Spectral Inequalities for Schrödinger Operators with Complex Potentials

Jean-Claude Cuenin

Loughborough University

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Setting :

$$\boxed{-\Delta + V \quad \text{on } L^2(\mathbb{R}^n)}$$

$V \in L^q(\mathbb{R}^n)$ complex , $q < \infty$ (decaying)

Interested in bounds on the

- "size"
- accumulation
- number

of discrete eigenvalues.

For real potentials,

$$\sum_j |z_j|^\gamma \lesssim \int_{\mathbb{R}^n} V_-(x)^{\frac{n}{2}+\gamma}\, dx$$

$\forall \gamma > 0$ if $n \geq 2$.

For complex potentials,

$$|z_j|^\gamma \lesssim \int_{\mathbb{R}^n} |V(x)|^{\frac{n}{2}+\gamma}\, dx$$

[$n=1$, $\gamma=\frac{1}{2}$ due to Abramov, Aslanyan, Davies]
[$n \geq 2$ due to Frank]

$\forall\ 0 < \gamma \leq \frac{1}{2}$ if $n \geq 2$.

Conjecture (Laptev – Safronov)

$$|z_j|^{\gamma} \lesssim \int_{\mathbb{R}^n} |V(x)|^{\frac{n}{2}+\gamma}\, dx$$

$\forall\ \frac{1}{2} \le \gamma < \frac{n}{2}$ if $n \geq 2$.

Counterexample (Bögli – C.)

$V_\lambda(x) \sim i\lambda \mathbb{1}\left(|x_1| < 1,\ |x'| < \lambda^{-\frac{1}{2}}\right)$

$\rightsquigarrow$ e.v. $z = (\lambda + i)^2$ (i.e. $\gamma \le \frac{1}{2}$ sharp)

Open problem 1

- Provide a purely constructive proof of the counterexample.
- Can one use a combination of many such "needles" to produce many eigenvalues?

Counterexample also shows optimality of :

a) $|z|^{\gamma} \lesssim \int |V|^{\frac{n}{2}+\gamma}$, $0<\gamma\le\frac{1}{2}$

b) $\mathrm{dist}(z,\mathbb{R}_+)^{\gamma-\frac{1}{2}}|z|^{\frac{1}{2}} \lesssim \int |V|^{\frac{n}{2}+\gamma}$, $\gamma>\frac{1}{2}$

c) $|z|^{\frac{1}{2}} \lesssim \sup_y \int \exp(-|\mathrm{Im}\sqrt{z}\,|x-y|)\,|V(x)|^{\frac{n}{2}+\frac{1}{2}}\,dx$

Q: Under what structural asepts. on V do we have improvements ?

Improvements under structural asspts. on V :

a) radial : $\gamma \le \frac{1}{2} \to \gamma < \frac{n}{2}$ (Frank-Simon)

b) sparse : $\int |V|^{\frac{n}{2}+\gamma} \to \sup_j \int |V_j|^{\frac{n}{2}+\gamma}$ (C.)

c) random : $\gamma \le \frac{1}{2} \to \gamma < \frac{n}{2}+1$ (C.-Merz)

Also expect improvements if "V does not look like the counterexample"

Open Problem 2: Prove

$$|Z|^{\frac{1}{2}} \lesssim \sup_{\ell} \int_{\ell} |V| =: \|XV\|_{\infty}$$

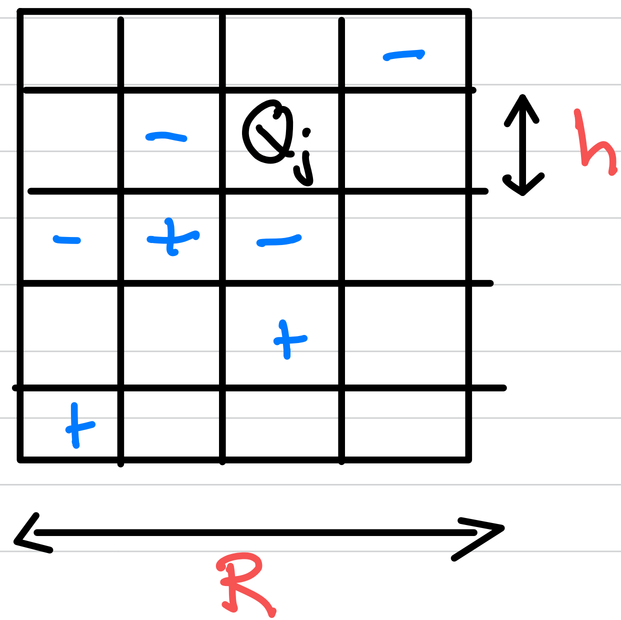
- True for radial V (Barcelo-Ruiz-Vega, Frank-Simon)
- Related to the "Mizohata-Takeuchi conjecture"

