalities
- 3 Polyharmonic operators on a subspace

Motivation

Classical Hardy inequality

$$\sum_{n=1}^{\infty} |u_n - u_{n-1}|^2 \geq \sum_{n=1}^{\infty} \rho_n |u_n|^2, \quad u \in \ell^2(\mathbb{N}), \quad u_0 := 0, \quad \rho_n = \frac{1}{4n^2}$$

- improvement to **optimal weight**

[Keller-Pinchover-Pogorzelski'18, Krejčířík-Štampach'22]

$$\rho_n^{\text{KPP}} = 2 - \sqrt{\frac{n-1}{n}} - \sqrt{\frac{n+1}{n}}$$

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$$-\Delta \geq \rho \quad \text{on} \quad \ell^2(\mathbb{N}), \quad -\Delta = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

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- no classical Rellich** inequality

$$(-\Delta)^2 \geq \frac{c}{n^4} \implies c \leq 0$$

[Huang-Ye'24]

- bilaplacian is **critical**

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discrete $\not\approx$ continuous

Spectral stability

- perturbation by **complex valued** bounded potential $\{v_n\}_n$
- **spectrum** of $-\Delta + v$ unchanged ?

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Theorem

[Krejčířik-Laptev-Štampach'22]

If $-\Delta \geq c|v|$ with $c \in (0, 1)$ then

$$\sigma(-\Delta + v) = \sigma_c(-\Delta + v) = \sigma(-\Delta) = \sigma_c(-\Delta) = [0, 4].$$

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- **Birman–Schwinger** principle
- possibly **embedded eigenvalues** if $c = 1$
- **subordination** condition in **continuous** setting

[Hansmann-Krejčířik'22]

[Fanelli-Krejčířik-Vega'18]

Criticality transition and Hardy type inequalities

Criticality transition

bounded $A \geq 0$ on $\ell^2(\mathbb{N})$ is **critical** if

$$\forall 0 \neq \rho = \{\rho_n\} \geq 0 : \inf \sigma(A - \rho) < 0$$

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The powers $(-\Delta)^\alpha$ with $\alpha > 0$ are critical if and only if $\alpha \geq 3/2$.

- $\alpha \in \mathbb{N}$: **Birman** inequalities in **continuous** setting

[Birman'61, Glazman'65]

$$(-\Delta)^\alpha \geq \frac{((2\alpha)!)^2}{16^\alpha (\alpha!)^2} \frac{1}{x^{2\alpha}}$$

[Rellich'56]

- powers of discrete Laplacian **on full line** \mathbb{Z} critical iff $\alpha \geq 1/2$

[Ciaurri-Roncal'18, Keller-Nietschmann'23]

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$$\forall m, n \in \mathbb{N} : \sup_{\lambda \in \mathbb{R}_-} ((-\Delta)^\alpha - \lambda)_{m,n}^{-1} \leq C_{m,n} \implies \text{subcriticality}$$

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- unitary **equivalence** to **multiplication** operator

$$(-\Delta)^\alpha = \mathcal{U}^{-1} M_{2^\alpha(1-x)^\alpha} \mathcal{U}$$

- matrix elements of **resolvent**

$$((-\Delta)^\alpha - \lambda)_{m,n}^{-1} = \frac{2}{\pi} \int_{-1}^1 \frac{U_m(x)U_n(x)}{2^\alpha(1-x)^\alpha - \lambda} \sqrt{1-x^2} dx$$

Theorem**Hardy type inequalities**There exists $\gamma = \gamma(\alpha) > 0$ such that

$$(-\Delta)^\alpha \geq \rho, \quad \rho_n = \begin{cases} \frac{\gamma}{n^{2\alpha}} & \alpha \neq 1/2 \\ \frac{\gamma}{n \ln(n+1)} & \alpha = 1/2. \end{cases}$$

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- **expected** optimal **decay** rate as $n \rightarrow \infty$
- $\alpha \in (0, 1)$: **continuous** setting

[Bogdan-Dyda'11]

$$(-\Delta)^\alpha \geq \rho, \quad \rho(x) = \frac{\kappa_\alpha}{x^{2\alpha}}$$

