The Generic Instability of Differential Operators

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3 June 2024

Local solvability

P is *locally solvable* near x_0 if

$$Pu = f$$

has a local weak (distribution) solution u near x_0 for all $f \in C^\infty$ in a set of finite codimension, thus P has locally a finite cokernel.

This is equivalent to *a priori* estimates for the L^2 adjoint P^*.

One can prove local nonsolvability by constructing local *approximate* solutions to $P^*u = 0$, which are called *pseudomodes*.

Observe that in the **analytic category** all nondegenerate PDO are locally solvable by the Cauchy-Kovalevsky theorem.

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Principal type

Definition

The operator P is of **principal type** if the *Hamilton field* $H_p = \partial_\xi p \partial_x - \partial_x p \partial_\xi$ is *nonradial* when $p = 0$, which is *generic*.

Then P has simple characteristics since $|dp| \neq 0$ when $p = 0$.

Theorem (Hörmander 1955)

PDO of principal type with principal symbol p such that $H_p\overline{p} = \{ p, \overline{p} \} = -2i \{ \operatorname{Re} p, \operatorname{Im} p \} \equiv 0$ are solvable.

For example PDO of principal type with **real** principal symbol are solvable, since then $\overline{p} = p$. This condition is *not invariant* under multiplication with non-vanishing factors (only when $p = 0$).

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The Lewy counterexample

Hans Lewy's counterexample (1957) The vector field

$$P = D_{x_1} + iD_{x_2} + i(x_1 + ix_2)D_{x_3} = D_{x_1} - x_2 D_{x_3} + i(D_{x_2} + x_1 D_{x_3})$$

is **not locally solvable anywhere** in $\mathbf{R}^3$. In fact, the range of P is a set of the *first category* in C^∞, a meagre set.

By the Cauchy-Kovalevsky theorem P is solvable for *analytic* functions.

P is the tangential Cauchy-Riemann operator on the boundary of the *strictly pseudoconvex set*

$$\Omega = \{\, (z_1, z_2) : \ |z_1|^2 + 2\,\mathrm{Im}\, z_2 < 0 \,\} \subset \mathbf{C}^2$$

Obs: $Pf\big|_{\partial\Omega} = 0$ for *any analytic function* $f(z_1, z_2)$ (large kernel).

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The bracket condition

Theorem (Hörmander 1960)

Local solvability implies that $\{\operatorname{Re} p, \operatorname{Im} p\} = H_{\operatorname{Re} p} \operatorname{Im} p = 0$ *on* $p^{-1}(0)$, *where* p *is the principal symbol of* P.

This is a necessary, invariant and nongeneric condition so **almost all** nonelliptic linear PDO are **not** locally solvable.

The principal symbol of $[P, P^*]$ is $\frac{1}{i}\{p, \overline{p}\} = -2\{\operatorname{Re} p, \operatorname{Im} p\}$, so a vanishing bracket means that the operator is *approximately normal*.

For the Lewy vector field we have $\{\operatorname{Re} p, \operatorname{Im} p\} = 2\xi_3$ and

$$p = 0 \Leftrightarrow \xi_1 = x_2\xi_3 \text{ and } \xi_2 = -x_1\xi_3$$

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Consequences of the bracket condition

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$$\{\operatorname{Re} p, \operatorname{Im} p\} = 0 \text{ on } p^{-1}(0)$$

has many consequences for the operator, for example:

- Local solvability
- Instability of quasilinear Cauchy problem
- Spectral instability - pseudospectrum
- Unique images of differential operators
- Finite dimensional kernels of differential operators
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Generalization: the Nirenberg-Treves conjecture

Condition (Ψ)**:** $\operatorname{Im} p$ does **not** change sign from $-$ to $+$ on the oriented bicharacteristics of $\operatorname{Re} p$. For PDO this condition gives:

Condition (P)**:** no change of sign, i.e., both (Ψ) and $(\overline{\Psi})$ hold. Then the bracket condition must hold. (Switch ξ and $-\xi$).