$$\int_{\mathbb{R}^n} |\widehat{g\,d\sigma}|^2 w \lesssim \|Xw\|_{\infty} \int_{S^{n-1}} |g|^2$$

Random Potentials

$$V_\omega(x) = \omega_j \, v_j \qquad \forall x \in Q_j$$

$$\boxed{P(\omega_j = \pm 1) = \tfrac{1}{2}} \qquad \boxed{\sum_j |v_j|^{\frac{n}{2}+\delta} < \infty}$$



Thm (C.–Merz)

For $\gamma \leq \frac{n}{2}+1$, every eigenvalue z of $-\Delta + V_\omega$ satisfies

$$\frac{|z|^{\gamma}}{\langle |z|^{1/2} h\rangle \left[\log \langle |z|^{1/2} R\rangle\right]^{7/2}} \lesssim M \sum_j |v_j|^{\frac{n}{2}+\gamma}$$

except for ω in a set of measure $\leq \exp(-cM^2)$

Proof uses an entropy bound due to Bourgain and ε-removal machinery due to Tao.

For $n=2$, $\gamma=2$: $\underbrace{|Z|^2}_{\cdots} \lesssim M \int |V|^3$

Open Problem 3 Prove for $n=2$

$$\underbrace{|Z|^3}_{\cdots} \lesssim M \int |V|^4$$

This is related to a conjecture of Bourgain

Consequence for Bögli-C. Counterexample:

"Randomization on suff. small scale h destroys counterex. almost surely."

Sums of eigenvalues

Consider bounds of the form

$$\boxed{\left[\sum |z_j|^{\alpha}\left(\frac{\operatorname{dist}(z_j,\operatorname{spec}(-\Delta))}{|z_j|}\right)^{\beta}\right]^{\frac{\gamma}{\alpha}} \lesssim \int_{\mathbb{R}^n} |V|^{\frac{n}{2}+\gamma}}$$

- Demuth-Hansmann-Katriel : $\alpha=\gamma\geq 1$, $\beta=\gamma+\frac{n}{2}+\varepsilon$
- Frank-Sabin / Frank : Bounds w. $\gamma\leq\frac{1}{2}$, $\boxed{\beta=1,\ \frac{\gamma}{\alpha}<1}$
- C.-Keedle-Isach : Bounds w. $\boxed{\gamma<\frac{n}{2}}$, $\beta=1$, $\frac{\gamma}{\alpha}<1$ for <u>radial</u> potentials

E.g. $n=1$: $\sum \operatorname{dist}(z_j, \operatorname{spec}(-\Delta)) \lesssim \left(\int_{\mathbb{R}} |V|\right)^2$

Optimality (?)

① Is $\beta = 1$ optimal?

② Is $\frac{\gamma}{\alpha}$ (<1) optimal?

Mostly open problems!

Re: ②

- $n=1$ bound optimal
- $\frac{\gamma}{\alpha}$ probably optimal in many cases

Potential counterexample:

Assume $\exists$ family $(V_R) \subseteq L^\infty_{comp}(\mathbb{R}^n)$ s.t.

a) $\|V_R\|_\infty \leq 1$, $\operatorname{supp} V \subseteq B(0,R)$

b) $\forall\, 0 < \delta \ll 1$ $\exists\, C_\delta > 0$ s.t. $-\Delta + V_R$ has at least

$C_\delta^{-1} R^{2n-\delta}$ eigenvalues in the set

$$\Sigma_{R,\delta} = \left\{ z \in \mathbb{C} : \begin{array}{c} C_\delta^{-1} R^{2-c\delta} \leq \operatorname{Re} z \leq C_\delta R^{2+c\delta} \\ C_\delta^{-1} \leq \operatorname{Im} z \leq C_\delta \end{array} \right\}$$

Then the best possible bound is

$$\frac{\gamma}{\alpha} \leq \frac{n}{2(n+\alpha-\beta)}$$

Open Problem 4

Prove $\exists$ such a counterex.

