The heat equation and its reachable space

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Mathematical aspects of the physics with non-self-adjoint operators
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I. Introduction : A control problem for a linear abstract system.

Abstract linear system

$$\begin{cases} y' = Ay + Bu, & t \geq 0. \\ y(0) = y_0. \end{cases}$$

- A generates a C^0 semigroup $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ on a Hilbert space H.
- $B \in \mathcal{L}(U, H)$ (or admissible) is the control operator.
- H, U Hilbert spaces.
  - $\to$ H : state space $\quad$ $y(t, \cdot)$ = state
  - $\to$ U : control space $\quad$ $u \in L^2(0, T; U)$: control

Goal: Understand how the control can act on the system

$$\begin{cases} y' = Ay + Bu \\ y|_{t=0} = y_0 \end{cases} \rightsquigarrow y(t) = \mathbb{T}_t y_0 + \int_0^t \mathbb{T}_{t-s} Bu(s)\,ds.$$

$\rightsquigarrow$ For $T > 0$, we want to describe the reachable space at time T starting from $y_0 = 0$

$$\mathcal{R}(T) = \left\{ \int_0^T \mathbb{T}_{T-s} Bu(s)\,ds, \quad u \in L^2(0,T;U) \right\}.$$

$$= \left\{ y(T), \text{ for } y \text{ solving } \begin{cases} y' = Ay + Bu \\ y(0) = 0 \end{cases}, \ u \in L^2(0,T;U) \right\}.$$

Goal: Describe $\mathcal{R}(T)$

Describing $\mathcal{R}(T)$

⇝ Completely characterized in finite dimension ($H = \mathbb{R}^n$).

$$\mathcal{R}(T) = \mathrm{Ran}\,(B, AB, \dots A^{n-1}B)$$

[Kalman]

⇝ For exactly controllable systems, $\mathcal{R}(T) = H$

Ex: Waves, Schrödinger under suitable geometric conditions

[Bardos Lebeau Rauch, ...].

⇝ For systems which are null controllable in any positive time, $\mathcal{R}(T)$ does not depend on T

$$\mathcal{R} = \mathcal{R}(T).$$

[Seidman]

Typical example: the heat equation.

II An abstract result

$$\begin{cases} y' = Ay + Bu \\ y|_{t=0} = 0. \end{cases}$$

Assumpt.: This system is null-controllable in any positive time.

Thm [S.E., Le Balc'h, Tucsnak '23]

The semi-group $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ defined on H is such that $\mathbb{T}_R = (\mathbb{T}_t|_R)_{t \geq 0}$ is a C^0 semi-group on R. $\leadsto$ Exactly controllable system on R.

Rk. Similar statement in [van Neerven '05].

A word on the proof.

- $\mathcal{R}$ is endowed with the norm (indexed by $T>0$)

$$\|z\|_{\mathcal{R}(T)} = \inf\left\{ \|u\|_{L^2(0,T;U)},\ u \in L^2(0,T;U),\ \left|\begin{array}{l} y' = Ay + Bu \\ y(0) = 0 \end{array}\right. \text{ and } y(T) = z \right\}.$$

Note that if $T_1 < T_2$, $\|z\|_{\mathcal{R}(T_2)} \leq \|z\|_{\mathcal{R}(T_1)}$.

and $\exists\, C(T_1, T_2)$, $\|z\|_{\mathcal{R}(T_1)} \leq C(T_1, T_2) \|z\|_{\mathcal{R}(T_2)}$.

- Let $\tau > 0$. For $t \in [0, \tau]$, and $z \in \mathcal{R}$

$$\|\mathbb{T}_t z\|_{\mathcal{R}(\tau)} \leq C(\tau) \|\mathbb{T}_t z\|_{\mathcal{R}(2\tau)} \leq C(\tau) \|z\|_{\mathcal{R}(2\tau - t)} \leq C(\tau) \|z\|_{\mathcal{R}(\tau)}.$$

$\boxtimes$

$\Rightarrow$ Perturbative arguments can be applied. cf [S.E, Le Balc'h, Tucsnak].

E.g. $\forall T > 0$, $\exists \varepsilon(T) > 0$, if $\boxed{Q \in \mathcal{L}(\mathcal{R})}$ satisfies $\|Q\|_{\mathcal{L}(\mathcal{R}(T))} \leq \varepsilon$

then the reachable space at time T for

$$\begin{cases} y' = (A+Q)y + Bu \\ y|_{t=0} = 0 \end{cases} \quad \text{and} \quad \begin{cases} y' = Ay + Bu \\ y|_{t=0} = 0 \end{cases}$$

are identical. $\mathcal{R}_{A+Q}(T) = \mathcal{R}_A(T) = \mathcal{R}$

Problems and open questions

- When $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ is an analytic semi-group in H, is $\mathbb{T}_{\mathcal{R}} = (\mathbb{T}_t|_{\mathcal{R}})$ only a C^0 semi-group on $\mathcal{R}$?
Can we get better regularity results? Analytic
Gevrey
Differentiable ...