Example For $P = D_t + if(t, x, D_x)$ with real first order operator f,

$$H_{\operatorname{Re} p} \operatorname{Im} p = \partial_t f(t, x, \xi) = 0 \text{ when } f = 0$$

is the bracket condition and (P) means that $t \mapsto f(t, x, \xi)$ cannot change sign when t varies.

The Nirenberg-Treves conjecture (1969)

A principal type ΨDO is locally solvable if and only if the principal symbol satisfies condition (Ψ).

Nirenberg and Treves proved this for ΨDOs with analytic symbols, which must vanish of finite order.

Generalization: the Nirenberg-Treves conjecture

Condition (Ψ)**:** Im p does **not** change sign from $-$ to $+$ on the oriented bicharacteristics of Re p. For PDO this condition gives:

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Resolution of the Nirenberg-Treves conjecture

Condition (P) is **sufficient** for local solvability for PDO of principal type. (Beals-Fefferman 1973)

Condition (Ψ) is **necessary** for local solvability for ΨDO of principal type. (Hörmander 1980)

Condition (Ψ) is **sufficient** for local solvability for ΨDO of principal type in **two** variables. (Lerner 1988)

Theorem (D. 2006)

Condition (Ψ) *is* **sufficient** *for local solvability for* Ψ*DO of principal type.*

This has been generalized to *systems* of principal type with constant characteristics (D. 2011), but not much is known in general for *nonprincipal type operators* .

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Nonprincipal type operators

If the principal symbol p_m vanishes of at least second order, then the *subprincipal symbol* $p_s \cong p_{m-1}$ is an important invariant.

Example Let

$$P = D_1 D_2 + B(x, D) \qquad \text{on } \mathbf{R}^n \qquad n \geq 3$$

where B is first order. The principal symbol $\xi_1 \xi_2$ vanishes of second order at the *double characteristic set* $\Sigma_2 = \{\, \xi_1 = \xi_2 = 0 \,\}$ which is *involutive*, i.e., if p and q vanish on Σ_2 then $\{p, q\}$ also vanishes.

The invariant *subprincipal symbol* is the principal symbol b of B **on** Σ_2.

If $\operatorname{Im} b$ *changes sign* on x_1 or x_2 lines, then P is *not solvable*. If $\operatorname{Im} b \neq 0$ then P is solvable. (Mendoza–Uhlmann 1983-84).

It was conjectured that P is solvable if $\operatorname{Im} b$ does not change sign in x, which is **not** true — also depends on $\operatorname{Re} b$ (see $Sub(\Psi)$ below).

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Necessary conditions: Limit characteristics on Σ_2

Definition $\Gamma \subset \Sigma_2$ is a **limit bicharacteristic** of real p if there exist bicharacteristics Γ_j of p that converge to Γ as *smooth curves*.

Example $p = \prod_j p_j$, p_j of real principal type, $p_j = \{ p_j, p_k \} = 0$ on the double characteristic set Σ_2 and $p_j \neq p_k$ on $p^{-1}(0) \setminus \Sigma_2$. Then the limit bicharacteristics of p are the bicharacteristics of p_j on Σ_2, $\forall j$.

Definition Condition $Lim(\Psi)$: the imaginary part of the subprincipal symbol $\operatorname{Im} p_s$ does not change sign from $-$ to $+$ on a limit bicharacteristic on Σ_2 (as for Mendoza–Uhlmann case). If principal symbol vanishes of even order, then no sign changes — $Lim(P)$.

Theorem (D. 2016)

If P has real principal symbol and does not satisfy condition $Lim(\Psi)$, then P is not locally solvable.

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Necessary conditions: Limit characteristics on Σ_2

Definition $\Gamma \subset \Sigma_2$ is a **limit bicharacteristic** of real p if there exist bicharacteristics Γ_j of p that converge to Γ as *smooth curves*.