In the random case $-\Delta + V_\omega$,

$$\sum_{1 \lesssim |z_j| \lesssim h^{-1/2}} \operatorname{dist}(z_j, \operatorname{spec}(-\Delta)) \lesssim \|(1+|x|)^{\frac{1}{2}+\varepsilon} V_\omega\|_\infty^{O(1)}$$

[C.-Merz]

$\forall \omega$ outside a set of small measure.

Open Problem 5: Prove this with $\|V_\omega\|_{L^{n+1-\varepsilon}}^{O(1)}$ the rhs.

Number of eigenvalues

One motivation for the counterex. is the following

upper bound* (C.) If $\|V\|_\infty \leq 1$, $\operatorname{supp} V \subset B(0,R)$, then

$$N(V) := \#\{\text{eigenvalues of } -\Delta + V\} \lesssim \boxed{R^{2n}}$$

Frank-Laptev-Safronov: $\forall\, \varepsilon > 0$

$$N(V) \lesssim \left(\varepsilon^{-1} \int_{\mathbb{R}^n} e^{\varepsilon|x|} |V(x)|^{\frac{n+1}{2}} dx \right)^2 \sim R^{2(n+1)}$$

* Also: Effective bounds on $\#\{\text{resonances in } B(0,r)\} = n_V(r)$

Zworski: $n_V(r) \leq C_V r^n$

$-\Delta_g$ on compact manifolds

(M,g) Riemannian mf. $\dim M = n \geq 2$

closed $(= \text{cpt.} + \partial M = \emptyset)$

$-\Delta_g \psi_j = \lambda_j^2 \psi_j \,, \quad 0 \leq \lambda_1 \leq \cdots \leq \lambda_j \longrightarrow +\infty$

Sogge:

$$\boxed{\|\psi_j\|_{L^p(M)} \lesssim \lambda_j^{\sigma(p)} \|\psi_j\|_{L^2(M)} \,, \quad 2 \leq p \leq \infty}$$

$$\sigma(p) = \max\left\{ \frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right),\; n\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{2} \right\}$$

Resolvent.

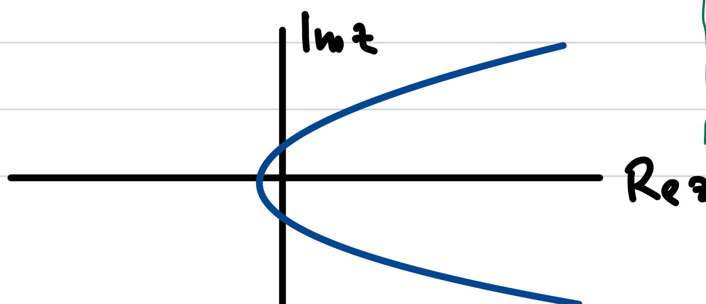
$$(-\Delta_g - z)^{-1} f = \sum_j (\lambda_j^2 - z)^{-1} \langle \psi_j, f \rangle \psi_j$$

Dos Santos Ferreira, Kenig, Salo:

$$\boxed{\| (-\Delta_g - z)^{-1} f \|_{L^{\frac{2n}{n-2}}(M)} \lesssim \| f \|_{L^{\frac{2n}{n+2}}(M)}}$$

$\forall z \in \mathbb{C}$ s.t. $\operatorname{Im}\sqrt{z} \gtrsim 1$.

[Bourgain - Sogge - Shao - Yao
Frank - Schimmer
Burq - Dos Santos Ferreira - Krupchyk]



Non-uniform bounds:

$$\|(-\Delta_g - z)^{-1}\|_{L^{p'}(M)\to L^{p}(M)} \lesssim \max\left\{\frac{|z|^{\sigma(p)}}{\operatorname{dist}(z,\operatorname{spec}(-\Delta_g))},\ |z|^{\sigma(p)-\frac{1}{2}}\right\}$$