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? **optimal weights** for $\alpha \neq 1$

Polyharmonic operators on a subspace

- discrete **Birman inequalities**

[Gupta'24, Huang-Ye'24]

$$(-\Delta)^\alpha \geq \frac{((2\alpha)!)^2}{16^\alpha (\alpha!)^2} \frac{1}{n^{2\alpha}} \quad \text{on } H_0^\alpha(\mathbb{N})$$

$$H_0^\alpha(\mathbb{N}) := \{u \in \ell^2(\mathbb{N}) : u_0 = \dots = u_{\alpha-1} = 0\}$$

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- **continuous** analogue on

$$H_0^\alpha(\mathbb{R}_+) = \{f \in H^\alpha(\mathbb{R}_+) : f^{(0)}(0) = \dots = f^{(\alpha-1)}(0) = 0\}$$

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Theorem

improved Rellich inequality

One has $(-\Delta)^2 \geq \rho$ on $H_0^2(\mathbb{N})$ with

$$\rho_n^{\text{GKS}} = 6 - 4 \left(\frac{n-1}{n}\right)^{3/2} - 4 \left(\frac{n+1}{n}\right)^{3/2} + \left(\frac{n-2}{n}\right)^{3/2} + \left(\frac{n+2}{n}\right)^{3/2} > \frac{9}{16} \frac{1}{n^4}$$

$$(-\Delta)^2 = \begin{pmatrix} 5 & -4 & 1 & & & \\ -4 & 6 & -4 & 1 & & \\ 1 & -4 & 6 & -4 & 1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & & & \ddots \end{pmatrix}$$

- **remainder factorisation ansatz**

$$(-\Delta)^2 - \rho = R^* R, \quad R = \begin{pmatrix} a_1 & b_1 & c_1 & & & \\ & a_2 & b_2 & c_2 & & \\ & & a_3 & b_3 & c_3 & \\ & & & \ddots & \ddots & \ddots \\ & & & & & & \ddots \end{pmatrix}$$

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- only **existence** and **asymptotics** of a, b, c
- weight ρ **not optimal**

[Huang-Ye'24]

Hardy :

[Keller-Pinchover-Pogorzelski'18]

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Conjecture

The **Birman** inequality $(-\Delta)^\alpha \geq \rho$ holds on $H_0^\alpha(\mathbb{N})$ with

$$\rho = \frac{(-\Delta)^\alpha \sqrt{g}}{\sqrt{g}}, \quad g_n = n^{2\alpha-1}$$

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- **remainder** difference operator of **order** α
- would **improve** upon **classical** weight

$$\rho_n > \frac{((2\alpha)!)^2}{16^\alpha (\alpha!)^2} \frac{1}{n^{2\alpha}}$$

Theorem

[Štampach-Waclawek'24]

The **Birman** inequality $(-\Delta)^\alpha \geq \rho$ holds on $H_0^\alpha(\mathbb{N})$ with the **optimal** weight

$$\rho = \frac{(-\Delta)^\alpha \sqrt{g}}{\sqrt{g}}, \quad g_n = n \prod_{j=1}^{\alpha-1} (n-j)^2$$

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- **improves** upon **conjectured weight** with $g_n = n^{2\alpha-1}$
- iterative application of **weighted remainder formula**

[Huang-Ye'24, Krejčířík-Laptev-Štampach'22]

$$\sum_{n=1}^{\infty} \omega_n |\nabla u_n|^2 + \sum_{n=1}^{\infty} \frac{\operatorname{div}(\omega \nabla g)_n}{g_n} |u_n|^2 = \sum_{n=1}^{\infty} \omega_{n+1} \left| \sqrt{\frac{g_n}{g_{n+1}}} u_{n+1} - \sqrt{\frac{g_{n+1}}{g_n}} u_n \right|^2$$

- discrete **gradient** and **divergence**

$$\nabla u_n = u_{n+1} - u_n, \quad \operatorname{div} u_n = u_n - u_{n-1}, \quad -\Delta = -\operatorname{div} \nabla$$

- abstract **remainder identity** for **higher order** Birman case

Thank you for your attention!

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