Example $p = \prod_j p_j$, p_j of real principal type, $p_j = \{ p_j, p_k \} = 0$ on the double characteristic set Σ_2 and $p_j \neq p_k$ on $p^{-1}(0) \setminus \Sigma_2$. Then the limit bicharacteristics of p are the bicharacteristics of p_j on Σ_2, $\forall j$.

Definition Condition $Lim(\Psi)$: the imaginary part of the subprincipal symbol $\operatorname{Im} p_s$ does not change sign from $-$ to $+$ on a limit bicharacteristic on Σ_2 (as for Mendoza–Uhlmann case). If principal symbol vanishes of even order, then no sign changes — $Lim(P)$.

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Necessary conditions: subprincipal characteristics

Assume Σ_2 is involutive with symplectic foliation given by the Hamilton fields, it is *nonradial* if *all* its Hamilton fields are.

Example $\Sigma_2 = \{\, \xi = 0 \,\}$ with leaves $L = \{\, (x, y_0; 0, \eta_0) : x \in \mathbf{R}^k \,\}$.

Definition P is of *subprincipal type* if $H_{p_s}\big|_{\Sigma_2} \subseteq T\Sigma_2$ and is transversal to the leaves L when $p_s = 0$.

Examples $\partial_x p_s = 0$ and $\partial_{y,\eta} p_s \neq 0$ when $p_s = 0$ on $\Sigma_2 = \{\, \xi = 0 \,\}$.

$\Delta_x + D_t$ is of *subprincipal type*, but $\Delta_x + D_x$ is *not*.

H_{p_s} is only defined modulo TL. To define condition (Ψ) for p_s we need

$$\left| dp_s\big|_{TL} \right| \le C|p_s| \qquad \text{for leaves } L \text{ of } \Sigma_2$$

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Condition $Sub(\Psi)$: $\operatorname{Im} p_s$ has constant sign on the leaves L of Σ_2 and p_s satisfies condition (Ψ) on $T^\sigma\Sigma_2 = T\Sigma_2/TL$, which is symplectic.

For PDE the condition means *no* sign changes, i.e., condition (P).

Example $T^\sigma\Sigma_2 \cong \{\,(w_0; y, \eta) : w_0 \in \Sigma_2\,\}$ for $\Sigma_2 = \{\,\xi = 0\,\}$.

Theorem (D. 2017)

If P has principal symbol vanishing of at least second order at a nonradial involutive manifold Σ_2, is of subprincipal type and does not satisfy condition $Sub(\Psi)$, then P is not locally solvable.

Also have extensions to the case when the Taylor expansion of the *refined principal symbol* $p + p_s$ does not satisfy (Ψ) (D. 2018).

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The refined principal symbol

Let $p_r = p + p_s$ be the **refined principal symbol**.

Definition

P is of *real subprincipal type* if the principal symbol is real and vanish of exactly second order at a nonradial involutive manifold Σ_2 and $H_{\operatorname{Re} p_s}\big|_{\Sigma_2} \subseteq T\Sigma_2$ is transversal to the leaves L when $p_s = 0$.

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Condition $Sub_r(\Psi)$: p_r satisfies condition (Ψ) on the limits at Σ_2.

$Sub_r(\Psi)$ means that $\operatorname{Im} p_s$ has constant sign on the leaves of Σ_2 and condition (Ψ) is satisfied for p_s on Σ_2, then $Lim(\Psi)$ and $Sub(\Psi)$ holds.

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Sufficient conditions

Theorem (D. 2024)

If P is of real subprincipal type and satisfies condition $Sub_r(\Psi)$ then P is locally solvable.

Thus, $Sub_r(\Psi)$ is necessary and sufficient for solvability of operators of real subprincipal type.

Conjecture: operators of subprincipal type with principal symbol vanishing of exactly second order at Σ_2 are solvable if and only if they satisfy both condition (Ψ) and $Sub_r(\Psi)$.

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