- Such result is useful if we know $\mathcal{R}$:
Do we know $\mathcal{R}$?

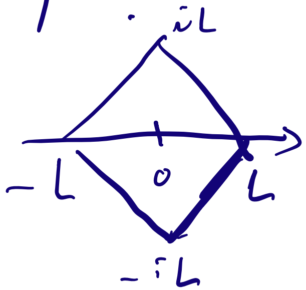
III. The heat eq.

$$\begin{cases}\partial_t y - \partial_{xx} y = 0 & \text{in } (0,T)\times(-L,L)\\ y(t,-L) = u_-(t), & y(t,L) = u_+(t).\end{cases}$$

- [Fattorini – Russell '71] on the Fourier basis
- [Martin – Rosier – Rouchon '16] in terms of holomorphy
- [Dardé – Ervedoza '18]
- [Kellay – Hartmann – Tucsnak '20], [Hartmann Orsoni] ...

$\to$ the reachable space is a space of holomorphic functions.

$\mathcal{R} = \{ z : (-L,L) \to \mathbb{C}$ which admits an holomorphic extension z_e in the square

$S =$ 

with $z_e \in L^2(S)\}$

$\mathcal{R} = L^2 \cap Hol(S)$. (Bergman space)

From our result, the heat semi-group $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ given by $\mathbb{T}_t z_0 = z(t)$, where z solves

$$\begin{cases} \partial_t z - \partial_{xx} z = 0 & \text{in } (0,T) \times (-L,L) \\ z(t,-L) = z(t,L) = 0 . & \text{in } (0,T) \\ z|_{t=0} = z_0 & \text{in } (-L,L) \end{cases}$$

is a C^0 semi-group in $\mathcal{R} = L^2 \cap \mathrm{Hol}(S)$.

Question: Is it better?

→ differentiable?

→ analytic?

→ Gevrey?

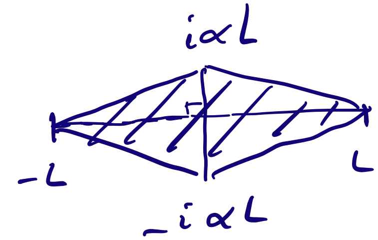
N.B. Proving directly that the heat semigroup is a C^0 semigroup in $\mathcal{R} = L^2 \cap Hol(S)$ is not an easy task.

$\rightsquigarrow$ Can be done by estimating the resolvent $(-\partial_{xx,D} - \lambda)^{-1}$ as an operator in $\mathcal{R}$.

Doing so, in 1d, we manage to prove that the heat semi-group is Gevrey in $L^2 \cap Hol(S)$

$$\left(\forall \lambda \in \mathbb{C},\ \mathrm{Re}\,\lambda \geq 0,\ \|(-\partial_{xx,D} - \lambda)^{-1}\|_{\mathcal{L}(\mathcal{R})} \leq \frac{C}{|\lambda|^{1/4}}\right)$$

and not analytic

For $\alpha \in (0,\infty)$, set $\Omega_\alpha =$ 

then the heat semigroup with Dirichlet boundary condition in $\pm L$ satisfies:

- $\forall t>0$, $\mathbb{T}_t : L^2(-L,L) \longrightarrow Hol(\mathbb{C})$.
- $\mathbb{T}^\alpha = \left(\mathbb{T}_t \big|_{L^2 \cap Hol(\Omega_\alpha)} \right)$ satisfies:

→ For $\alpha < 1$, $\mathbb{T}^\alpha$ is an analytic semigroup in $L^2 \cap Hol(\Omega_\alpha)$

→ For $\alpha > 1$, $\mathbb{T}^\alpha$ is not a C^0 semigroup in $L^2 \cap Hol(\Omega_\alpha)$.

Further comments and open questions.

- For heat equations, the reachable space is very likely given by a space of holomorphic functions.
- What are the regularity properties of the corresponding semigroup restricted to this space?
- More generally, can we describe all spaces of holomorphic functions for which the heat semigroup is a C^0 semigroup?

In higher dimensions.

$\Omega \subset \mathbb{R}^d$, Ω smooth bounded domain.

$$\begin{cases} \partial_t y - \Delta y = 0 & \text{in } (0,T)\times\Omega. \\ y|_{(0,T)\times\partial\Omega} = u(t,x) \\ y|_{t=0} = y_0. \end{cases}$$

[Strohmaier – Waters 2022] suggests that the reachable space should be a space of holomorphic functions in $\mathcal{E}(\Omega) = \{ a+ib,\ a\in\Omega,\ b\in\mathbb{R}^d \text{ with } |b| < \operatorname{dist}(a,\partial\Omega)\}$

$\rightsquigarrow$ Proved in the case of a ball.

$\rightsquigarrow$ Currently revisiting this work with A. Tendani-Soler.

Thank you for your attention!

Based on

- S.E., K. Le Balc'h, M. Tucsnak, 2022, JFA.
- S.E. & A. Tendani-Soler, ongoing work