$\forall\, p \in \left[2, \frac{2n}{n-2}\right]$, $\operatorname{Re} z \geq 1$

If $1/q = 1/p' - 1/p \in \left(n/2, \infty\right) \Rightarrow$

$$\boxed{\operatorname{spec}(-\Delta_g + V) = \bigcup_{j=1}^{\infty} B(\lambda_j^2, r_j)}\ ,\quad r_j = C\|V\|_{L^q(M)}\,\lambda_j^{\sigma(p)} \quad (j>1)$$

e.g. $n=2$: $r_j = C\|V\|_{\frac{3}{2}}\,\lambda_j^{1/6}$

For manifolds with negative curvature

one can improve r_j by a logarithm of λ_j

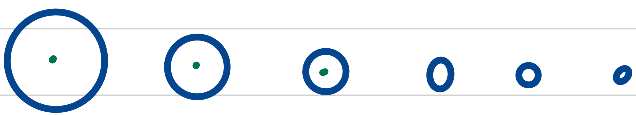
and on the torus by a power of λ_j.

Compare e.g. to Landau Hamiltonian in 2d:

$$L = \left(-i\partial_x + \frac{y}{2}\right)^2 + \left(-i\partial_y - \frac{x}{2}\right)^2$$

$$\operatorname{Spec}(L) = \{ \lambda_k = 2k+1 \ , \ k = 0,1,\dots \}$$

$$\operatorname{Spec}(L+V) \subseteq \bigcup_k B\left(\lambda_k, C\|V\|_{L^{3/2}(\mathbb{R}^2)} \lambda_k^{-\frac{1}{3}}\right)$$



shrinking clusters

Some general proof techniques

- Bounds for "size" usually based on Birman–Schwinger principle:

$$\boxed{z \in \operatorname{spec}(-\Delta+V) \iff -1 \in \operatorname{spec}\big(\underbrace{|V|^{\frac12}(-\Delta-z)^{-1}V^{\frac12}}_{=:BS(z)}\big)}$$

$$1 \leq \|BS(z)\| \leq \|V\|_q \, \|(-\Delta-z)^{-1}\|_{p' \to p} \qquad \left(\frac1q = \frac1{p'} - \frac1p\right)$$

- Reduced to resolvent bounds

- Localize in frequency: $R_0^{loc}(z) = \int_{\lambda \sim 1} (\lambda - z)^{-1} \frac{dE(\lambda)}{d\lambda}\, d\lambda$
and scale to $|z| = 1$

- If $\mathrm{supp}(V) \subset B_R$ $\rightsquigarrow$ Can blur in freq. on $\frac{1}{R}$ scale:

$$\int_{\lambda \sim 1} \left(|\lambda - z| + \tfrac{1}{R} \right)^{-1} \left\| |V|^{\frac{1}{2}} \frac{dE_\lambda}{d\lambda} V^{\frac{1}{2}} \right\| \, d\lambda$$

$$\lesssim \underline{\log R} \sup_{\lambda \sim 1} \left\| |V|^{\frac{1}{2}} \frac{dE_\lambda}{d\lambda} V^{\frac{1}{2}} \right\|$$

- $\frac{dE_\lambda}{d\lambda} = c(\lambda) \, \underline{E E^*}$, $\quad Eg(x) = \int_{S^{n-1}} e^{ix\cdot\xi} g(\xi) \, d\sigma(\xi)$

$\rightsquigarrow$ $\| E^* |V| E \|_{L^2(S^{n-1}) \to L^2(S^{n-1})}$

- $\underline{\varepsilon\text{-removal}}$ to get rid of $\log R$ and $R \to \infty$.

- Bounds for sums / number of e.v. usually based on analysis of determinants, e.g.

$$\det\left(\mathbb{1} + [BS(z)]^{\alpha}\right).$$

- Need good bounds on Schatten norms of $BS(z)$, e.g.

$$\|BS(z)\|_{\mathfrak{S}^{n+1}(\mathcal{B}(L^2(\mathbb{R}^n)))} \lesssim |z|^{-\frac{1}{n+1}} \|V\|_{L^{n+1}}$$

[Frank-Sabin]

- These are dual to bounds for ortho-normal functions, $\langle f_i, f_j \rangle = \delta_{ij}$:

$$\Big\| \Big(\sum_{j=1}^{N} |Ef_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L^{\frac{2(n+1)}{n-1}}} \lesssim N^{\frac{1}{2}-\varepsilon}$$

- Analogues on manifolds: replace E by

$$\Pi_\lambda = \mathbb{1}\left(\sqrt{-\Delta_g} \in [\lambda, \lambda+1]\right)$$

Thank you for your

attention !